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Pseudo Rho-Algebras in Fields with their Applications Using Polynomials with Two Variables

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Abstract
The main purpose of this work is to present the concept of pseudo Rho-algebra as a generalization of the Rho-algebra. We investigate this type behaves nicely in terms of common mappings like homomorphisms and direct products. In fact, we can show identify kernel objects and the fundamental theorem of homomorphisms using Rho-ideals in this theory. Also, some important classes of pseudo Rho-algebra and their applications are investigated and studied. Moreover, the necessary characteristics for or tantamount to membership in major subtypes of pseudo Rho/d-algebras under these cases are identified. The Rho/d-algebras and pseudo Rho/d-algebras are generated using some new classes of polynomials with two variables that are introduced in this work, like commutative identical, associative identical, regular identical, proper conditionally identical, and improper conditionally identical. Finally, the category of pseudo Rho/d-algebras can be classified using a conditionally couple, giving us additional flexibility in dealing with two different fields.

Keywords: ρ −algebras, homomorphisms, pseudo d −algebras, polynomials.

جبر–رو المستعار في الحقول مع تطبيقاته باستخدام متعددات الحدود ذات المتغيرين

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الخلاصة
الغرض الرئيسي من هذا العمل هو تقديم مفهوم جبر–رو المستعار كتعميم لمفهوم جبر–رو, كذلك تم التحقق من سلوك هذا النوع الجديد ضمن التطبيقات الشائعة مثل الهومومورفزم و الضرب الديكارتيزي, في الواقع ممكن ان نبين تحديد هيئة النواة و النظرية الاساسية للهومومورفزم بأستخدام امثلية –رو في هذه النظرية , كذلك العديد من الاصناف المهمة من جبر– رو المستعار تم استكشاف و توضيح الأمثلة التي تحدث في المواقف المختلفة وتم تحديد الخصائص الضرورية أو التي ترقى إلى العضوية في الأنواع الفرعية الرئيسية من الجبر– Roh/d المستعار تحت هذه الحالات . حيث في هذا العمل تم توليد جبر– Roh/d و كذلك جبر–Roh/d الملحقة بواسطة بعض انواع متعددات الحدود ذات المتغيرين حيث هذه الانواع تم تقديمها في هذا البحث مثل المتطابق الابدالي و المتطابق التجميعي و المتطابق المنتظم و المتطابق المشروط الفعلي و المتطابق المشروط الغير فعلي. أخيراً, صنف من جبور– Roh/d ممكن تحديدها بواسطة الزوج المشروط وهذا يعطينا مرونة اضافية اكثر في التعامل مع حقلين مختلفين.

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1. Introduction

BCK −algebras and BCI −algebras are two types of abstract algebras introduced by Iseki and Tanaka [1,2]. A broad family of abstract algebras known as BCH −algebras [3, 4] are defined by Hu and Li. Also, they demonstrated the BCI −algebra type which is a legitimate subtype of the BCH −algebra type.

The introduction of MV −algebras [5] gave an algebraic semantics for the Lukasiewicz logics. BCK −algebras [6] are related to various other areas of study, including lattice ordered groups, MV −algebras. Georgescu and Iorgulescu [7] developed a pseudo MV −algebras. Next, they defined pseudo BCK −algebras as extensions of BCK −algebras [8]. After that, Jun [9] study pseudo BCK −algebras. He discovered criteria that allow pseudo BCK −algebras to be $\wedge$ −semi−lattice ordered. In 2018, the notion of Rho-algebras (ρ −algebras) is investigated and studied [10]. Next, some applications in algebra are discussed [11-20]. In recent years, some studies are applied in many classes of algebras [21-32].

As a result, in this study, we present the concepts of pseudo ρ −algebras and connected ρ −algebras. Also, we mentioned that any ρ −algebra is pseudo ρ −algebras. It is worth noting that the class of pseudo ρ −algebras behaves well in comparison to normal constructs, standard homomorphism results, kernels specified as ideals, and the fundamental isomorphism theorem are examples. We present three substantial types of these algebras. We discover both necessary and sufficient criteria for the formation of pseudo ρ −algebras. Conditionally identical, commutative identical, proper conditionally identical, and improper conditionally identical, are found in examples given below, as well as a lot of minor examples in other situations. Additional observations concerning ρ −algebras have found when we discuss these groups of cases. As a result, the concept of necessarily identical polynomials $\mathcal{A}(t,r)$ over fields $\mathcal{R}$ manifest to be of relevance apart from the theory connected with polynomially appointed ρ −algebras over fields.

2.Preliminaries

Definition 2.1. [18]

Let F be a set with a constant $0 \in F$, and a binary operation ($\star$). We say $(F,\star,0)$ is a d-algebra if such that:

i. $t \star t = 0$,

ii. $0 \star t = 0$,

iii. $t \star r = 0$ and $r \star t = 0$ imply that $t = r$, $\forall\, t, r \in F$.

Definition 2.2. [10]

Let $(F,\star,0)$ be d-algebra. We say $(F,\star,0)$ is a ρ-algebra (ρ–A) if such that $t \star r = r \star t \neq 0$, $\forall\, t \neq r \in F - \{0\}$.

Example 2.3. Let $F = \{0, \sigma_1, \sigma_2, \sigma_3\}$ be a ρ-algebra, see Table 1.

Table 1: $(F,\star,0)$ is a(ρ–A)

$\star$	0	σ_1	σ_2	σ_3
0	0	0	0	0
σ_1	σ_1	0	σ_2	σ_3
σ_2	σ_2	σ_2	0	σ_3
σ_3	σ_3	σ_3	σ_3	0

Definition 2.4. [10] Let $L:(W,o,\propto_W)\to(C,*,\propto_C)$ be a mapping of ρ-algebras. L is said to be a ρ −homomorphism if $L(sot)=L(s)*L(t), \forall s,t\in W$.

3. Pseudo ρ −algebras and pseudo ρ-ideals

The homomorphism of pseudo ρ-algebras with some results of their kernels and images are shown in this section.

Definition 3.1.

Let $0\in\mathcal{F}$ with two binary operations ($\star$) and ($\triangle$). We say $(\mathcal{F},\star,\triangle,0)$ is a pseudo ρ-algebra if such that:

i. $t\star t=t\triangle t=0$,
ii. $0\star t=0\triangle t=0$,
iii. $t\star r=r\triangle t=0$ imply that $t=r, \forall t,r\in\mathcal{F}$,
iv. $t\star r=r\triangle t\neq 0, \forall\, t\neq r\in\mathcal{F}-\{0\}$.

Note that if $(\mathcal{F},\star,0)$ is a δ-algebra , then letting $t\triangle r=t\star r$, produces a pseudo ρ-algebra $(\mathcal{F},\star,\triangle,0)$. Hence, every ρ-algebra is a pseudo ρ-algebra in a natural way.

Example 3.2. Let $\mathcal{F}=\{0,x,y,z\}$ be a set. Define two binary operations ($\star$) and ($\triangle$) on $\mathcal{F}$ as following tables (2) and (3):

Table 2: $(\mathcal{F},\star,0)$ is not (ρ–A)

$\star$	0	x	y	z
0	0	0	0	0
x	x	0	y	x
y	y	y	0	y
z	z	x	x	0

Table 3: $(\mathcal{F},\triangle,0)$ is not (ρ–A)

$\triangle$	0	x	y	z
0	0	0	0	0
x	y	0	y	x
y	x	y	0	x
z	z	x	y	0

Hence $(\mathcal{F},\star,0)$ and $(\mathcal{F},\triangle,0)$ are not ρ-algebras, however $(\mathcal{F},\star,\triangle,0)$ is a pseudo ρ-algebra.

Definition 3.3. Let $(\mathcal{F},\star,\triangle,0)$ be a pseudo ρ-algebra and $\emptyset\neq V\subseteq\mathcal{F}$ is called a pseudo ρ-subalgebra of $\mathcal{F}$ if $t\star r, t\triangle r\in V$ whenever $t\in V$ and $r\in V$.

Definition 3.4. Let $(\mathcal{F},\star,\triangle,0)$ be a pseudo ρ-algebra and $\emptyset\neq V\subseteq\mathcal{F}$ is called a pseudo ρ -ideal of $\mathcal{F}$ if it such that:

(i) $t,r\in V$ imply $t\star r\in V$ and $t\triangle r\in V$,
(ii) $t\star r, t\triangle r\in V$ and $r\in V$ imply $t\in V, \forall\, t,r\in V$.

The following example explains the definitions above.

Example 3.5. Let $\mathcal{F}=\{0,x,y,z\}$ be a set. Define two binary operations ($\star$) and ($\triangle$) on $\mathcal{F}$ as following tables (4) and (5):

Table 4: $(\mathcal{F},\star,0)$ is not (ρ–A)

$\star$	0	x	y	z
0	0	0	0	0
x	x	0	y	x
y	y	y	0	x
z	z	x	y	0

Table 5: $(\mathcal{F},\triangle,0)$ is not (ρ–A)

$\triangle$	0	x	y	z
0	0	0	0	0
x	x	0	y	x
y	y	y	0	y
z	z	x	x	0

Then $(\mathcal{F},\star,0)$ and $(\mathcal{F},\triangle,0)$ are not ρ-algebras, but $(\mathcal{F},\star,\triangle,0)$ is a pseudo ρ-algebra, where $V=\{0,x\}$ is pseudo ρ-algebra and $V=\{0,z\}$ is pseudo ρ-ideal.

Lemma 3.6. Assume that $(\mathcal{F},\star,\triangle,0)$ is a pseudo ρ-algebra. Then any pseudo ρ-ideal is pseudo ρ-subalgebra.

Proof: Let $(\mathcal{F},\star,\triangle,0)$ be a pseudo ρ-algebra and V be pseudo ρ-ideal. Then condition (i) of Definition 3.4 is hold, since V is a pseudo ρ-ideal. Hence V is a pseudo ρ-subalgebra.■

Remark 3.7: The convers of Lemma 3.6 is not true in general from Example 3.5, $V=\{0,x\}$ is a pseudo ρ-subalgebra, but not a pseudo ρ-ideal, since $z\star x=x\in V, z\notin V, z\triangle x=x\in V, z\notin V$.

Lemma 3.8: Let V be a pseudo ρ-ideal of a pseudo ρ-algebra $\mathcal{F}$. If $t\in V$ with $r\star t=0$ and $r\triangle t=0$, then $r\in V$.

Proof: It is clear that $0\in V$. Since $V\neq\emptyset$, then there exists x in V and hence from (i) of Definition 3.4, we get $x\star x=x\triangle x\in V$, but $x\star x=x\triangle x=0$ by (i) of Definition 3.1. Now, let $t\in V$ with $r\star t=0$ and $r\triangle t=0$. Therefore, we have $r\star t\in V$ and $r\triangle t\in V$ and from (ii) in Definition 3.4, we consider that $r\in V$. ■

Definition 3.9. Assume that $(\mathcal{F},\star,0)$ is a ρ-algebra. We say $\mathcal{F}$ is a regular ρ-algebra if $t\star 0=t$, for any $t\in\mathcal{F}$.

Definition 3.10. Let $(\mathcal{F},\star,\triangle,0)$ be a pseudo ρ-algebra. We say it is a regular pseudo ρ-algebra, if it is a regular.

Lemma 3.11. Let $T:(\mathcal{F},\star_1,\triangle_1,0)\to(P,\star_2,\triangle_2,0)$ be a homomorphism from ρ-algebra into regular pseudo ρ-algebra, then KerT is a pseudo ρ-ideal of $\mathcal{F}$.

Proof. We need to show that conditions (i) and (ii) in Definition 3.4 are hold. Let $t,r\in$ Ker T, thus$T(t)=T(r)=0$ and hence $T(t\star_1 r)=T(t)\star_2 T(r)=0$ and $T(t\triangle_1 r)=T(t)\triangle_2 T(r)=0$ by using (i) of Definition 3.1. Then $t\star_1 r$, $t\triangle_1 r\in$ Ker T and hence condition (i) is hold. Now, let $t\star_1 r, t\triangle_1 r\in$ Ker T and $r\in$ Ker T, we need only to show that $t\in$

Ker T. $\mathrm{T}(t \star_1 r) = \mathrm{T}(t) \star_2 \mathrm{T}(r) = 0$ and $\mathrm{T}(t \triangle_1 r) = \mathrm{T}(t) \triangle_2 \mathrm{T}(r) = 0$, as $t \star_1 r, t \triangle_1 r \in$ Ker T. Also, $\mathrm{T}(r) = 0$, because $r \in$ Ker T. But $\mathrm{T}(t) \star_2 0 = \mathrm{T}(t)$ and $\mathrm{T}(t) \triangle_2 0 = \mathrm{T}(t)$, as $(\mathrm{P},\star_2,\triangle_2, 0)$ is regular pseudo ρ-algebra. Therefore, $0 = \mathrm{T}(t) \star_2 \mathrm{T}(r) = \mathrm{T}(t) \star_2 0 = \mathrm{T}(t)$ and $0 = \mathrm{T}(t) \triangle_2 \mathrm{T}(r) = \mathrm{T}(t) \triangle_2 0 = \mathrm{T}(t)$. This implies $\mathrm{T}(t)$=0 and hence $t \in$ Ker T. So, condition (ii) is hold. Then KerT is a pseudo ρ-ideal of $\mathcal{F}$.

Lemma 3.12. Suppose that $\mathrm{T}: (\mathcal{F},\star_1,\triangle_1, 0) \to (\mathrm{P},\star_2,\triangle_2, 0)$ is a homomorphism of pseudo ρ-algebras. Then, $t \star_1 r, r \triangle_1 t \in$ KerT if and only if $\mathrm{T}(t) = \mathrm{T}(r)\ \forall\ t, r \in \mathcal{F}$.

Proof. If $t \star_1 r, r \triangle_1 t \in$ KerT, then $\mathrm{T}(t) \star_2 \mathrm{T}(r) = \mathrm{T}(t \star_1 r) = 0$ and $\mathrm{T}(r \triangle_1 t) = \mathrm{T}(r) \triangle_2 \mathrm{T}(t) = 0$ and thus, by (iii) of Definition 3.1, $\mathrm{T}(t) = \mathrm{T}(r)$.
Conversely, let $\mathrm{T}(t) = \mathrm{T}(r)$, and $t \star_1 r, r \triangle_1 t \notin$ KerT, then $0 \neq \mathrm{T}(t \star_1 r) = \mathrm{T}(t) \star_2 \mathrm{T}(r)$ and $0 \neq \mathrm{T}(t \triangle_1 r) = \mathrm{T}(t) \triangle_2 \mathrm{T}(r)$. However, $\mathrm{T}(t) = \mathrm{T}(r)$. Hence, $\mathrm{T}(t) \star_2 \mathrm{T}(t) \neq 0$ and $\mathrm{T}(t) \triangle_2 \mathrm{T}(t) \neq 0$. But this is a contradiction with (i) in Definition 3.1.

Lemma 3.13. Suppose that $\mathrm{T}: (\mathcal{F},\star_1,\triangle_1, 0) \to (\mathrm{P},\star_2,\triangle_2, 0)$ is a homomorphism of pseudo ρ-algebras, and $(\mathrm{P},\star_2,\triangle_2, 0)$ is regular pseudo ρ-algebra. If $r \in$ KerT, Then $t \star_1 (t \star_1 r)$, $(t \star_1 r) \star_1 t$, $t \triangle_1 (t \star_1 r)$, $(t \star_1 r) \triangle_1 t$, $t \star_1 (t \triangle_1 r)$, $t \triangle_1 (t \triangle_1 r)$, $(t \triangle_1 r) \triangle_1 t \in$ KerT.
Proof. Straightforward.

Lemma 3.14. Suppose that $\mathrm{T}: \mathcal{F} \to \mathcal{G}$ is a homomorphism of pseudo ρ-algebras**.** Then T is a monomorphism if and only if KerT $= \{0\}$.

The proofs for the following propositions are omitted since they are simple.

Proposition 3.15. Suppose that $\mathcal{F}$, P and $\mathcal{D}$ are three pseudo ρ-algebras, and $\mathcal{K}: \mathcal{F} \to \mathrm{P}$, $\mathcal{T}: \mathcal{F} \to \mathcal{D}$ are two homomorphisms of pseudo $\delta -$ algebras, where $\mathcal{K}$ is onto. If Ker$\mathcal{K} \subseteq$ Ker$\mathcal{T}$, then $\mathrm{T} \circ \mathcal{K} = \mathcal{T}$ for some $\mathrm{T}: \mathrm{P} \to \mathcal{D}$ and T is unique.

Proposition 3.16. Suppose that $\mathcal{F}$, P and $\mathcal{D}$ be three pseudo ρ-algebras, and $\mathcal{T}: \mathcal{F} \to \mathcal{D}$, $\mathcal{K}: \mathrm{P} \to \mathcal{D}$ are two homomorphisms of pseudo $\rho -$ algebras, where $\mathcal{K}$ is one to one. If Img$\mathcal{T} \subseteq$ Img$\mathcal{K}$, then $\mathcal{K} \circ \mathrm{T} = \mathcal{T}$ for some $\mathrm{T}: \mathcal{F} \to \mathrm{P}$ and T is unique.

It should be noted that typical projections from the sum or direct product of pseudo ρ-algebras to their components are homomorphisms of pseudo $\rho -$ algebras with the usual form kernels.

4. Polynomials with its applications in $d/\rho -$algebras over fields

Suppose that E is a field and $\mathcal{A}(t, r)$ is a polynomial with coefficients in the field E and in two variables. For example, let $\mathrm{E} = \mathcal{R}$ be the field of real numbers, hence $\mathcal{A}(t, r) = t^5 - r^2$ and $B(t, r) = tr + a$ for some $a \in$ E are such polynomials. Moreover, for $(1, -2) \neq (1,2)$, we have $\mathcal{A}(1, -2) = \mathcal{A}(1,2)$. Also, $B(1,2) = B(2,1)$, but $(1,2) \neq (2,1)$. Hence, $\mathcal{A}(t, r)$ and $B(t, r)$ are not injective functions. However, $C(t, r) = 3^t 2^r$ is an injective function.

Definition 4.1. Let $\mathcal{A}(t, r)$ be a polynomial. We say it is conditionally identical if there is an $\mathfrak{a} \in$ E such that $\mathcal{A}(\mathfrak{a}t, r) = \mathcal{A}(\mathfrak{a}r, t)$, for $t, r \in$ E. … (1)

Example 4.2. If $\mathcal{A}(t,r) = 2t^2r^2 + 1$, the Eq. (1) becomes $\mathcal{A}(\mathfrak{a}t,r) = 2(\mathfrak{a}t)^2r^2 + 1 = 2(\mathfrak{a}r)^2t^2 + 1 = \mathcal{A}(\mathfrak{a}r,t)$. Then the polynomial $\mathcal{A}(t,r)$ is conditionally identical.

Definition 4.3. Let $\mathcal{A}(t,r)$ be a conditionally identical for $\mathfrak{a}$. We say it is a semi injective for $\mathfrak{a}$ if such that $\mathcal{A}(\mathfrak{a},t) = \mathcal{A}(\mathfrak{a},r) \rightarrow t = r$, for any $t,r \in$ E. …(2)

Definition 4.4. let $\mathcal{A}(t,r)$ be a conditionally identical for $\mathfrak{a}$. We say

(i) $\mathcal{A}(t,r)$ is commutative identical for $\mathfrak{a}$ if such that $\mathcal{A}(\mathfrak{a}t,\mathfrak{a}r) = \mathcal{A}(\mathfrak{a}r,\mathfrak{a}t)$, for any $t,r \in$ E. …(3)

(ii) $\mathcal{A}(t,r)$ is associative identical for $\mathfrak{a}$ if such that $\mathcal{A}(\mathfrak{a}t,\mathfrak{a}r) = \mathcal{A}(\mathfrak{a}tr,\mathfrak{a}) \rightarrow t = r$, for any $t,r \in$ E. …(4)

(iii) $\mathcal{A}(t,r)$ is regular identical for $\mathfrak{a}$ if $\mathcal{A}(t,r)$ such that Eq.(3) and Eq.(4).

Example 4.5. The polynomial $\mathcal{A}(t,r) = (t)^2(2r)^2 + 1$ in Example 4.2 is a commutative identical and an associative identical for any $\mathfrak{a} \in \mathrm{E} = \mathcal{R}$, the real field. Then $\mathcal{A}(t,r) = (t)^2(2r)^2 + 1$ is a regular identical. Also, let $\mathcal{A}(t,r) = 1 + t + r$, for any $t,r \in \mathrm{E} = \mathcal{R}$, Eq. (1) becomes $\mathcal{A}(\mathfrak{a}t,r) = \mathcal{A}(\mathfrak{a}r,t) \rightarrow 1 + \mathfrak{a}t + r = 1 + \mathfrak{a}r + t$. Therefore, $\exists \mathfrak{a} = 1 \in \mathrm{E} = \mathcal{R}$ satisfies Eq.(1). In other side, it is semi injective, since it satisfies Eq.(2) for $\mathfrak{a} = 1$. So, it is precisely the equality permitted. Nevertheless, $t,r \in \mathrm{E} = \mathcal{R}$ may be taken arbitrary and hence Eq.(3) also is hold. Then $\mathcal{A}(t,r) = 1 + t + r$ is commutative identical as well. Consider another polynomial $\mathcal{A}(t,r) = t^2 + 4r^2 - 1$ over $\mathrm{E} = \mathcal{R}$, the real field. Hence $\exists \mathfrak{a} = 2 \in \mathrm{E} = \mathcal{R}$ satisfies Eq.(1). Also, the Eq. (3) becomes $(\mathfrak{a}t)^2 + 4(\mathfrak{a}r)^2 - 1 = (\mathfrak{a}r)^2 + 4(\mathfrak{a}t)^2 - 1$, but at $\mathfrak{a} = 2$ Eq. (3) does not hold for any $t \notin \{r, -r\}$. Hence $\mathcal{A}(t,r) = t^2 + 4r^2 - 1$ is not commutative identical for $\mathfrak{a} = 2$. Now suppose that for the field E, if $\partial \neq 0$, we set $t\#r = \partial(t-a)(a-r)[\mathcal{A}(\mathfrak{a},t) - \mathcal{A}(\mathfrak{a},r)]^2 + \mathfrak{a}$, where $\mathcal{A}(t,r)$ is a polynomial in two variables. This notation will be applied in lemma 4.6.

Lemma 4.6. Assume that $\mathcal{A}(t,r)$ is a semi-injective for $\mathfrak{a} \in$ E. If $t\#r = \mathfrak{a} = r\#t$, then $r = t$.

Proof. Assume $t\#r = \mathfrak{a} = r\#t$ and $\mathcal{A}(t,r)$ is a semi-injective for $\mathfrak{a} \in$ E with $t \neq r$. Then $t\#r = \partial(t-\mathfrak{a})(\mathfrak{a}-r)[\mathcal{A}(\mathfrak{a},t) - \mathcal{A}(\mathfrak{a},r)]^2 + \mathfrak{a} = \mathfrak{a}$. Therefore, $\partial(t-\mathfrak{a})(\mathfrak{a}-r)[\mathcal{A}(\mathfrak{a},t) - \mathcal{A}(\mathfrak{a},r)]^2 = 0$. Also, $\partial(r-\mathfrak{a})(\mathfrak{a}-t)[\mathcal{A}(\mathfrak{a},r) - \mathcal{A}(\mathfrak{a},t)]^2 = 0$, but $\mathcal{A}(\mathfrak{a},r) \neq \mathcal{A}(\mathfrak{a},t)$ (since $t \neq r$), and hence $\partial[\mathcal{A}(\mathfrak{a},r) - \mathcal{A}(\mathfrak{a},t)]^2 \neq 0$. We consider that

$$(t-\mathfrak{a})(\mathfrak{a}-r) = 0. \qquad \text{……. (4)}$$

and

$$(r-\mathfrak{a})(\mathfrak{a}-t) = 0. \qquad \text{……. (5)}$$

Then from (4) and (5), we get $r = t$. ■

Lemma 4.7. Assume that $\mathcal{A}(t,r)$ is semi-injective for $\mathfrak{a} \in$ E. Then (E, #, $\mathfrak{a}$) is d −algebra.

Proof: We need to show the following:

(i) $t \# t = \mathfrak{a}$, (ii) $\mathfrak{a} \# t = \mathfrak{a}$, (iii) $t \# r = \mathfrak{a}$, and $r \# t = \mathfrak{a} \Rightarrow t = r$, $\forall t,r \in$ E.

$(\boldsymbol{i})$ $t \# t = \partial(t-\mathfrak{a})(\mathfrak{a}-t)[\mathcal{A}(\mathfrak{a},t) - \mathcal{A}(\mathfrak{a},t)]^2 + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$,

$(\boldsymbol{ii})$ $\mathfrak{a} \# t = \partial(\mathfrak{a}-\mathfrak{a})(\mathfrak{a}-r)[\mathcal{A}(\mathfrak{a},\mathfrak{a}) - \mathcal{A}(\mathfrak{a},r)]^2 + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$,

$(\boldsymbol{iii})$ Let $t \# r = \mathfrak{a}$ and $r \# t = \mathfrak{a}$. Then by Lemma 4.6, we have $t = r$. Therefore, (E, #, $\mathfrak{a}$) is d −algebra. ■

Note that, it is not necessary (E, #, $\mathfrak{a}$) be ρ −algebra in general for any $\mathcal{A}(t,r)$ semi-injective for $\mathfrak{a} \in$ E. Let $\mathcal{A}(t,r) = 1 + t + r$, in Example 4.5, then $\mathcal{A}(t,r)$ is a semi injective for $\mathfrak{a} =$

1.Take $t \notin \{r, -r\}$ in E $= \mathcal{R} - \{1\}$, then $t\#r \neq r\#t$. That means there are $t \neq r \in$ E $= \mathcal{R} - \{1\}$ but $t\#r \neq r\#t$, and hence Definition 2.1 is not hold. Thus (E, #, 1) is not ρ–algebra.

Remark 4.8. If $\partial \neq 0$, we set $t\zeta r = \partial(t-r)^2[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] + \mathfrak{a}$, where $\mathcal{A}(t,r)$ is a polynomial in two variables.

Lemma 4.9. Assume that $\mathcal{A}(t,r)$ is a conditionally identical for $\mathfrak{a} \in$ E. If $t\zeta r = r\zeta t$, for any $t \neq r \in$ E $- \{\mathfrak{a}\}$. Then $\mathcal{A}(t,r)$ is commutative identical.

Proof. Assume $t\zeta r = r\zeta t$ and $\mathcal{A}(t,r)$ are conditionally identical for $\mathfrak{a} \in$ E with $\neq r$. Then $\partial(t-r)^2[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] + \mathfrak{a} = \partial(r-t)^2[\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] + \mathfrak{a} \rightarrow \partial(t-r)^2[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] = \partial(r-t)^2[\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}rt, \mathfrak{a})]$. But $\partial(t-r)^2 = \partial(-1)^2(r-t)^2 = \partial(r-t)^2 \neq 0$ [since $t \neq r$ and $\partial \neq 0$]. Then $[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] = [\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}rt, \mathfrak{a})]$. We consider that $[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}r, \mathfrak{a}t)] = [\mathcal{A}(\mathfrak{a}tr, \mathfrak{a}) - \mathcal{A}(\mathfrak{a}rt, \mathfrak{a})] = 0$, Therefore, $\mathcal{A}(at, ar) = \mathcal{A}(ar, at)$. Hence $\mathcal{A}(t,r)$ is commutative identical. ■

Lemma 4.10. Assume that $\mathcal{A}(t,r)$ is an associative identical for $\mathfrak{a} \in$ E. If $t\zeta r = \mathfrak{a} = r\zeta t$, then $t = r$.

Proof. Assume $\mathcal{A}(t,r)$ is an associative identical for $\mathfrak{a} \in$ E. Let $t\zeta r = \mathfrak{a} = r\zeta t$, we get $t\zeta r = \partial(t-r)^2[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] + \mathfrak{a} = \partial(r-t)^2[\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}rt, \mathfrak{a})]$
$+\mathfrak{a} = \mathfrak{a}$. Then we consider the following equations.

$\partial(t-r)^2[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] = 0.$ …(6)

$\partial(r-t)^2[\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}rt, \mathfrak{a})] = 0.$ ...(7)

From Eq.(6) and Eq.(7) we get $\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) = \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})$, then $t = r$ since $\mathcal{A}(t,r)$ is an associative identical for $\mathfrak{a} \in$ E. ■

Lemma 4.11. Assume that $\mathcal{A}(t,r)$ is regular identical for $\mathfrak{a} \in$ E. Then $t\zeta r = r\zeta t \neq \mathfrak{a}$, for any $t \neq r \in$ E $- \{\mathfrak{a}\}$.

Proof. Assume $\mathcal{A}(t,r)$ is regular identical for $\mathfrak{a} \in$ E. Then $\mathcal{A}(t,r)$ is associative and commutative identical for $\mathfrak{a} \in$ E. If $t \neq r \in$ E $- \{\mathfrak{a}\}$, we get $t\zeta r = \partial(t-r)^2 [\mathcal{A}(\mathfrak{a}t, \mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})] + \mathfrak{a} = \partial(-1)^2(r-t)^2[\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}rt, \mathfrak{a})] + \mathfrak{a}$,since $\mathcal{A}(t,r)$ is commutative identical, and hence $t\zeta r = r\zeta t$. Also, if $t\zeta r = \mathfrak{a}$, then $r\zeta t = \mathfrak{a}$. Then we get $\mathcal{A}(\mathfrak{a}r, \mathfrak{a}t) = \mathcal{A}(\mathfrak{a}tr, \mathfrak{a})$. Therefore, $t = r$ [since $\mathcal{A}(t,r)$ is associative identical], but this contradiction with $t \neq r \in$ E $- \{\mathfrak{a}\}$. Then $t\zeta r = r\zeta t \neq \mathfrak{a}$, for any $t \neq r \in$ E $- \{\mathfrak{a}\}$.■

Lemma 4.12. Assume that $\mathcal{A}(t,r)$ is regular identical for $\mathfrak{a} \in$ E. Then (E, $\zeta, \mathfrak{a}$) is ρ −algebra.

Proof:
We need to show the following:

(i) $t\,\zeta\,t = \mathfrak{a}$,

(ii) $\mathfrak{a}\,\zeta\,t = \mathfrak{a}$,

(iii) $t\,\zeta\,r = \mathfrak{a}$ and $r\zeta\,t = \mathfrak{a} \Rightarrow t = r$, $\forall t, r \in$ E,

(iv) For any $t \neq r \in$ E $- \{\mathfrak{a}\} \Rightarrow t\,\zeta\,r = r\,\zeta t \neq \mathfrak{a}$.

($\boldsymbol{i}$) $t\,\zeta\,t = \partial(t-t)^2[\mathcal{A}(\mathfrak{a}t, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}t^2, \mathfrak{a})] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$.

($\boldsymbol{ii}$) $\mathfrak{a}\,\zeta\,t = \partial(\mathfrak{a}-t)^2[\mathcal{A}(\mathfrak{a}^2, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}(\mathfrak{a}t), \mathfrak{a})] + \mathfrak{a} = \partial(\mathfrak{a}-t)^2[\mathcal{A}(\mathfrak{a}^2, \mathfrak{a}t) - \mathcal{A}(\mathfrak{a}^2, \mathfrak{a}t)]$
$+\mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$.

(***iii***) Assume $t\zeta r = \mathfrak{a} = r\zeta t$, since every regular identical for $\mathfrak{a} \in$ E is an associative identical for $\mathfrak{a} \in$ E, then by Lemma (4.10), we get $t = r$.
(***iv***) Let $t \neq r \in \Upsilon - \{\mathfrak{a}\}$. Then by Lemma 4.11, we have $t\zeta r = r\zeta t \neq \mathfrak{a}$. Therefore (E, ζ, 𝔞) is ρ −algebra. ■

Note that, if $\mathcal{A}(t,r)$ is conditionally identical for $\mathfrak{a} \in$ E, but it is not regular identical, then (E, ζ, 𝔞) is not necessary to be ρ −algebra. For example, let $\mathcal{A}(t,r)$ be a polynomial over real field, and defined by $(t,r) = t - r + 2$, $\forall t,r \in$ E = R . Then it is a conditionally identical for $\mathfrak{a} = -1 \in$ E. Since $\mathcal{A}(-t,r) = \mathcal{A}(-r,t)$, for any $t,r \in$ E, but $\mathcal{A}(-t,-r) \neq \mathcal{A}(-r,-t)$ for some $2 = t \neq r = 3 \in$ E, so it is not commutative identical and hence it is not regular identical. Also, (ϒ, ζ, 𝔞) is not ρ −algebra since for some $2 = t \neq r = 3 \in$ E $- \{-1\}$, we have $t\zeta r \neq r\zeta t$. However, $\mathcal{A}(t,r)$ is semi injective for $\mathfrak{a} = -1 \in$ E. Then we consider that (E, #, 𝔞) is d −algebra. In other side, let $B(t,r) = 3^{tr}$ be conditionally identical for $\mathfrak{a} = 1 \in$ E and defined # and ζ as following: $t\#r = \partial(t-1)(1-r)[B(1,t) - B(1,r)]^2 + 1$ and $t\zeta r = \partial(t-r)^2[B(t,r) - B(tr,1)] + 1$. Then we consider that (E, #, ζ, 1) is a pseudo d −algebra.

Proposition 4.13. Assume that $\mathcal{A}(t,r)$ is an associative identical for $\mathfrak{a} \in$ E. Then (E, #, ζ, 𝔞) is a pseudo d −algebra.

Proof: We need to show the following:
(i) $t \# t = t\zeta t = \mathfrak{a}$, (ii) $\mathfrak{a} \# t = \mathfrak{a}\zeta t = \mathfrak{a}$, (iii) $t \# r = \mathfrak{a}$, and $r\zeta t = \mathfrak{a} \Rightarrow t = r$, $\forall t,r \in$ E.
(***i***) $t \# t = \partial(t-\mathfrak{a})(\mathfrak{a}-t)[\mathcal{A}(\mathfrak{a},t) - \mathcal{A}(\mathfrak{a},t)]^2 + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$, Also, we get $t\,\zeta\,t = \partial(t-t)^2[\mathcal{A}(\mathfrak{a}t,\mathfrak{a}t) - \mathcal{A}(\mathfrak{a}t^2,\mathfrak{a})] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$. Then $t \# t = t\zeta t = \mathfrak{a}$.

(***ii***) $\mathfrak{a} \# t = \partial(\mathfrak{a}-\mathfrak{a})(\mathfrak{a}-r)[\mathcal{A}(\mathfrak{a},\mathfrak{a}) - \mathcal{A}(\mathfrak{a},r)]^2 + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$, Also, we get $\mathfrak{a}\,\zeta\,t = \partial(\mathfrak{a}-t)^2[\mathcal{A}(\mathfrak{a}^2,\mathfrak{a}t) - \mathcal{A}(\mathfrak{a}(\mathfrak{a}t),\mathfrak{a})] + \mathfrak{a} = \partial(\mathfrak{a}-t)^2[\mathcal{A}(\mathfrak{a}^2,\mathfrak{a}t) - \mathcal{A}(\mathfrak{a}\mathfrak{a},\mathfrak{a}t)] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$. Thus, $\mathfrak{a} \# t = \mathfrak{a}\zeta t = \mathfrak{a}$.

(***iii***) Let $t \# r = \mathfrak{a} = r\zeta t$. Then $\zeta\,t = \partial(t-r)^2$ $[\mathcal{A}(\mathfrak{a}t,\mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr,\mathfrak{a})] + \mathfrak{a} = \mathfrak{a}$, thus $\partial(t-r)^2$ $[\mathcal{A}(\mathfrak{a}t,\mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr,\mathfrak{a})] = 0$. Assume that $t \neq r$, then $\partial(t-r)^2 \neq 0$ and hence $[\mathcal{A}(\mathfrak{a}t,\mathfrak{a}r) - \mathcal{A}(\mathfrak{a}tr,\mathfrak{a})] = 0$, so $\mathcal{A}(\mathfrak{a}t,\mathfrak{a}r) = \mathcal{A}(\mathfrak{a}tr,\mathfrak{a})$. Thus $t = r$ [since $\mathcal{A}(t,r)$ is an associative identical], but this but this contradiction with $t \neq r$. Therefore $t = r$. Hence, (E, #, ζ, 𝔞) is pseudo d −algebra. ■

5. Applications on (proper/improper) conditionally identical in algebras.

This section discusses the concepts of proper conditionally identical and improper conditionally identical, as well as their algebraic results.

Definition 5.1. Let $\mathcal{A}(t,r)$ be a polynomial with coefficients in the field E and satisfies Eq. (1) for some $\mathfrak{a} \in$ E. We say it is a proper conditionally identical for 𝔞 if such that:

$$\mathcal{A}(\mathfrak{a}t,r) = \mathcal{A}(\mathfrak{a}r,t) \neq 0, \text{ for any } t,r \in \text{E}. \quad ...(8)$$

Example 5.2. If $\mathcal{A}(t,r) = 4t^2r^2 + 7$, the Eq. (1) becomes $\mathcal{A}(\mathfrak{a}t,r) = 4(\mathfrak{a}t)^2r^2 + 7 = 4(\mathfrak{a}r)^2t^2 + 7 = \mathcal{A}(\mathfrak{a}r,t)$. Also, for any $\mathfrak{a}$, $t,r \in$ E we have $4(\mathfrak{a}r)^2t^2 \geq 0$, then $0 < 4(\mathfrak{a}r)^2t^2 + 7$ and this implies $\mathcal{A}(\mathfrak{a}t,r) = \mathcal{A}(\mathfrak{a}r,t) \neq 0$, for any $t,r \in$ E. Hence the polynomial $\mathcal{A}(t,r)$ is a proper conditionally identical.

Remark 5.3. Every proper conditionally identical is conditionally identical, but the converse is not true in general.

Example 5.4.
If $\mathcal{A}(t,r) = 3(t - \mathfrak{a}r)^2$, the Eq. (1) becomes $\mathcal{A}(\mathfrak{a}t,r) = 3(\mathfrak{a}t - \mathfrak{a}r)^2 = 3(-1)^2(\mathfrak{a}r - \mathfrak{a}t)^2 = 3(\mathfrak{a}r - \mathfrak{a}t)^2 = \mathcal{A}(\mathfrak{a}r,t)$. Therefore, $\mathcal{A}(t,r)$ is conditionally identical. But, $3(\mathfrak{a}r - \mathfrak{a}t)^2 = 0$, when $t = r \in$ E. Then $\mathcal{A}(t,r)$ dose not satisfy Eq. (8) and hence it is not proper conditionally identical. Now suppose that for the field E, if $\partial \neq 0$, we set $t\mathcal{Z}r = \partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(t - r)\mathcal{A}(\mathfrak{a}t,r) + \mathfrak{a}$, where $\mathcal{A}(t,r)$ is a polynomial in two variables.

Lemma 5.5. Assume that $\mathcal{A}(t,r)$ is proper conditionally identical for $\mathfrak{a} \in$ E. Then if $t\mathcal{Z}r = r\mathcal{Z}t = \mathfrak{a}$, then $t = r \in$ E.
Proof. Assume $\mathcal{A}(t,r)$ is proper conditionally identical for $\mathfrak{a} \in$ E. Let $t\mathcal{Z}r = r\mathcal{Z}t = \mathfrak{a}$ and $t \neq r \in$ E, we get $\partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(t - r)\mathcal{A}(\mathfrak{a}t,r) + \mathfrak{a} = \partial(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - rt)(r - t)\mathcal{A}(\mathfrak{a}r,t) + \mathfrak{a} = \mathfrak{a}$. Also, $\mathcal{A}(\mathfrak{a}t,r) = \mathcal{A}(\mathfrak{a}r,t) \neq 0$ for any $t,r \in$ E (since $\mathcal{A}(t,r)$ is proper conditionally identical). In other side, $\partial(t - r) \neq 0$ and $\partial(r - t) \neq 0$ (since $t \neq r$), so $\partial(t - r)\mathcal{A}(\mathfrak{a}t,r) \neq 0$ and $\partial(r - t)\mathcal{A}(\mathfrak{a}r,t) \neq 0$ and hence we consider the following equations:
$(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr) = 0.$ ...(9)
$(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - rt) = 0.$ …(10)
By subtracting Eq. (9) from Eq. (10), we get $2(\mathfrak{a}t - \mathfrak{a}r) = 0$, and hence $t = r$. But this is a contradiction with our hypotheses $t \neq r$. Then $t = r \in$ E. ■

Lemma 5.6. Assume that $\mathcal{A}(t,r)$ is a proper conditionally identical for $\mathfrak{a} \in$ E. Then $(\text{E}, \mathcal{Z}, \mathfrak{a})$ is d −algebra.

Proof: We need to show the following:
(i) $t\,\mathcal{Z}\,t = \mathfrak{a}$, (ii) $\mathfrak{a}\,\mathcal{Z}\,r = \mathfrak{a}$, (iii) $t\,\mathcal{Z}\,r = \mathfrak{a}$ and $r\mathcal{Z}\,t = \mathfrak{a} \Rightarrow t = r, \forall t,r \in$ E.
$(\boldsymbol{i})$ $t\,\mathcal{Z}\,t = \partial(\mathfrak{a}t - \mathfrak{a}t + \mathfrak{a}^2 - t^2)(t - t)\mathcal{A}(\mathfrak{a}t,t) + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$,
$(\boldsymbol{ii})$ $\mathfrak{a}\,\mathcal{Z}\,t = \partial(\mathfrak{a}t - \mathfrak{a}^2 + \mathfrak{a}^2 - \mathfrak{a}t)(\mathfrak{a} - t)\mathcal{A}(\mathfrak{a}^2,t) + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$,
$(\boldsymbol{iii})$ Assume $t\mathcal{Z}r = \mathfrak{a} = r\mathcal{Z}t$. Then from Lemma 5.5, we have $t = r$. Hence $(\text{E}, \mathcal{Z}, \mathfrak{a})$ is d −algebra. ■

Lemma 5.7. Assume that $\mathcal{A}(t,r)$ is proper conditionally identical for $\mathfrak{a} \in$ E. Then $(\text{E}, \mathcal{Z}, \mathfrak{a})$ is ρ −algebra, if $(\mathfrak{a}^2 - tr) = (tr - \mathfrak{a}^2)$, $\forall t,r \in$ E.

Proof: From Lemma 5.6, we have $(\text{E}, \mathcal{Z}, \mathfrak{a})$ is d −algebra. Then we need only to show that for any $t \neq r \in \text{E} - \{\mathfrak{a}\} \Rightarrow t\mathcal{Z}r = r\mathcal{Z}t \neq \mathfrak{a}$. $t\mathcal{Z}r = \partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(t - r)\mathcal{A}(\mathfrak{a}t,r) + \mathfrak{a} = \partial(-1)[\mathfrak{a}t - \mathfrak{a}r + (rt - \mathfrak{a}^2)](-1)(r - t)\mathcal{A}(\mathfrak{a}r,t) + \mathfrak{a} = \partial(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - rt)(r - t)\mathcal{A}(\mathfrak{a}r,t) + \mathfrak{a} = r\mathcal{Z}t$. Then $t\mathcal{Z}r = r\mathcal{Z}t$. Now, if $t\mathcal{Z}r = \mathfrak{a}$, then $r\mathcal{Z}t = \mathfrak{a}$, so $\partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(t - r)\mathcal{A}(\mathfrak{a}t,r) = 0$ and $\partial(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - rt)(r - t)\mathcal{A}(\mathfrak{a}r,t) = 0$. Also, $\mathcal{A}(\mathfrak{a}t,r) = \mathcal{A}(\mathfrak{a}r,t) \neq 0$ for any $t,r \in$ E since $\mathcal{A}(t,r)$ is proper conditionally identical. In other side, $\partial(t - r) \neq 0$ and $\partial(r - t) \neq 0$ (since $t \neq r$), so $\partial(t - r)\mathcal{A}(\mathfrak{a}t,r) \neq 0$ and $\partial(r - t)\mathcal{A}(\mathfrak{a}r,t) \neq 0$ and hence we consider the following equations:
$(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr) = 0.$ …(11)
$(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - tr) = 0$. …(12)
By subtracting Eq. (11) from Eq. (12), we get $2(\mathfrak{a}t - \mathfrak{a}r) = 0$, and hence $t = r$. But this is a contradiction with our hypotheses $t \neq r$. Then $t = r \in$ E. ■

Remark 5.8. Suppose that for the field E, if $\partial \neq 0$, we set $t\mathcal{S}r = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a}$, where $\mathcal{A}(t, r)$ is a polynomial in two variables.

Definition 5.9 Let $\mathcal{A}(t, r)$ be a conditionally identical for some $\in$ E. We say it is an improper conditionally identical for $\mathfrak{a}$ if such that

$$\mathcal{A}(\mathfrak{a}t, t) \neq \mathcal{A}(\mathfrak{a}r, r), \text{ for any } t \neq r \in \text{E}. \quad \ldots(13)$$

Example 5.10.
Let $\mathcal{A}(t, r) = 1 + t + r$, for any $t, r \in \text{E} = \mathcal{R}$, Eq. (1) becomes $\mathcal{A}(\mathfrak{a}t, r) = \mathcal{A}(\mathfrak{a}r, t) \to 1 + \mathfrak{a}t + r = 1 + \mathfrak{a}r + t$. Therefore, $\exists \mathfrak{a} = 1 \in \text{E} = \mathcal{R}$ satisfies Eq.(1). So, it is precisely the equality permitted. Let $t \neq r \in \text{E} = \mathcal{R}$ may be taken arbitrary and hence Eq. (13) also is hold, since $1 + \mathfrak{a}t + t \neq 1 + \mathfrak{a}r + r$, for any $t \neq r \in$ E. In other side, let $\mathcal{A}(t, r)$ be a proper conditionally identical for $\mathfrak{a}$ in Example (5.2). Hence, for some $t \neq r = -t \in$ E we get $\mathcal{A}(\mathfrak{a}t, t) = 4(\mathfrak{a})^2 t^4 + 7 = 4(\mathfrak{a})^2 r^4 + 7 = \mathcal{A}(\mathfrak{a}r, r)$. Then the polynomial $\mathcal{A}(t, r)$ is not improper conditionally identical.

Lemma 5.11. Assume that $\mathcal{A}(t, r)$ is an improper conditionally identical for $\mathfrak{a} \in$ E. If $t\mathcal{S}r = r\mathcal{S}t = \mathfrak{a}$, then $t = r \in$ E.

Proof.
Assume $\mathcal{A}(t, r)$ is an improper conditionally identical for $\mathfrak{a} \in$ E. Let $t\mathcal{S}r = r\mathcal{S}t = \mathfrak{a}$. Then, we get $\partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] = \partial(\mathfrak{a}r - \mathfrak{a}t - \mathfrak{a}^2 + rt)[\mathcal{A}(\mathfrak{a}r, r) - \mathcal{A}(\mathfrak{a}t, t)] = 0$. If $t \neq r \in$ E we have $\mathcal{A}(\mathfrak{a}t, t) \neq \mathcal{A}(\mathfrak{a}r, r)$, since $\mathcal{A}(t, r)$ is improper conditionally identical, and hence we consider the following equations

$$(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr) = 0. \quad \ldots(14)$$

and

$$(\mathfrak{a}r - \mathfrak{a}t - \mathfrak{a}^2 + rt) = 0. \quad \ldots(15)$$

By subtracting Eq. (14) from Eq. (15), we get $2(\mathfrak{a}t - \mathfrak{a}r) = 0$, and hence $t = r$. But this is a contradiction with our hypotheses $t \neq r$. Then $t = r$. ∎

Lemma 5.12. Assume that $\mathcal{A}(t, r)$ is an improper conditionally identical for $\mathfrak{a} \in$ E. Then (E, $\mathcal{S}, \mathfrak{a}$) is d −algebra.

Proof: We need to show the following: $(i)\ t\mathcal{S}t = \mathfrak{a}$, $(ii)\ \mathfrak{a}\mathcal{S}r = \mathfrak{a}$, $(iii)\ t\,\mathcal{S}\,r = \mathfrak{a}$, and $r\mathcal{S}t = \mathfrak{a} \Rightarrow t = r, \forall t, r \in$ E.
$(\boldsymbol{i})\ t\,\mathcal{S}t = \partial(\mathfrak{a}t - \mathfrak{a}t - \mathfrak{a}^2 + t^2)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}t, t)] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$,
$(\boldsymbol{ii})\ \mathfrak{a}\mathcal{S}\,r = \partial(\mathfrak{a}^2 - \mathfrak{a}r - \mathfrak{a}^2 + \mathfrak{a}r)[\mathcal{A}(\mathfrak{a}^2, \mathfrak{a}) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$,
$(\boldsymbol{iii})$ Assume $= \mathfrak{a} = r\mathcal{S}t$. Then from Lemma 5.11, we get $t = r$. Then (E, $\mathcal{S}, \mathfrak{a}$) is d −algebra. ∎

Lemma 5.13. Assume that $\mathcal{A}(t, r)$ be an improper conditionally identical for $\mathfrak{a} \in$ E. Then (E, $\mathcal{S}, \mathfrak{a}$) is ρ −algebra, if $(\mathfrak{a}^2 - tr) = (tr - \mathfrak{a}^2)$, $\forall t, r \in$ E.

Proof: From Lemma 5.12, we have (E, $\mathcal{S}, \mathfrak{a}$) is d −algebra. Then we need only to show that for any $t \neq r \in \text{E} - \{\mathfrak{a}\} \Rightarrow t\mathcal{S}r = r\mathcal{S}t \neq \mathfrak{a}$. Now, we consider that $t\mathcal{S}r = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a} = \partial(-1)[\mathfrak{a}r - \mathfrak{a}t + (\mathfrak{a}^2 - rt)]\,(-1)[\mathcal{A}(\mathfrak{a}r, r) - \mathcal{A}(\mathfrak{a}t, t)] + \mathfrak{a} = \partial(\mathfrak{a}r - \mathfrak{a}t - \mathfrak{a}^2 + rt)[\mathcal{A}(\mathfrak{a}r, r) - \mathcal{A}(\mathfrak{a}t, t)] + \mathfrak{a} = r\mathcal{S}t$. Therefore, $t\mathcal{S}r = r\mathcal{S}t$. Now, if $tor = \mathfrak{a}$, then $r\mathcal{S}t = \mathfrak{a}$, so $\partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] = 0$ and $\partial(\mathfrak{a}r - \mathfrak{a}t -$

$\mathfrak{a}^2 + rt)[\mathcal{A}(\mathfrak{a}r, r) - \mathcal{A}(\mathfrak{a}t, t)] = 0$. Also, $\mathcal{A}(\mathfrak{a}t, t) \neq \mathcal{A}(\mathfrak{a}r, r)$, for any $t \neq r \in$ E, since $\mathcal{A}(t, r)$ is improper conditionally identical. That means, $[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] \neq 0$ and $[\mathcal{A}(\mathfrak{a}r, r) - \mathcal{A}(\mathfrak{a}t, t)] \neq 0$, and hence we consider the following equations:
$(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr) = 0.$ …(16)
$(\mathfrak{a}r - \mathfrak{a}t - \mathfrak{a}^2 + rt) = 0.$ …(17)
By subtracting Eq. (16) from Eq. (17), we get $2(\mathfrak{a}t - \mathfrak{a}r) = 0$, and hence $t = r$. But this is a contradiction with our hypotheses $t \neq r$. Then $t = r \in$ E. Then (E, $\mathcal{S}, \mathfrak{a}$) is ρ −algebra. ■

Lemma 5.14. Let E be a field, and $\partial \neq 0$. Define $t\mathcal{S}r = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a}$ and $t\mathcal{Z}r = \partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(t - r)B(\mathfrak{a}t, r) + \mathfrak{a}$. Then (E, $\mathcal{S}, \mathcal{Z}, \mathfrak{a}$) is a pseudo d −algebra, if $\mathcal{A}(t, r)$ is improper and $B(t, r)$ is proper for $\mathfrak{a} \in$ E.

Proof:
We need to show the following:
$(i) t\, \mathcal{S}\, t = t\, \mathcal{Z}\, t = \mathfrak{a}$,
$(ii) \mathfrak{a}\, \mathcal{S}\, t = \mathfrak{a}\, \mathcal{Z}\, t = \mathfrak{a}$,
$(iii) t\mathcal{S}\, r = r\mathcal{Z}\, t = \mathfrak{a} \Rightarrow t = r, \forall t, r \in$ E,
$(\boldsymbol{i})$ $t\mathcal{S}t = \partial(\mathfrak{a}t - \mathfrak{a}t - \mathfrak{a}^2 + t^2)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}t, t)] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$. On other side, t ◎ $t = \partial(\mathfrak{a}t - \mathfrak{a}t + \mathfrak{a}^2 - t^2)(t - t)B(\mathfrak{a}t, t) + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$. Hence $t\, \mathcal{S}\, t = t\, \mathcal{Z}\, t = \mathfrak{a}$.
$(\boldsymbol{ii}) \mathfrak{a}\mathcal{S}t = \partial(\mathfrak{a}^2 - \mathfrak{a}t - \mathfrak{a}^2 + \mathfrak{a}t)[\mathcal{A}(\mathfrak{a}^2, \mathfrak{a}) - \mathcal{A}(\mathfrak{a}t, t)] + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$. Moreover, $\mathfrak{a}\, \mathcal{Z}\, t = \partial(\mathfrak{a}t - \mathfrak{a}^2 + \mathfrak{a}^2 - \mathfrak{a}t)(\mathfrak{a} - t)B(\mathfrak{a}^2, t) + \mathfrak{a} = 0 + \mathfrak{a} = \mathfrak{a}$. Hence $\mathfrak{a}\, \mathcal{S}\, t = \mathfrak{a}\, \mathcal{Z}\, t = \mathfrak{a}$.
$(\boldsymbol{iii})$ Assume $t\mathcal{S}\, r = r\mathcal{Z}\, t = \mathfrak{a}$, thus $t\mathcal{S}r = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a} = \mathfrak{a}$....(18) and $r\mathcal{Z}t = \partial(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - tr)(r - t)B(\mathfrak{a}r, t) + \mathfrak{a} = \mathfrak{a}$…(19). Assume $t \neq r$, then from Eq. (18), we get $(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr) = 0$, since $\mathcal{A}(t, r)$ is improper, and from Eq. (19), we get $(\mathfrak{a}t - \mathfrak{a}r + \mathfrak{a}^2 - tr) = 0$, since $B(t, r)$ is proper. This implies $t = r$, but this contradiction with $t \neq r$. Then $t = r$. Hence (E, $\mathcal{Z}, \mathcal{S}, \mathfrak{a}$) is a pseudo d −algebra. ■

Definition 5.15. Let $\mathcal{A}(t, r)$ and $B(t, r)$ be two polynomials with coefficients in the field E. We say they are conditionally couple, if there is $\mathfrak{a} \in$ E such that

$$[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] = (r - t)B(\mathfrak{a}t, r), \text{ for any } t \neq r \in \text{E}. \quad \ldots(20)$$

Example 5.16. Let $\mathcal{A}(t, r) = t + (1 - \mathfrak{a})r$ and $B(t, r) = 1$, $\forall t, r \in$ E be two polynomials with coefficients in the real field E = R. Since $[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] = [[\mathfrak{a}t + (1 - \mathfrak{a})t] - [\mathfrak{a}r + (1 - \mathfrak{a})r]] = (r - t)$. Moreover, $(r - t)B(\mathfrak{a}t, r) = (r - t)$. Hence $\mathfrak{a} \in$ E such that Eq. (20), and then $\mathcal{A}(t, r)$ and $B(t, r)$ are conditionally couple.

Lemma 5.17. Let $\mathcal{A}(t, r)$ and $B(t, r)$ be conditionally couple for $\mathfrak{a} \in$ E and for any $\partial \neq 0$, $t, r \in$ E define $t\mathcal{S}r = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a}$ and $t\mathcal{Z}r = \partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(t - r)B(\mathfrak{a}t, r) + \mathfrak{a}$. Then (E, $\mathcal{S}, \mathcal{Z}, \mathfrak{a}$) is a pseudo ρ −algebra, if $\mathcal{A}(t, r)$ and $B(t, r)$ are improper and proper, respectively for $\mathfrak{a} \in$ E with $(\mathfrak{a}^2 - tr) = (tr - \mathfrak{a}^2)$, $\forall t, r \in$ E.

Proof:
From Lemma 5.14, we have (E, $\mathcal{S}, \mathcal{Z}, \mathfrak{a}$) is a pseudo d −algebra. Then we need only to show that $t\mathcal{S}r = r\mathcal{Z}\, t \neq \mathfrak{a}$, for any $t \neq r \in$ E − {$\mathfrak{a}$}. If $t \neq r \in$ E − {$\mathfrak{a}$}, then $t\mathcal{S}r = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)[\mathcal{A}(\mathfrak{a}t, t) - \mathcal{A}(\mathfrak{a}r, r)] + \mathfrak{a} = \partial(\mathfrak{a}t - \mathfrak{a}r - \mathfrak{a}^2 + tr)\, (r - t)B(\mathfrak{a}t, r) + \mathfrak{a}$….(21) [since $\mathcal{A}(t, r)$ and $B(t, r)$ be conditionally couple for $\mathfrak{a} \in$ E]. From Eq. (21), we have $t\mathcal{S}r =$

$(-1)\,\partial(\mathfrak{a}r - \mathfrak{a}t + \mathfrak{a}^2 - tr)(-1)(t-r)B(\mathfrak{a}t, r) + \mathfrak{a} = t\mathcal{Z}r$, but $t\mathcal{Z}r = r\mathcal{Z}t \neq \mathfrak{a}$, see Lemma 5.7. Hence, $t\mathcal{S}r = r\mathcal{Z}t \neq \mathfrak{a}$. Then $(\mathrm{E}, \mathcal{S}, \mathcal{Z}, \mathfrak{a})$ is a pseudo ρ −algebra. ■

Apparently, the theory of polynomially defined ρ/d −algebras and pseudo ρ/d −algebras may be easily extended to wider classes of rings than only fields, where the concept of "conditionally speciality" becomes a ring theoretical notion that may be of importance in and of itself.

6. Conclusions

Some notions and results of pseudo Rho/d-algebra are shown. Also, we proposed several new concepts in this study and investigated their applicability using polynomials with coefficients in the field and in two variables. These new notions such as conditionally identical, semi injective, commutative identical, associative identical, regular identical, proper conditionally identical, and improper conditionally identical are linked algebras with polynomials in two variables to generate pseudo algebras for some classes of algebras such as Rho/d-algebras, and this method is based on our four operations $(\#, \zeta, \mathcal{S}, \mathcal{Z})$that are given in this paper. In future work, we will investigate the requirements for our ideas that are required to consider that each of $(\Upsilon, \#, \zeta, \mathfrak{a})$, $(\Upsilon, \#, \mathcal{Z}, \mathfrak{a})$, $(\Upsilon, \#, \mathcal{S}, \mathfrak{a})$, $(\Upsilon, \zeta, \mathcal{Z}, \mathfrak{a})$, and $(\Upsilon, \zeta, \mathcal{S}, \mathfrak{a})$ is pseudo Rho/d-algebra. The concept of conditionally couple will guide us in opening the door to debate our results in two distinct fields in order to examine more new results that apply in two different fields rather than the same field.

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