Pure-Hollow Modules and Pure-Lifting Modules

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Abstract
Let \( R \) be a commutative ring with identity, and \( W \) be a unitary left \( R \)-module. In this paper we introduce and study a new class of modules called pure hollow (Pr-hollow) and pure-lifting (Pr-lifting). We give a fundamental, properties of these concepts also, we introduce some conditions under which the quotient and direct sum of Pr-lifting modules is Pr-lifting.

Keywords: small submodule, hollow module, lifting module, Pure hollow, pure lifting

Introduction
Throughout this paper \( R \) is a commutative ring with unity and all modules are unitary \( R \)-modules. In [1],[2] also in [3],[4],[5] researchers introduced the concepts of large-hollow module and large-lifting module with some conditions. Let \( W \) be an \( R \)-module, a submodule \( K \) of a module \( W \) is called small in \( W \), denoted by \( K \ll W \), if every submodule \( H \) of \( W \), we have \( K+H=W \), then \( H=W \) in [6],[7],[8] also recall that in [9],[10]. In [11] A proper submodule \( K \) of \( W \) is called pure small (Pr-small) submodule of \( W \) denoted by \( K\ll_{pr} W \), if \( K+H=W \), then \( H \) is a pure submodule of \( W \). In [12] \( H \) is named pure submodule denoted \( (H \leq_{pr} W) \) if \( H \cap IW=IH \) for every ideal \( I \) of \( R \). An \( R \)-module \( W \) is called hollow \( R \)-module if whole proper submodule \( K \) of \( R \)-module \( W \) is small submodule in \( W \) [13],[14]. Bethinks that, every small be Pr-small there for every hollow \( R \)-module is Pr-hollow, where an \( R \)-module \( W \) is called Pr-hollow \( R \)-module if every proper submodule \( K \) of \( R \)-module \( W \) is Pr-small submodule in \( W \). In [15] An \( R \)-module \( W \) is called lifting if for every submodule \( K \) of \( W \) there is a decomposition

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W = H ⊗ L and K ∩ L ≪ L, where L is a submodule of W, equivalently W is called lifting module in [16,17] if and only if for every submodule A of W there exists a submodule K of A, (K ≪ A) such that W = K ⊗ X with A ∩ X ≪ W, in [18, 19]. In [20] Let W be an R-module W, is called large-lifting (L-lifting), if for every submodule A of W there exists a submodule K of A, such that W = K ⊗ X and A ∩ X ≪ L, W where X is a submodule of W. in [21],[22], with [23], and [14], M be an R-module, a proper submodule N of M, is called large-small (L-small) submodule of M denoted by (N <L M), if N + K = M where K ≤ M, then K is essential (Large) submodule of M (K ≤ e M). Every lifting module is Large lifting module. This paper consists two sections, in section one we introduce the concept of pure hollow (Pr-hollow) modules, and we give some properties that we need it in this paper. In section two we give the concept of pure lifting (Pr-lifting) modules and some of its properties, such that an R-module W is said to be Pr-lifting, if for each submodule K of W there, exists a submodule L of K such that W = L ⊗ H and K ∩ H ≪ pr W where H submodule of W.

2. Pure Hollow Modules.

In this, we introduce pure-hollow modules as generalization of hollow modules, also We give some rules properties of these modules.

Definition 2.1:
An R-module W is called pure-hollow (Pr-hollow) module, if all submodule of W is Pr-small submodule of W.
Remarks and Examples 2.2:
1. Since every small is, Pr-small in [11], then every hollow module is Pr-hollow module.
2. The converse of (1) is not true in general, for example in Z6 as Z-module is not hollow modules but Z6 is Pr-hollow.
3. Every simple module is hollow, so is Pr-hollow module.
4. If W be semi simple module, then W is Pr-hollow module.
This clear since recall that in [11] every semisimple is Pr-small, then is Pr-hollow.
5. If W is Pr-hollow module, then a submodule of W need not Pr-hollow for example Zp∞ as Z-module is Pr-hollow since it is hollow in [15], but Z a submodule of Zp∞ is not Pr-hollow.
6. If W is Pr-hollow R-module, then it is decomposable.
Proof: Suppose K, H are submodule of W such that W = K + H either H or K is pure which means W decomposable.

Proposition 2.3: If W is direct summand, then every submodule module of W is Pr-hollow.
Proof: Suppose H, K are a proper, submodules of W, such that K ≤ H and H is direct summand of W. Since W is Pr-hollow, then H ≪ pr W and since H is direct summand of W by [7], if K ≪ pr W then K ≪ pr H implies H is, Pr-hollow.

Proposition 2.4: If W1 and W2 are R-modules such that φ: W1 → W2 is an epimorphism if W2 is Pr-hollow then W1 is Pr-hollow.
Proof: Let K be a proper submodule of W1, thus φ(K) is a proper submodule of W2, if not, then φ(K) = W2, so K = W1 and this contradiction. Since W2 is Pr-hollow, then φ(K) ≪ pr W2, and hence by [11], φ⁻¹(φ(K)) ≪ pr W1, so K ≪ pr W1 and hence W1 is Pr-hollow.

Corollary 2.5: If W is an R-module and H is a submodule of W if \( \frac{W}{H} \) is Pr-hollow, then W is Pr-hollow.
Proof: Let \( \frac{W}{H} \) be a submodule of \( \frac{W}{H} \) such that H < K ≤ W since \( \frac{W}{H} \) is Pr-hollow, then \( \frac{K}{H} \approx \frac{W}{H} \), by [11], we get K ≪ pr W we get W is Pr-hollow.

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Corollary 2.6: If \( W \) is Pr-hollow then \( \frac{W}{K} \) is Pr-hollow for \( K \leq W \).

Theorem 2.7: If \( W_1 \) and \( W_2 \) are R-modules and if \( W=W_1 \oplus W_2 \) such that \( W \) is duo modules, then \( W \) is Pr-hollow if and only if, \( W_1 \) and \( W_2 \) are Pr-hollow with \( H \cap W_i \neq W_i \) for \( i=1,2 \) and \( H \leq W \).

Proof: \( \rightarrow \) Clearly by [11].

\( \leftarrow \) Suppose \( H \) is a proper submodule of \( W \) and \( W_1 \) , \( W_2 \) are Pr-hollow. Since \( W \) is duo module, then \( H=(H \cap W_1) \oplus (H \cap W_2) \), hence \( H \cap W_1 \) and \( H \cap W_2 \) are proper submodule of \( W_1 \) and \( W_2 \), also since \( W_1 \) and \( W_2 \) are Pr-hollow, then \( H \cap W_1 \lhd_{pr} W_1 \) and \( H \cap W_2 \lhd_{pr} W_2 \) thus by [11] we get \((H \cap W_1) \oplus (H \cap W_2) \lhd_{pr} W_1 \oplus W_2 \) and cause \( H \lhd_{pr} W \).

Proposition 2.8: Let \( W=W_1 \oplus W_2 \) be an R- modules with \( W_1 \) and \( W_2 \) are submodules of \( W \) and \( W \) is distributive, then \( W \) is Pr-hollow if and only if, \( W_1 \) and \( W_2 \) are Pr-hollow, such that \( K \cap W_i \neq W_i \), \( i=1,2 \) and \( K \leq W \).

Proof: \( \rightarrow \) Clearly by Theorem 2.7.

\( \leftarrow \) Since \( W \) is distributive, then \( K=(K \cap W_1)+(K \cap W_2) \), where \( K \) be a proper submodule of \( W \). So, by proof of Theorem 2.7, we get \( W \) is Pr-hollow.

3. Pr-Lifting Modules.

We introduce Pr-lifting module as generalization of lifting module.

Recall that an R-module \( W \) is called lifting module if for every submodule \( K \) of \( W \), then exists a submodule \( L \) of \( K \) such that \( W=L \oplus H \) and \( K \cap H \lhd H \) in [18] and [15].

Definition 3.1: An R-module \( W \) is called Pure-lifting (Pr-lifting). If for each submodule \( K \) of \( W \) there exists a submodule \( L \) of \( K \) such that \( W=L \oplus H \) and \( K \cap H \lhd_{pr} W \), then \( H \) is submodule of \( W \).

Remarks 3.2: 1. All lifting module is Pr-lifting

Proof: Suppose \( W \) lifting R-module clearly every submodule \( K \) of \( W \) there exists submodule \( L \) such that \( W=L \oplus H \) and \( K \cap H \lhd W \) by [15], by [11] \( K \cap H \lhd_{pr} W \). The convers of remark is not true since \( K \cap H \lhd_{pr} W \) need not true in general \( K \cap H \lhd W \) by [11].

2. \( Z_6 \) is Pr-lifting module since the Pr-small submodule of \( Z_6 \) is \( 3Z_6 \) and \( 2Z_6 \), \( Z_6=2Z_6 \oplus 3Z_6 \) thus \( K=2Z_6 \) then \( L=\{0\} \), \( Z_6=\{0\} \oplus 2Z_6 \), \( Z_6 \cap 2Z_6=2Z_6 \lhd_{pr} Z_6 \), and \( K=3Z_6 \), \( L=\{0\} \), \( H=Z_6 \), \( Z_6=\{0\} \oplus 2Z_6 \lhd_{pr} 3Z_6 \), \( Z_6 \lhd_{pr} Z_6 \) but not lifting module since \( 2Z_6 \) not small in \( Z_6 \) and \( 3Z_6 \) not small of \( Z_6 \).

3. Every semisimple R-module is Pr-lifting. For, example \( Z_6 \) is semisimple implies \( Z_6 \) is Pr-lifting.

4. The convers is not true in general, for example: If \( W=Z_8 \) as Z-module the submodule \( 2Z_8=\{0,2,4,6\} \) of \( Z_8 \), \( 4Z_8 \) is a submodule of \( 2Z_8 \) such that \( Z_8=4Z_8 \oplus Z_8 \) and \( Z_8 \cap 2Z_8=2Z_8 \lhd_{pr} Z_8 \) also \( 4Z_8 \) has only \( \{0\} \) a submodule of \( Z_8 \) such that \( Z_8=\{0\} \oplus Z_8 \), \( Z_8 \cap 4Z_8=4Z_8 \lhd_{pr} Z_8 \). Thus \( Z_8 \) is Pr-lifting but not semi-simple

Proposition 3.3: Every Pr-hollow is Pr-lifting module.

Proof: Let \( W \) be Pr-hollow module and \( H \) be a submodule of \( W \) and suppose \( W=\{0\} \oplus W \) and \( H \cap W = H \lhd_{pr} W \) since \( W \) is Pr-hollow, then \( W \) is Pr-lifting module by Definition 3.1. Example for this, \( Z_6 \) as Z-module is Pr-hollow since all submodule of \( Z_6 \) is Pr-small implies Pr-lifting.

Remark 3.4: The converse of Proposition 3.3 is not true in general for example \( Z_{12} \) is Pr-lifting but not Pr-hollow.

Remark 3.5: Every local module is, Pr-lifting.

Proof: Since every local is hollow hence it is Pr-hollow by remarks and examples (2.2) (1), there, for it is Pr-lifting by Proposition 3.3.

Proposition 3.6: If \( W \) is an indecomposable, then \( W \) is Pr-hollow if and only if is, Pr-lifting.
Proposition 3.9: Any direct summand of $W$ is $Pr$-lifting module.

Theorem 3.7: If $W$ is an $R$-module, then the following are equivalent:
1. $W$ is $Pr$-lifting module.
2. All submodule $H$ of $W$ can be written as $H = K \oplus M$ where $K$ direct summand of $W$ and $M$ $\ll_{Pr} W$.
3. All submodule $H$ of $W$ there exists a direct summand $K$ of $W$ such that $K \leq H$ and $\frac{H}{K} \ll_{Pr} \frac{W}{K}$.

Proof: $\Rightarrow$ Let $W'$ be $Pr$-lifting and $H$ be a proper submodule of $W$ and suppose $K \leq H$ such that $W = K \oplus L$ where $L \leq W$ and $L \cap H \ll_{Pr} W$, since $W$ is indecomposable, then either $K = 0$ or, $K = W$. If $K = W$, then $H = W$ and this contradiction, so $K = 0$ and hence $W = L$, so $H = H \cap W = H \cap L \ll_{Pr} W$ hence $H \ll_{Pr} W$ and so $W$ is $Pr$-hollow.

Recall that in [17], if $H$ is a direct summand of $W$, then $H$ is pure in $W$.

Now, we give characterization of $Pr$-lifting module

Theorem 3.7: If $W$ is an $R$-module, then the following are equivalent:
1. $W$ is $Pr$-lifting module.
2. All submodule $H$ of $W$ can be written as $H = K \oplus M$ where $K$ direct summand of $W$ and $M$ $\ll_{Pr} W$.
3. All submodule $H$ of $W$ there exists a direct summand $K$ of $W$ such that $K \leq H$ and $\frac{H}{K} \ll_{Pr} \frac{W}{K}$.

Proof: $1 \Rightarrow 2)$ Suppose $W$ is $Pr$-lifting, let $H$ be a submodule of $W$, then there exists $K$ a submodule of $H$ such that $W = K \oplus L$ and $L \cap H \ll_{Pr} W$ where $L$ be submodule of $W$. Now $H = H \cap W$, $H = H \cap (K \oplus L) = K \oplus (H \cap L)$ by Modular Law. Suppose $V = K$ and $U = H \cap U$, then $V$ direct summand of $W$ and $U \ll_{Pr} W$.

$2 \Rightarrow 3)$ Assume $H \leq W$, and $V = \bigoplus \cup U$ such that $V$ direct summand of $W$ and $U \ll_{Pr} W$, impose $K \triangleleft V$ with $H \triangleleft V$ and $\frac{V}{K} \triangleleft \frac{V}{V}$ implies $V = V + U + K = U + K$ since $U \ll_{Pr} W$ so $V \leq_{Pr} W$, since $V$ direct summand of $W$, then $V$ is pure in $W$, hence $\frac{V}{K} \triangleleft \frac{V}{V}$ by [11], so $\frac{H}{K} \ll_{Pr} \frac{W}{K}$.

$3 \Rightarrow 1)$ Suppose $H$ be a submodule of $W$, there exist a submodule $K$ of $H$ such that $W = K \oplus L$ and $\frac{H}{K} \ll_{Pr} \frac{W}{K}$ by (3) hence $H \ll_{Pr} W$ by [7] since $H \cap L \leq H \leq W$, we get $H \cap L \ll_{Pr} W$ by [11]

Proposition 3.8: If $W$ is $Pr$-lifting module and $K, L$ are submodules of $W$ such that $W = K \oplus L$, then there exist $H$ is direct summand of $W$ such that $(H + K) \leq_{Pr} W$.

Proof: Suppose $W$ be $Pr$-lifting, then by theorem (3.6) $L = H + C$, where $H$ is a direct summand of $W$ and $C \ll_{Pr} W$, since $W = K \oplus L$, so $W = H + C + K$ but $C \ll_{Pr} W$, hence $H + K \leq_{Pr} W$.

Proposition 3.9: Any direct summand of $Pr$-lifting module is $Pr$-lifting module.

Proof: Suppose $W$ is $Pr$-lifting and $W = W_1 \bigoplus W_2$ let $H$ be a submodule of $W_1$ so $H \leq W$ since $W$ is $Pr$-lifting by Theorem 3.7 implies $H = H \bigoplus L$ where $V$ direct summand of $W$ and $L \ll_{Pr} W$.

By [7] we get $L \ll_{Pr} W_1$ since $L \leq W_1 \leq W$ since $W_1$ direct summand of $W$ and $L \ll_{Pr} W$, then $L \ll_{Pr} W_1$ to prove $V$ is direct summand of $W_1$. Since $W_1 \cap W = W_1 \cap (V \bigoplus U) = V \bigoplus (W_1 \cap U)$ by modular law clearly $V$ direct summand of $W_1$ there for $W_1$ is $Pr$-lifting.

Theorem 3.10: If $W$ is an $R$-module, then the following statements, are equivalent:
1. $W$ is $Pr$-lifting module.
2. For each submodule $H$ of $W$, there exists $\phi \in End(W)$ such that $\phi^2 = \phi$, $\phi(W) \leq H$ and $(1-\phi)(H) \ll_{Pr} W$.

Proof: $1 \Rightarrow 2)$ let $H$ be a submodule of $W$, then there exists a submodule $K$ of $H$ such that $W = K \oplus L$ and $L \ll_{Pr} W$ where $L$ be a submodule of $W$. Let $\phi: W \rightarrow K$ be a projection map, clearly $\phi^2 = \phi$, and $W = K \oplus L = \phi(W) \bigoplus (1-\phi)(W)$, $\phi(W) \leq H$. Now $(1-\phi)(H) = H \cap (1-\phi)(W) = H \cap L \ll_{Pr} W$, so $(1-\phi)(H) \ll_{Pr} W$.

$2 \Rightarrow 1)$ Let $H$ be a submodule of $W$, then there exists $\phi \in End(W)$ such that $\phi^2 = \phi$, $\phi(W) \leq H$ and $(1-\phi)(H) \ll_{Pr} W$. Clear that $W = \phi(W) \bigoplus (1-\phi)(W)$, now, let $K = \phi(W)$ and $L = (1-\phi)(W)$, hence $H \cap L = H \cap (1-\phi)(W)$, to show that, $H \cap (1-\phi)(W) = (1-\phi)(H)$, put $x = (1-\phi)(m) \in H \cap (1-\phi)(W)$, since $(1-\phi)^2 = (1-\phi)$ so $x = (1-\phi)^2(m) = (1-\phi)(m) \in (1-\phi)(H)$. 

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Now let $x = (1 - \varphi)(m) \in (1 - \varphi)(H)$; $m \in H$, then $x \in (1 - \varphi)(W)$, $x = (1 - \varphi)(m) \in H$, hence $x \in H \cap (1 - \varphi)(W)$ so $H \cap L = H \cap (1 - \varphi)(W) = (1 - \varphi)(H) \ll_{pr} W$, hence $H \cap L \ll_{pr} W$, so $W$ is Pr-lifting.

Proposition 3.11: If $W$ is indecomposable module, then $W$ is not Pr-lifting for every non trivial submodule $K$ of $W$.

Proof: Let $W$ be Pr-lifting, for every non trivial submodule $K$ of $W$, we have $K = H + L$ by Theorem 3.7, where $L \ll_{pr} W$ and $H$ direct summand of $W$, since $W$ is indecomposable this, contradiction, hence $W$ is not Pr-lifting.

Proposition 3.12: If $W$ is Pr-lifting module and $K$ is a submodule of $W$, then $\frac{W}{K}$ need not be Pr-lifting module for example $Z_4$ as Z-module is Pr-lifting and $2Z_4$ is submodule of $Z_4$ but $\frac{Z_4}{2Z_4} \approx Z_2$ since $Z_2$ is indecomposable this not Pr-lifting by Proposition 2.11.

Proposition 3.13: If $W$ is Pr-lifting module and $K$ is a submodule of $W$ such that for each direct summand $H$ of $W$ , $\frac{W}{K}$ direct summand of $\frac{W}{K}$, then $\frac{W}{K}$ is Pr-lifting.

Proof: Impose $L \leq \frac{W}{K}$ since $W$ is Pr-lifting there exist $H \leq L$ such that $W = H \oplus B$, $B \leq W \frac{L}{H}$, hence $\frac{W}{K}$ direct summand of $\frac{W}{K}$. implies $H + K$ direct summand of $W$, implies $\frac{L}{H + K} \ll_{pr} \frac{W}{H + K}$, then $H + K \leq L$ and then $\frac{H + K}{K}$, since $\frac{W}{K}$ direct summand of $\frac{W}{K}$ now, $\frac{L}{W + K} \ll_{pr} \frac{W}{H + K}$ implies $\frac{W}{K}$ is Pr-lifting assist Theorem 3.6.

Corollary 3.14: If $W$ is Pr-lifting and distributive module and $H$ is a submodule of $W$ then $\frac{W}{H}$ is Pr-lifting.

Proof: Impose $K$ is a direct summand of $W$, such that $W = K \oplus L$ for some submodule of $W$, hence $\frac{W}{K} \oplus \frac{L}{K} = \frac{K + H}{K} + \frac{L + H}{K}$ and since $W$ is distribution module, then $(K + H) \cap (L + H) = ((K + H) \cap L) + (K \cap L) + (K \cap H) + H = H$ hence $\frac{W}{H} = \frac{K + H}{K} \oplus \frac{L + H}{K}$ and assist Proposition 3.12, $\frac{W}{H}$ is Pr-lifting.

Recall that if $W = W_1 \oplus W_2$ be an R-module, then $\frac{W}{A} = \frac{W_1}{A} \oplus \frac{W_2}{A}$ for every fully invariant, submodules $A$ of $W$ [8].

Corollary 3.15: If $W$ is Pr-lifting module if $K$ is fully invariant submodule of $W$ then $\frac{W}{K}$ is Pr-lifting.

Proof: Impose $W = W_1 \oplus W_2$ since $W$ is fully invariant then $\frac{W}{A} = \frac{W_1}{A} \oplus \frac{W_2}{A}$ for all submodule $A$ of $W$ is full invariant that assist Proposition 3.13 we have $\frac{W}{A}$ is Pr-lifting.

Proposition 3.16: If $W = W_1 + W_2$ is an R-module, such that $=\text{Ann}_R(W_1) + \text{Ann}_R(W_2)$, if $W_1$ and $W_2$ are Pr-lifting then $W$ is Pr-lifting.

Proof: Impose $H = H_1 \oplus H_2$ is a submodule of $W$ since $R = \text{Ann}_R(W_1) + \text{Ann}_R(W_2)$ for some $H_1 \leq W_1$, $H_2 \leq W_2$ assist theorem (3.6) we have $H_1 = K_1 + L_1$ and $H_2 = K_2 + L_2$ since $H_1$ and $H_2$ are Pr-lifting such that $K_1$ direct summand of $W_1$ and $K_2$ direct summand of $W_2$, $L_1 \ll_{pr} W_1$, $L_2 \ll_{pr} W_1$, now $K_1 \oplus K_2$ is direct summand of $W_1 \oplus W_2 = W$, nackedly $L_1 \ll_{pr} W_1$, $L_2 \ll_{pr} W_2$ implies $L_1 \oplus L_2 \ll_{pr} W_1 \oplus W_2 = W$ hence $W$ is Pr-lifting assist definition of Pr-lifting.

Corollary 3.17: If $W = W_1 \oplus W_2$ is duo module if $W_1$ and $W_2$ are Pr-lifting then $W$ is Pr-lifting.

Proof: Impose $W$ is duo module and $K$ is a submodule of $W$, then $K$ is fully invariant, hence $K = K \cap W = K \cap (W_1 \oplus W_2) = (K \cap W_1) \oplus (K \cap W_2)$ and assist Proposition 3.15 we have $W$ is Pr-lifting.
Proposition 3.18: Let \( W = \bigoplus_{i \in I} W_i \) be a fully stable module, if \( W_i \) is Pr-lifting for each \( i \in I \) then \( W \) is Pr-lifting.

Proof: Suppose \( K \) is a submodule of \( W \), \( K = \bigoplus_{i \in I} (K \cap W_i) \) [7] since \( K \cap W_i \leq W_i \) and \( W_i \) is Pr-lifting then \( K \cap W_i = L_i \cap H_i \) where \( L_i \) and \( H_i \) are direct summand of \( W_i \) and \( H_i \subseteq_{pr} W_i \) hence \( K = \bigoplus_{i \in I} (K \cap W_i) = \bigoplus_{i \in I} (L_i \cap H_i) = \bigoplus_{i \in I} (L_i) + \bigoplus_{i \in I} (H_i) \), we can readily that \( \bigoplus_{i \in I} (L_i) \) is direct summand of \( \bigoplus_{i \in I} W_i = W \) and since \( H_i \subseteq_{pr} W_i \) then \( \bigoplus_{i \in I} (H_i) \subseteq_{pr} \bigoplus_{i \in I} W_i = W \) assist [21] hence assist Theorem 3.6), we have \( W \) is Pr-lifting.

Proposition 3.19: If \( W \) is faithful, finitely generated and multiplication \( R \)-module then \( W \) is Pr-lifting if and only if \( R \) is Pr-lifting.

Proof: \( \Rightarrow \) Impose \( W \) is Pr-lifting and \( J \) is an ideal of \( R \), assist, Theorem 3.6, there exists \( L \) direct summand of \( W \) and \( K \subseteq_{pr} W \) such that \( H = JW = L + K \), \( K \leq W \) and \( H \leq W \), since \( W \) is multiplication, then there exists \( I \) and \( F \) are ideal of \( R \) such that \( K = IW \) and \( L = FW \), so \( JW = IW + FW = (J + F)W \), and since \( W \) is faithful, finitely generated and multiplication module, then we get \( J = IF \) assist [18], to show \( I \) direct summand of \( R \), impose \( W = L \oplus B \) and \( B = AW \) for some \( A \) is ideal of \( R \) so \( RW = W = IW \oplus AW = (I + A)W \) and since \( W \) is cancelation [14] hence \( R = I + A \), now to show \( I \cap A = 0 \), since \( W \) is faithful, finitely generated and multiplication, then \( 0 = IW \cap AW = (I \cap A)W \) so \( I \cap A = 0 \), and hence \( I \) is direct summand of \( R \) and since \( K = FW \subseteq_{pr} W \) then \( F \subseteq_{pr} R \) assist Theorem 3.12, so \( R \) is Pr-lifting.

\( \Leftarrow \) If \( R \) is Pr-lifting and \( H \) is a submodule of \( W \) where \( W \) is \( R \)-module, since \( W \) is multiplication then there exists \( J \) an ideal of \( R \) such that \( H = JW \), and there exist \( I \) is direct summand of \( R \) and \( K \subseteq_{pr} R \) such that \( J = I + K \) hence \( JW = (I + K)W = IW + KW \) so, \( H = IW + KW \), to show \( IW \) is direct summand of \( W \), impose \( R = I + U \) for some \( U \) is ideal of \( R \), hence \( W = RW = (I + U)W = IW + UW \) and since \( W \) is faithful finitely generated and multiplication, then \( IW \cap UW = (I \cap U)W = 0W \), so \( IW \) is direct summand of \( W \) and since \( K \subseteq_{pr} R \) then \( U \subseteq_{pr} W \) assist Theorem 3.12, hence \( W \) is Pr-lifting.

4. Conclusions

We will try to generalize the twiggled of Pr-hollow \( R \)-module to some other concepts in future works. In this lucubration, the twiggled of Pr-hollow \( R \)-modules is studied as a generalization of hollow submodule and some properties of this concepts are investigated also, we lucubration Pr-lifting as generalization of lifting module with some properties such as:

1. Every hollow submodule of \( R \)-module \( W \) is Pr-hollow.
2. Every simple module is Pr-hollow.
3. If \( W \) be semisimple module, then \( W \) is Pr-hollow module.
4. If \( W \) is Pr-hollow module, then not all submodule is Pr-hollow module.
5. Every Pr-hollow module is decomposable.
6. All lifting module is Pr-lifting module.
7. Every semisimple \( R \)-module is Pr-lifting module.
8. Every hollow module is Pr-lifting.
9. Every Pr-local is, Pr-lifting.
10. If \( W \) is an indecomposable, then \( W \) is Pr-hollow if and only if \( W \) is Pr-lifting.
11. Any direct summand of Pr-lifting module is Pr-lifting module.
12. If \( W \) is indecomposable module, then \( W \) is not Pr-lifting for every non-trivial submodule \( K \) of \( W \).

References


