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Existence Results for A Nonlinear Degenerate Parabolic Equation involving $p(x)$-Laplacian Type Diffusion Process

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Abstract.

In this study, a nonlinear degenerate parabolic equation is used to describe a nonlinear -Laplacian equation process that arises in many areas of science and engineering in mechanics, quantum physics, and chemical design. This work has the objective of proving the existence of the local weak solution of a nonlinear *p(x)*-Laplacian equation by the compactness theorem. The uniformly local characteristics of the solutions for the gradients by estimating the regularization problem and using the Moser iterative techniques. Moreover, some properties of the local solutions depend on uniformly bounded situations and the $L^{p(x)}$-norm to the gradient is considered.

Keywords: Nonlinear $p(x)$-Laplacian; regularization technique;local weak local solution.

وجود حلول للمعادلة المكافئة غيرالخطية المنفردة تتضمن عملية الانتشار من نوع $p(x)$ لابلاسياً

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الخلاصة

تركز دراستنا على مناقشة المعادلة المكافئة المنفردة غير الخطية كنموذج لعملية الانتشارمن نوع $p(x)$ لابلاسياً والتي تتميز بالعديد من التطبيقات في مجالات العلوم والهندسة الميكانيكية و فيزياء الكم والتصميم الكيميائي و غيرها. أيضاً تم مناقشة وجود حل ضعيف لمعادلة الانتشار من نوع $p(x)$ لابلاسياً بالاعتماد على نظرية التراص تمثل الهدف الاساس لهذا البحث. الصفة المحلية المنتظمة للحلول تحتاج الى تقدير مشكلة جديدة اخرى تسمى المشكلة القياسية او المنتظمة بالاعتماد على التقنية التكرارية (Moser). بالإضافة الى ما ذكرنا، دراسة خواص الحلول المحلية المقيدة بانتظام لهذه المعادلة و كذلك مشتقاتها الجزئية التي تنتمي للفضاء المعياري $L^{p(x)}$.

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1. **Introduction**

The asymptotic behaviour of nonlinear degenerate parabolic equations, that includes a $p(x)$-Laplacian operator in regular or irregular domains, has been explored by several authors [1-5]. The global attractor on the natural weak energy space is established by the general theory, namely the existence, uniqueness, and regularity, see [6-9]. In addition to the typical questions of the general theory, this paper investigates the existence and stability of the solutions to the equation associated with the $p(x)$-Laplacian type with variable or constant exponents. More specifically, the investigation will focus on the existence and uniqueness of the weak solution and the global attractor. The consideration of the Cauchy Dirichlet problem (CDP) for the nonlinear parabolic Laplacian equation is the following:

$$u_t - \mathrm{div}\left(\varrho^{\gamma}|\nabla u^m|^{p(x)-2}\nabla u^m\right) - \alpha(u^m)\int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|ds = 0\,, \text{ in } \mathcal{D}_{\mathrm{t}} = \mathcal{D}\times(0,\infty) \quad (1)$$

$$u(x,0) = \phi(x), \quad in \ \ \mathcal{D}, \quad (2)$$

$$u(x,t) = 0, \quad on \quad \partial\mathcal{D}\times(0,\infty), \quad (3)$$

where $\alpha = \alpha(u^m), \phi(x) \geq 0,\ \mathcal{D} \subset \mathbb{R}^N$ is an open domain with a smooth boundary $\partial\mathcal{D}$ which is bounded. Equation(1) has the term$\int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|ds$, which spatially depends on a nonlocal function and $\Lambda(s)$ is a bounded function. Also, the function $\varrho = \varrho(x) = dist(x,\partial\mathcal{D})$ represents the distance function from the boundary. Suppose that $p(x), m > 1, N \geq 1, \gamma > 0$,

$$\phi^m \in L^{(mq(x)-m+1)\ /m}(\mathcal{D}), \quad q(x) > 1, \quad |\alpha(\xi)| \leq c|\xi|^{1/m},\ \ \xi \in \mathbb{R} \quad (4)$$

This problem has received much attention due to various applications in mechanics, quantum physic, chemical reaction design, and biophysics models. It has been extensively studied whether equation (1) is linear or nonlinear, uniformly parabolic or degenerate parabolic. We just give a cursory review of what follows.

The existence of a nonnegative solution to the CDP (1)- (3), stated in the weak sense, is established if $\alpha(u^m) \equiv 0$, see [10, 11]. The following problem was considered in [12] and [13].

$$u_t = F(u) + div(|\nabla u|^{p-2}\nabla u), \quad (5)$$

where $F(u) = \alpha(u) - |u|^{\gamma}u$ and it is shown the existence of a global attractor in $L^2(\mathcal{D}_{\mathrm{t}})$, which is a bounded set in $W_0^{1,p}(\mathcal{D}_{\mathrm{t}}) \cap L^{\gamma+2}(\mathcal{D}_{\mathrm{t}})$. In [12], the lengthy behavior of solutions to the next equations below was investigated

$$u_t = div(|\nabla u|^{p-2}\nabla u) + \alpha\int_{\mathcal{D}}\Lambda(s)|u(s,t)|^{\beta}ds - \mathcal{A}(x)\ |u|^{\gamma}u, \quad (6)$$

in $L^p(\mathcal{D}_{\mathrm{t}})$ space.

In this work, we will study the weak local solution to the problem by using the regularized method that is typically used (1) with the initial and boundary values (2),(3), respectively. Also, we should use the test function which is smooth and has a compact support. In this technique, the compactness theorem is applied. The main techniques are motivated by [10, 12] . However, due to the local and nonlocal nonlinearity of the equation we investigated, we must restrict the exponents m, p to show the purpose of showing the existence of the problem (1)-(3) of the initial value. At the same time, in comparison with [6], equation (1) is more complicated then that in [6] which makes it more difficult to estimate the gradient term of the solution and to prove the continuity of the solution etc. Also, we did not impose any restrictions on the derivative $\alpha'(\xi)$ of the function $\alpha(\xi)$, which is clearly a promotion, however, it must satisfy that $|\alpha'(\xi)| \leq c|\xi|^{r-1}$ in [3, 6].

2. **The aim and objectives of the study**

The objective of the work is to study the existence of the local weak solution of the $p(x)$-Laplacian equation by using the theorems of comparison and compactness. The locally uniform characteristics of the solutions for gradients by estimating the regularization problem and using the Moser iterative techniques. Moreover, some properties of the local solutions depend on uniformly bounded situations, The $L^{p(x)}$-norm to the gradient is considered.

3. **Materials and methods**

In this section, we begin to give some fundamental definitions as well as characteristics of functional spaces containing variable exponents. for more details, see [14-17]. In this work, we will extend the previous results over the following spaces. $L^{p(x)}(D)$ is different from the classical $L^{p}(D)$ space in that the exponent p is not constant but a function. The spaces $L^{p(x)}(D)$ fit into the framework of the Musielak–Orlicz spaces, therefore they are also semi-modular spaces. Consider the space of all measurable real-valued functions that are measurable on $L^{p(x)}(D)$, such that

$$\|u\|_{L^{p(x)}(\mathcal{D})} = \left(\int_{\mathcal{D}} |u(x)|^{p(x)}dx\right)^{1/p(x)} < \infty.$$

The space $\left(L^{p(x)}(\mathcal{D}), \|u\|_{p(x)}\right)$ is a separable, uniformly convex Banach space. Also, the significant Sobolev space, which is denoted by $W^{1,p(x)}(\mathcal{D})$ is defined as follows:

$$W^{1,p(x)}(\mathcal{D}) = \{u: u \ \& \ |\nabla u| \ in \ L^{p(x)}(\mathcal{D})\} \quad (7)$$

with the norm:

$$\|\nabla u\|_{W^{1,p(x)}} = \|u\|_{L^{p(x)}(\mathcal{D})} + \|\nabla u\|_{L^{p(x)}(\mathcal{D})}, \ \ \forall u \in W^{1,p(x)}(\mathcal{D}).$$

The closure of $C_0^{\infty}(\mathcal{D})$ in $W^{(1,p(x))}(\mathcal{D})$ becomes $W_0^{1,p(x)}(\mathcal{D})$. A significant property of the functional spaces, together with variable exponents that were considered in [18],

$$W_0^{1,p(x)}(\mathcal{D}) \neq \{v \in W^{1,p(x)}(\mathcal{D}) : v|_{\partial\mathcal{D}} = 0\} = \mathring{W}^{1,p(x)}(\mathcal{D}).$$

The next lemma gives some basic properties of the variable exponent Sobolev space.

Lemma 3.1.

(i) A spaces $L^{p(x)}(\mathcal{D})$, $W^{1,p(x)}(\mathcal{D})$ and $W_0^{1,p(x)}(\mathcal{D})$ are Banach spaces that are reflexive.

(ii) Having $p(x)$-Hölder's inequality. Suppose that $s_1(x)$ and $s_2(x)$ be real functions with $1/s_1(x) + 1/s_2(x) = 1$ and $q_1(x) > 1$. Then the conjugate space of $L^{s_1(x)}(\mathcal{D})$ is $L^{s_2(x)}(\mathcal{D})$. For all $u \in L^{s_1(x)}(\mathcal{D})$ and $\in L^{s_2(x)}(\mathcal{D})$, then

$$\left|\int_{\mathcal{D}} u\, v\, dx\right| \leq 2\|u\|_{L^{s_1(x)}(\mathcal{D})} \|u\|_{L^{s_2(x)}(\mathcal{D})}.$$

(iii) $\int_{\mathcal{D}} |u|^{p(x)}dx = 1 \quad if \ \ \|u\|_{L^{p(x)}(\mathcal{D})} = 1$.

If $\|u\|_{L^{p(x)}(\mathcal{D})} < 1$, then $\|u\|^{p^+}_{L^{p(x)}(\mathcal{D})} \leq \int_{\Omega} |u|^{p(x)}dx \leq \|u\|^{p^-}_{L^{p(x)}(\mathcal{D})}$.

If $\|u\|_{L^{p(x)}(\mathcal{D})} > 1$, then $\|u\|^{p^-(x)}_{L^{p(x)}(\mathcal{D})} \leq \int_{\Omega} |u|^{p(x)}dx \leq \|u\|^{p^+(x)}_{L^{p(x)}(\mathcal{D})}$.

When $p_2(x) \geq p_1(x)$ then $L^{p_1(x)}(\mathcal{D}) \supset L^{p_2(x)}(\mathcal{D})$.

When $p_2(x) \geq p_1(x)$ then $W^{1,p_2(x)}(\mathcal{D}) \hookrightarrow W^{1,p_1(x)}(\mathcal{D})$.

(iv) Then there is a constant C such that if $p(x) \in C(\mathcal{D})$

$$\|u\|_{L^{p(x)}(\mathcal{D})} \leq C\|\nabla u\|_{L^{p(x)}(\mathcal{D})}, \ \forall u \in W_0^{1,p(x)}(\mathcal{D}).$$

It is called $p(x)$-Poincarés inequality.

Thus the norms $\|\nabla u\|_{L^{p(x)}(\mathcal{D})}$ and $\|u\|_{W^{1,p(x)}(\mathcal{D})}$ of the exponent Sobolev space of $W_0^{1,p(x)}(\mathcal{D})$ are equivalent. (see[19]). However, if the exponent $p(x)$ is required to satisfy a logarithmic Hölder continuity condition

$$|p(x) - p(s)| \leq \omega(|x - s|)$$

$\forall x, s \in Q_T, |x - s| < 1/2$ with $\overline{\lim_{s\to 0^+}} \omega(\xi) ln(1/\xi) = C < \infty,$

then

$$W_0^{1,p(x)}(\mathcal{D}) = W^{1,p(x)}(\mathcal{D})\,. \tag{8}$$

Now we consider the following significant definitions from [19].

Definition 3.2. A positive function u satisfies the conditions (1)-(3); it is referred to be a weak solution.

$$u \in L^{\infty}_{\text{loc}}\left((0,\infty); L^{\infty}(\mathcal{D})\right), \tag{9}$$

$$u^m \in L^{\infty}_{\text{loc}}\left((0,\infty); W_0^{1,p(x)}(\mathcal{D})\right),\;\; u_t \in L^2_{\text{loc}}\left((0,\infty); L^2(\mathcal{D})\right) \tag{10}$$

$$\iint_{\mathcal{D}_t}\left[uf_t - \varrho^{\gamma}(x)|\nabla u^m|^{p(x)-2}\nabla u^m \cdot \nabla f\right]dxdt \; +$$

$$\iint_{\mathcal{D}_t}\left[\alpha(u^m)\int_{\mathcal{D}} \Lambda(s)|\nabla u^m(s,t)|ds\right] f\, dxdt = 0, \forall f \in C_0^1(\mathcal{D}_t). \tag{11}$$

To find the solution to the problem. (1)-(3) by taking into consideration the regularized equation

$$u_t - div\left((\varrho^{\gamma}(x)|\nabla u^m|^2 + 1/k)^{(p(x)-2)/2}\nabla u^m\right) - \alpha(u^m)\int_{\mathcal{D}} \Lambda(s)|\nabla u^m(s,t)|ds = 0\,, \tag{12}$$

with (2) as the initial and homogeneous boundary values (3).

Definition 3.3. If u_k is a solution to the initial boundary problem of (12)-(2)-(3), $\lim_{k\to\infty} u_k = u$ in $\mathcal{D}_{\text{t}}$, (1)-(3) has a weak solution u, then u is described to be a local weak solution.

This study requires some important auxiliary results [17] [20, 21]. They are as follows.

Lemma 3.4. If $n > l \geq 1,\; q(x) \geq 2, 2nl/(n-l) \geq q(x) \geq r \geq 1,\; u^2 \; in \; W^{1,l}(\mathcal{D})$, then

$$\|u\|_{L^{q(x)}} \leq C^{1/2}\|u\|_r^{1-\theta}\|u^2\|_{1,l}^{\theta} \tag{13}$$

Where $\theta = 2(r^{-1} - q(x)^{-1})/(n^{-1} - l^{-1} + 2r^{-1})$.

This lemma is a generalization of the Gagliardo-Nirenberg inequality.

Lemma 3.5. Let $s = s(t)$ be a positive function on $(0,T]$. If it has the form

$$Bt^{-k}s + Ct^{-\delta} \geq s' + At^{\lambda\theta-1}s^{1+\theta}, \quad 0 < t \leq T, \tag{14}$$

with$A, \theta > 0, \lambda\theta \geq 1, B, C \geq 0, k \leq 1$, and then

$$s \leq A^{-\frac{1}{\theta}}(2\lambda + 2BT^{1-k})^{\frac{1}{t}}t^{-\lambda} + 2C(\lambda + BT^{1-k})^{-1}t^{1-\delta},\;\; T \geq t > 0. \tag{15}$$

Lemma 3.6. Suppose that $A_1 \geq 1, r, R, M > 0, \lambda_1 > 0$. For $N = 2,3,\cdots, let$

$$A_N = RA_{N-1} - M, \theta_N = nR(1 - A_{N-1}A_N^{-1})(N(R-1)+r)^{-1},$$

$$B_N = (A_N + M)\theta_N^{-1} - A_N, \lambda_N = \left(1 + \lambda_{n-1}(B_N - M)\right)B_N^{-1}.$$

Then

$$\lim_{N\to\infty} \lambda_N = \frac{A_1\lambda_1 r + n}{a_1 + Mn} \tag{16}$$

This lemma was also proved in [12] .

Assume in the study that $\text{p(x)} > 1 + 1/\text{m}$, so equation (1) is a double degeneracy equation. Using the technique of the Moser iteration and the regularized problem (12)-(2)-(3), we obtain the local boundary properties of the solution u_k as well as the local boundary properties of the $\text{L}^{\text{p(x)}}$-norm for the gradient ∇u_k. Estimating and proving the main results of our study are given in the following section.

4. The Main Results

Theorem 4.1.

It is assumed that Λ is a suitable smooth bounded function and that α is satisfied (4). If $p(x) > 1 + 1/m,\ \phi(x) \geq 0,$

$$\phi^m(x) \in L^{q(x)-1+1/m}(\mathcal{D}),\ \ 2 - 1/m < q(x) < 3, \tag{17}$$

$$1 < p(x),\ 1 < max\{p(x) - 1 - 1/m/m, (q(x) - 1 + 1/m\}, \tag{18}$$

$$\epsilon = max\{D_1 + D_2,\ D_3\} < 1, \tag{19}$$

where

$$D_1 = mn/(nm(p(x) - 1) - n + mq(x))\ ,$$
$$D_2 = (m(p(x) - 1) + m - 2)/(m(p(x) - 1) - 1)$$

and

$$D_3 = (n(1 + m))/(mn(p(x) - 1) - n + mq(x))$$

then the problems (1)-(3) have a solution with a weak local weak u, which satisfies

$$u^m \in L^\infty_{loc}\left(0, \infty; L^{q(x)-1+1/m}(\mathcal{D})\right) \cap L^\infty_{loc}\left(0, \infty; W_0^{1,p(x)}(\mathcal{D})\right) \tag{20}$$

and

$$\|u^m\|_{L^\infty} \leq c\big(1 + t^{-\lambda}\big)(1 + t)^{-1/(q(x)-1+1/m)}, t > 0, \tag{21}$$

where $\lambda = n(p(x)q(x) + (p(x) - 1 - 1/m)n)^{-1}$. Moreover, if $p(x) > 2$, then

$$\|\nabla u^m\|_{L^{p(x)}} \leq c\big(1 + t^{-\delta_1}\big)(1 + t)^{-\sigma},\ \ t > 0 \tag{22}$$

where

$$\delta_1 = max\{1 + (m - 1)/(m(p(x) - 1) - 1),\ \delta - 1\}\ , \delta = max\ \{(m + 1)/m, 2\}$$

and

$$\sigma = (p(x)[3m - 1] + m)/\big([m(p(x) - 1) - 1](p(x) - 1)\big).$$

Theorem 4.2. let u be a nonnegative weak solution to the problem (1)-(3). If v satisfies,

$$v_t \geq div\big(\varrho^\gamma(x)|\nabla v^m|^{p(x)-2}\nabla v^m\big) + \alpha(v^m)\int_{\mathcal{D}} \Lambda(s)|\nabla v^m(s,t)|ds \tag{23}$$
$$in\ \ \mathcal{D}_t = \mathcal{D} \times (0, \infty)$$

$$v(x, 0) \geq \phi(x), \qquad x \in \mathcal{D} \tag{24}$$

$$v(x, t) = 0, \quad (x, t) \in \partial\mathcal{D} \times (0, \infty) \tag{25}$$

then

$$u(x, t) \geq v(x, t),\ \forall (x, t) \in \mathcal{D}_t\,. \tag{26}$$

Theorem 4.3. Let u and v be two weak local solutions to equation (1) with $\gamma < p(x) - 1$, the same partial homogeneous boundary value

$$u|_{\Gamma_{p(x)\times(0,T)}} = 0 = v|_{\Gamma_{p(x)\times(0,T)}},$$

with the initial data
$u(x, 0) = v(x, 0)$.
Moreover

$$|\nabla u^m| \leq c\varrho^{-\gamma}(x), \qquad |\nabla v^m| \leq c\varrho^{-\gamma}(x)$$

then

$$\int_{\mathcal{D}} |u^m(x,t) - v^m(x,t)|\, dx \leqslant \int_{\mathcal{D}} |\phi - v_0^m| dx + c \int_{\Gamma'_{p(x)}} |u^m - v^m| d\Gamma$$
$$+ \lim_{n\to\infty}\ \sup_{\Gamma'_{p(x)}} \tau_N(u^m - v^m)|u^m - v^m| d\Gamma,\ \forall t \in [0, T).$$

where $N > 0$ is a natural number. The details of the definition and the properties of the test function $\tau_N(s)$ are given in the following section.

5. Discussion of results

Instead of trying to deal with the regularized problem (12)-(2)-(3) directly, one can deal with the case $m = 1$, we must consider the approximate problem. For small $s > 0$

$$u_t = div\left(\left(\varrho^{\gamma}(x)|\nabla u^m|^2 + (1/k)\right)^{(p(x)-2)/2}\nabla u^m\right) + \alpha(u^m)\int_{\mathcal{D}} \Lambda(y) \mid \nabla u^m(s,t) \mid ds \qquad (27)$$

$$u(x,0) = \phi_k(x) + s, \ \ x \in \mathcal{D} \qquad (28)$$

$$u(x,t) = s, \quad x \in \partial\mathcal{D}, t \geq 0 \qquad (29)$$

with a condition $\phi_k(x) \geq 0$ that is an appropriate smooth function such that $\phi_k(x) \in L^{\infty}(\mathcal{D}), lim_{k\to\infty}\phi_k^m{}_{q(x)-1+1/m} = \phi^m{}_{q(x)-1+1/m}$.

We know that problem (27)-(29) has a nonnegative classical solution u_{ks} since the existence of the initial boundary value problem of the quasilinear equation in the divergent form is obtained using the Leray-Schauder fixed point theory. For more details See [22]. Assume that $s \to 0$. Following the same procedure in [23-25] , we can prove it.

$$u_{ks} \to u_k, \text{ in } C(\mathcal{D}_t),$$
$$\nabla u_{ks}^m \to \nabla u_k^m, \text{ in } L^{p(x)}(\mathcal{D}_t),$$
$$u_{kst} \to \nabla u_{kt}, \text{ in } L^2(\mathcal{D}_t),$$
$$\varrho^{\gamma}(x)|\nabla u_{ks}^m|^{p(x)-2}\nabla u_{ksx_i}^m \to * \ \varrho^{\gamma}(x)|\nabla u_k^m|^{p(x)-2}\nabla u_{kx_i}^m, \text{weakly star}$$
$$\text{in } L_{loc}^{\infty}\left(0,\infty; L^{p(x)/(p(x)-1)}(\mathcal{D})\right)$$

and u_k is the solution to equation (27) with the following initial boundary values

$$u(x,0) = \phi_k(x), \quad x \in \mathcal{D}, \qquad (30)$$

$$u(x,t) = 0, x \in \partial\mathcal{D}, \quad t \geq 0. \qquad (31)$$

Lemma 5.1. Assume that

(H_1) $\alpha(z) \in C(R^1), |\alpha(z)| \leq \Lambda_0|z|^{1/m}$, for some$\Lambda_0 > 0$.

(H_2) $\Lambda(x) \in L^{\infty}$.

In addition, $p(x) < 2 + (1/m)$, $2 - 1/m \leq q(x) < 3$, then $u_k^m \in L_{\text{loc}}^{\infty}\left(0,\infty; L^{q(x)-1+1/m}(\mathcal{D})\right)$ and

$$\|u_k^m\|_{q(x)-1+1/m} \leq c(1+t)^{-1/q(x)-1+1/m}, t \geq 0 \qquad (32)$$

Proof. Just for simplicity, we denote u_k as u in the following proof. Only provide case proof $q(x) > 2 - 1/m$, if $q(x) = 2 - 1/m$, the conclusion can be obtained in a minor version. Let $A_n = (q(x) - 2)N^{3-q}, B_n = (3 - q(x))N^{(2-q(x))}$,

$$f_N(z) = \begin{cases} z^{q-1}, & \text{if } \quad z \geq \ 1/N \\ A_n z^2 + B_n z, & \text{if } \quad 0 \leq z < 1/N. \end{cases}$$

and suppose that $N > k$, we multiply (30) with $f_N(u^m)$ and integrate the result on $\mathcal{D}$. Since $f'(z) > 0$, we have

$$\int_{\mathcal{D}} f_N(u^m)\varrho^{\gamma} div\left(|\nabla u^m|^2 + (1/k)\right)^{(p(x)-2)/2}\nabla u^m dx$$
$$= \int_{\mathcal{D}} \nabla\varrho^{\gamma}\left(|\nabla u^m|^2 + (1/k)\right)^{(p(x)-2)/2}|\nabla u^m|^2 f_N'(u^m)dx$$
$$\leq -\int_{\mathcal{D}} \nabla\varrho^{\gamma}|\nabla u^m|^{p(x)} f_N'(u^m)dx = -\int_{\mathcal{D}} \nabla\varrho^{\gamma}\left|\nabla(\int_0^{u^m}\left(f_N'(s)\right)^{1/p(x)}ds\right|^{p(x)}) \, dx \qquad (33)$$

Suppose that $|\alpha(z)| \leq \Lambda_0 z^r$. Then

$$\left|\int_{\mathcal{D}\cap\{u^m \leq 1/N\}} \alpha(u^m) f_N(u^m) \ \int_{\mathcal{D}} \Lambda(s)|\nabla u^m(s,t)|dsdx\right|$$

$$\leq c(\Lambda)N^{1-q(x)-r}\int_{\mathcal{D}}|\nabla u|^m ds \leq c(\Lambda)N^{1-q(x)-r}\|\nabla u^m\|_{q(x)-1+1/m}.$$

If $r = 1/m$,

$$\left|\int_{\mathcal{D}\cap\{u^m>1/N\}}\alpha(u^m)f_N(u^m)\ \int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|\ dsdx\right|$$
$$\leq c(\Lambda)\ \int_{\mathcal{D}} u^{m(r+q(x)-1)}dx\int_{\mathcal{D}}|\nabla u|^m ds$$

Using Poincare inequality for the second integral so that

$$\left|\int_{\mathcal{D}\cap\{u^m>1/N\}}\alpha(u^m)f_N(u^m)\ \int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|\ dsdx\right| \leq c\|u^m\|_{q(x)-1+1/m}^{q(x)-1+1/m},$$

then

$$\left|\int_{\mathcal{D}}\alpha(u^m)f_N(u^m)\int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|\ dsdx\right|$$
$$\leq c\|\nabla u^m\|_{(m(q(x)-1)+1)/m}\left[n^{1-q(x)-\frac{1}{m}}+\|u^m\|_{q(x)-1+1/m}^{q(x)-1+1/m}\right] \qquad (34)$$

Based on the previous calculations , we get

$$\int_{\mathcal{D}} f_N(u^m)u_t dx+\int_{\mathcal{D}}\nabla\varrho^\gamma\left|\nabla\int_0^{u^m}\left(f_N'(s)\right)^{1/p(x)}ds\right|^{p(x)}dx \leq c\|u^m\|_{q(x)-1+1/m}^{q(x)-1+1/m}+O\left(\frac{1}{N^{q(x)-1}}\right)$$
(35)

By using Holder inequality and then we apply Poincare inequality for the second integral, we have

$$\int_{\mathcal{D}} f_N(u^m)u_t dx+c\int_{\mathcal{D}}\left|\int_0^{u^m}\left(f_N'(s)\right)^{1/p(x)}ds\right|^{p(x)}dx \leq c\|u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)+1}+O\left(\frac{1}{N^{q(x)-1}}\right).$$
(36)

Let $N\to\infty$ in (36). Conclude that

$$\frac{d}{dt}\int_{\mathcal{D}} u^{m(q(x)-1)+1}dx+c\int_{\mathcal{D}} u^{m[q(x)-1+(1/m)+p-1-(1/m)]}dx \leq c\|u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)+1} \qquad (37)$$

By Jessen's inequality, from (37), we obtain

$$\frac{d}{dt}\|u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)}+c\|u^m\|_{q(x)-1+1/m}^{q(x)-1+1/m+p(x)-1-(1/m)} \leq c\|u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)+1}$$

If $2+\ (1/m) < p(x)$, young inequality,

$$\frac{d}{dt}\|u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)}+c\|u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)+p(x)-1-(1/m)} \leq c,$$

then

$$\|u^m\|_{q(x)-1+1/m} \leq c(1+t)^{-1/p(x)-1-1/m}\ .$$

The desired result is satisfied.

Lemma 5.2. If $p(x) > 1+\frac{1}{m}$, and u_k are the solutions to the problem (27)-(30)-(31), then

$$\|u_k^m\|_\infty \leq ct^{-\lambda}, 0 < t \leq 1, \qquad (38)$$
$$\|u_k^m\|_\infty \leq c(t+1)^{-1/(mp(x)-m-1)/m},\ 1 \leq t, \qquad (39)$$

with $\lambda = n/((mp(x)-m-1)/m)n+q(x)p(x)$.

Proof. Multiply (27) by $u^{m(l-1)}$, then integrate the result on D ,

$$\int_{\mathcal{D}} u^{m(l-1)}u_t dx = \int_{\mathcal{D}} div\left(\varrho^\gamma(x)(|\nabla u^m|+(1/k))^{(p(x)-2\ )/2}\nabla u^m\right)u^{m(l-1)}dx$$
$$+\int_{\mathcal{D}}\alpha(u^m)u^{m(l-1)}\int_{\mathcal{D}}\Lambda(y)|u^m(s,t)|^\beta dsdx$$
$$= -(l-1)\int_{\mathcal{D}}\varrho^\gamma(x)(|\nabla u^m|+(1/k))^{(p(x)-2)/2}|\nabla u^m|^2u^{m(l-2)}dx$$
$$+\int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|\ ds\int_{\mathcal{D}}\alpha(u^m)u^{m(l-1)}dx$$
$$\leq -(l-1)\int_{\mathcal{D}}\varrho^\gamma(x)(|\nabla u^m|+(1/k))^{(p(x)-2)/2}|\nabla u^m|^2u^{m(l-2)}dx$$
$$+c(\Lambda)\int_{\mathcal{D}}|\nabla u^m(s,t)|\ ds\int_{\mathcal{D}} u^{m(l-1)+1}dx$$

Using Holder inequality and then we apply Poincare inequality for the second integral, which leads to the conclusion

$\frac{d}{dt}\|u^m\|_{l-1+(1/m)}^{l-1+(1/m)} + c((ml-m+1)/$
$m)^{2-p(x)}\int_{\mathcal{D}}\left|\nabla u^{p(x)+1-1+(1/m)-1-(1/m)\,/p(x)}\right|^{p(x)}dx$
$\leq c\|u^m\|_{l-1+(1/m)}^{l-1+(1/m)}\|\nabla u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)+1} + c\|u^m\|_{l-1+(1/m)}^{l-1},$

from Poincare inequality again and by (26), we get
$c\|u^m\|_{l-1+(1/m)}^{l-1+(1/m)}\|\nabla u^m\|_{q(x)-1+(1/m)}^{q(x)-1+(1/m)+1} + c\|u^m\|_{l-1+(1/m)}^{l-1} \leq \|u^m\|_{l-1+(1/m)}^{l-1+(1/m)} +$
$c\|u^m\|_{l-1+(1/m)}^{l-1}$.

Assume that $= (ml-m+1)/m$. So

$$\frac{d}{dt}\|u^m\|_L^L + cL^{2-p(x)}\int_{\mathcal{D}}\left|\nabla u^{m(p(x)+L-1-(1/m))/p}\right|^{p(x)}dx \leq c\|u^m\|_L^{L+1} + c\|u^m\|_L^{L-(1/m)} \quad (40)$$

Such that c is independent of l. Take
$L_1 = q(x)-1+(1/m), L_N = rL_{N-1} - (p(x)-1-(1/m)), \theta_n = (n(r-1)+p(x))^{-1}rN(1-L_{N-1}L_N^{-1}),$
where $N = 2,3,\cdots$and
$\mu_N = (p(x)+L_N-1-(1/m))\theta_n^{-1} - L_N, r > 1 + (p(x)-1-(1/m))q(x)^{-1}$, and from Lemma 3.6, then

$$\|u^m\|_{L_n} \leq c^{p(x)/(L_N+p(x)-1-(1/m))}\|u^m\|_{L_{N-1}}^{1-\theta_N}\left\|\nabla u^{m(L_N+p(x)-1-(1/m))/p}\right\|_{p(x)}^{p(x)\theta_N/(p(x)-1-(1/m)+L_n)} \quad (41)$$

By choosing $L = L_N$ in (40), and from (41), it becomes

$$\frac{d}{dt}\|u^m\|_{L_N}^{L_N} + c^{-\frac{p(x)}{\theta_N}}L_N^{2-p(x)}\|u^m\|_{L_N}^{L_N+\mu_N}\|u^m\|_{L_{N-1}}^{p(x)-1-\frac{1}{m}-\mu_N} \leq c\|u^m\|_{L_N}^{L_N+1} + c\|u^m\|_{L_N}^{L_N-(1/m)} \quad (42)$$

The bounded sequences $\{\lambda_N\}$ and $\{\xi_N\}$ are shown in the following

$$u^m{}_{L_N} \leq \xi_N t^{-\lambda_N}, 0 < t \leq 1. \quad (43)$$

Without losing the generality, assume that $\|u^m\|_{L_N} \geq 1$. Otherwise, picking $\xi_n \equiv 1$, (41) is true. As a result, from (40), we have the following formula
$\frac{d}{dt}\|u^m\|_{L_N}^{L_N} + c^{-p(x)/\theta_N}L_N^{2-p(x)}\|u^m\|_{L_N}^{L_N+\mu_N}\|u^m\|_{L_{N-1}}^{p(x)-1-(1/\mathrm{m})-\mu_N} \leq c\|u^m\|_{L_N}^{L_N+1} \cdot 0 < t \leq 1.$

If $N = 1$, by Lemma 5.1, $\lambda_1 = 0, \xi_1 = sup_{t\geq 0}\, u^m$ then it makes (43) true. If (43) is true for $N-1$, from (42),

$$\frac{d}{dt}\|u^m\|_{L_N}^{L_N} + c^{-\frac{p(x)}{\theta_N}}L_N^{2-p(x)}\|u^m\|_{L_N}^{L_N+\mu_N}\xi_{N-1}^{p(x)-1-\left(\frac{1}{m}\right)-\mu_N}t^{-\left(p(x)-1-\left(\frac{1}{m}\right)-\mu_N\right)\lambda_{N-1}} \leq c\|u^m\|_{L_N}^{L_N+1} \quad (44)$$

we can pick

$$\lambda_N = \left(\lambda_{N-1}\left(\mu_N - p(x) + 1 + \left(\frac{1}{m}\right)\right) + 1\right)\mu_N^{-1}, \xi_N = \xi_{N-1}\left(c^{p/\theta_N}L_N^{p(x)-1}\lambda_N\right)^{1/\mu_N},$$

$N = 2,3,\ldots$

$$\frac{d}{dt}\|u^m\|_{L_N}^{L_N} + c\|u^m\|_{L_N}^{L_N+\lambda_N} \leq cu^m{}_{L_N}^{L_N+1}. 0 < t \leq 1 \quad (45)$$

assume that

$$1 < n\ /\ (p(x)-1-(1/m))n + q(x)p(x), \quad (46)$$

Now let $N \to \infty$,

$$\lambda_n \to \lambda = n/\ (p(x)-1-(1/m))n + p(x)q(x).$$

$$\frac{d}{dt}\|u^m\|_{L_N}^{L_N} + c\|u^m\|_{L_N}^{L_N+\lambda_N} \leq 0. \quad 0 < t \leq 1, \quad (47)$$

equation (43) is true because of Lemma 3.5 and (38). Furthermore, it is clear that $\{\xi_n\}$ is bounded. As a result, according to Lemma 3.5,(38) is true. To prove (39), simply set

$$\tau = \log(1+t), t \geq 1, \ w(\tau) = (1+t)^{(m(p(x)-1)-1)/m^2} u^m(t).$$

By (40), we get

$$\frac{d}{d\tau}\|w(\tau)\|_L^L + cL^{2-p(x)}\left\|\nabla w^{(L+p-1-(1/\text{m}))/p(x)}\right\|_{p(x)}^{p(x)} \leq (L/p(x) - 1 - (1/m))\|w(\tau)\|_L^L$$
$$+c\|w(\tau)\|_L^{L+1}, \tau \geq log2$$

By using the lemma 3.1 in [16], we can get (39), however, the details are omitted here.

Lemma 5.3. If $p(x) > max\{2, 1+(1/m)\}$, u_k is the solution to problems (27)-(30)-(31) then

$$\|\nabla u_k^m\|_{p(x)} \leq ct^{-(1+\ (m-1)/(m(p(x)-1)-1))} + ct^{1-\delta}, 0 < t \leq 1, \tag{48}$$
$$\|\nabla u_k^m\|_{p(x)} \leq c(1+t)^{-(p(x)(2m-1)+m)/(m(p(x)-1)-1)(p(x)-1)}, t \geq 1, \tag{49}$$

and

$$\int_t^T \int_{\mathcal{D}} u_k^{m-1}(u_{kt})^2 dx\, ds \leq ct^{-\left(1+\frac{m-1}{m(p(x)-1)-1}\right)} + ct^{-\left(\lambda\rho+\frac{m-1}{m(p(x)-1)-1}\right)} + ct^{-((1+m)/\text{m})\lambda} \tag{50}$$

$0 < t < T$.

Here $\delta = max\{(m-1)/m, 2\}$.

Proof: It follows the same technique as in Lemma 5.2.

Proof of Theorem 4.1. Using the compactness theorem (see[15, 26]), theorem 1.1, and from Lemmas 5.1, 5.2, and 5.3, we consider a sequence $\{u_k\}$ such that $u_k \to u$, a.e. in $\mathcal{D}_t$ if $k \to \infty$, so that

$$\lim_{k\to\infty} \alpha(u_k^m)\int_{\mathcal{D}} \Lambda(s)|\nabla u_k^m(s,t)|\, ds = \alpha(u^m)\int_{\mathcal{D}} \Lambda(s)|\nabla u_k^m(s,t)|ds.$$

Furthermore, the sequence $\{u_k\}$

$$u_k \to u,\ weak *\ in\ L_{loc}^{\infty}\left(0,\infty; L^{(q(x)-1)m+1}(\mathcal{D})\right) \tag{51}$$
$$u_{kt} \to u_t,\ weakly\ in\ L^2\big(0,\infty; L^2(\Omega)\big), \tag{52}$$
$$\nabla u_k^m \to \nabla u^m,\ weakly\ in\ L_{loc}^{p(x)}\left(0,\infty; L^{p(x)}(\mathcal{D})\right) \tag{53}$$
$$\varrho_k^{\gamma}|\nabla u_k^m|^{p(x)-2}u_{kx_i}^m \to \xi_i,\ \ a\ weak *\ in\ L_{loc}^{\infty}\left(0,\infty; L^{p(x)/(p(x)-1)}(\mathcal{D})\right)$$

where $\xi = \{\xi_i : 1 \leq i \leq N\}$ and every ξ_i is a function in $L_{loc}^{\infty}\left(0,T; L^{p(x)/(p(x)-1)}(\mathcal{D})\right)$.

(51) and (52) are true. It remains to prove that

$$\xi = \varrho^{\gamma}|\nabla u^m|^{p(x)-2}\nabla u^m, \qquad \text{in } L_{loc}^{\infty}\left(0,\infty; L^{p(x)/(p(x)-1)}(\mathcal{D})\right). \tag{54}$$

It is easy to know that

$$\iint_{\mathcal{D}_t}\big(u\varphi_t - \xi\cdot\nabla\varphi + \alpha(u^m)\int_{\mathcal{D}} \Lambda(s)|\nabla u^m(s,t)|ds\ \varphi\big)dxdt = 0, \forall\varphi \in C_0^{\infty}(\mathcal{D}_t), \tag{55}$$

Thus, Let us assume that

$$\iint_{\mathcal{D}_t} \xi\cdot\nabla\varphi dxdt, \forall\varphi \in C_0^1(\mathcal{D}_t) = \iint_{\mathcal{D}_t} \varrho^{\gamma}|\nabla u^m|^{p(x)-2}\nabla u^m\cdot\nabla\varphi dxdt\,; \tag{56}$$

then (59) and (10) are true.

First, for any $\psi \in C_0^{\infty}(\mathcal{D}_t), 0 \leq \psi \leq 1$, we have

$$\iint_{\mathcal{D}_t} \psi\varrho_k^{\gamma}\big(|\nabla u_k^m|^{p(x)-2}\nabla u_k^m - |\nabla v^m|^{p(x)-2}\nabla v^m\big)\cdot\nabla(u_k^m - v^m)dxdt \geq 0, \tag{57}$$

If we multiply (27) by $u_k^m\psi$ two sides, then

$$\iint_{\mathcal{D}_t} \psi\varrho_k^{\gamma}(|\nabla u_k^m|^2 + (1/k))^{(p(x)-2)/2}|\nabla u_k^m|^2 dxdt = 1/m\ \iint_{\mathcal{D}_t} \psi_t u_k^{m+1} dxdt$$
$$-\iint_{\mathcal{D}_t} \varrho_k^{\gamma} u_k^m(|\nabla u_k^m|^2 + (1/k))^{(p(x)-2)/2}\nabla u_k^m\cdot\nabla\psi dxdt$$
$$+\iint_{\mathcal{D}_t}\left[\alpha(u_k^m)\int_{\mathcal{D}} \Lambda(y)|\nabla u_k^m(s,t)|ds\right]u_k^m\psi dxdt, \tag{58}$$

Note that when $1 < p(x) < 2$,we get

$$|\nabla u_k^m|^2 \geq \left(|\nabla u_k^m|^2 + (1/k)\right)^{p(x)/2} - (1/k)^{p(x)/2},$$

$$\left(|\nabla u_k^m|^2 + (1/k)\right)^{(p(x)-2)/2} |\nabla u_k^m| \leq \left(|\nabla u_k^m|^2 + (1/k)\right)^{(p(x)-1)/2}$$

and when $p(x) \geq 2$, we obtain

$$(|\nabla u_k^m|^2 + (1/k))^{((p(x)-2)/2} |\nabla u_k^m|^2 \geq |\nabla u_k^m|^{p(x)}$$

$$(|\nabla u_k^m|^2 + (1/k))^{((p(x)-2)/2} |\nabla u_k^m| \leq \left(|\nabla u_k^m|^{p(x)-1} + 1\right)$$

By (57), (58), we have

$$\frac{1}{m+1}\iint_{\mathcal{D}_t} \psi_t u_k^{m+1} dxdt - \iint_{\mathcal{D}_t} \varrho_k^\gamma u_k^m (|\nabla u_k^m|^2 + (1/k))^{((p(x)-2)/2} \nabla u_k^m \cdot \nabla\psi dxdt + (1/k)^{((p(x)-2)/2} mes\mathcal{D}$$
$$+ \iint_{\mathcal{D}_t} \left[\Lambda(u_k^m) \int_{\mathcal{D}} \Lambda(s) |\nabla u_k^m(s,t)| dy\right] u_k^m \psi dxdt - \iint_{\mathcal{D}_t} \psi \varrho_k^\gamma |\nabla u_k^m|^{p(x)-2} \nabla u_k^m \cdot \nabla v^m dxdt$$
$$- \iint_{\mathcal{D}_t} \psi \varrho^\gamma |\nabla v^m|^{p(x)-2} \nabla v^m \cdot \nabla(u_k^m - v^m) dxdt \geq 0 \qquad (59)$$

$$\frac{1}{m+1}\iint_{\mathcal{D}_t} \psi_t u_k^{m+1} dxdt - \iint_{\mathcal{D}_t} \varrho_k^\gamma u_k^m (|\nabla u_k^m|^2 + (1/k))^{((p(x)-2)/2((p(x)-2)/2} \nabla u_k^m \cdot \nabla\psi dxdt + (1/k)^{((p(x)-2)/2} mes\mathcal{D}$$
$$+ \iint_{\mathcal{D}_t} \left[\Lambda(u_k^m) \int_{\mathcal{D}} \Lambda(s) |\nabla u_k^m(s,t)| dy\right] u_k^m \psi dxdt - \iint_{\mathcal{D}_t} \psi \varrho_k^\gamma |\nabla u_k^m|^{p(x)-2} \nabla u_k^m \cdot \nabla v^m dxdt$$
$$- \iint_{\mathcal{D}_t} \psi \varrho^\gamma |\nabla v^m|^{p(x)-2} \nabla v^m \cdot \nabla(u_k^m - v^m) dxdt$$
$$- \iint_{\mathcal{D}_t} \psi (\varrho^\gamma - \varrho_k^\gamma) |\nabla v^m|^{p(x)-2} \nabla v^m \cdot \nabla(u_k^m - v^m) dxdt \geq 0 \qquad (60)$$

Since

$$(|\nabla u_k^m|^2 + (1/k))^{(p(x)-2)/2} \nabla u_k^m = |\nabla u_k^m|^{p(x)-2} \nabla u_k^m + (p(x)-2)/2k \int_0^1 (|\nabla u_k^m|^2 + (z/k))^{(p(x)-4)/2} dz\, \nabla u_k^m,$$

Noticing

$$\left|\iint_{\mathcal{D}_t} \psi (\varrho^\gamma - \varrho_k^\gamma) |\nabla v^m|^{p(x)-2} \nabla v^m \cdot \nabla(u_k^m - v^m) dxdt\right|$$
$$\sup_{(x,t)\in\mathcal{D}_t} \left|\psi(\varrho^\gamma - \varrho_k^\gamma)\right| / \varrho^\gamma \iint_{\mathcal{D}_t} \varrho^\gamma |\nabla v^m|^{p(x)-1} |\nabla u_k^m - \nabla v^m| dx\, dt$$
$$\leqslant \sup_{(x,t)\in\mathcal{D}_t} \left|\psi(\varrho^\gamma - \varrho_k^\gamma)\right| / \varrho^\gamma \left(\iint_{\mathcal{D}_t} \left|\nabla v^m\right|^{p(x)} dx\, dt + \iint_{\mathcal{D}_t} \varrho^\gamma \nabla v^m|^{p(x)-1} |\nabla u_k^m| dx\, dt\right|\right) \quad (61)$$

and

$$\lim_{k\to\infty} \iint_{\mathcal{D}_t} \int_0^1 (|\nabla u_k^m|^2 + (z/k))^{(p(x)-4)/2} dz \nabla u_k^m \cdot \nabla\psi u_k^m dxdt = 0,$$

By Hölder inequality, there holds

$$\iint_{\mathcal{D}_t} \varrho^\gamma |\nabla v^m|^{p(x)-1} |\nabla u_k^m| dx\, \mathrm{d}t \leqslant$$
$$\left(\iint_{\mathcal{D}_t} \left(\varrho^m |\nabla v^m|^{p(x)-1}\right)^{q(x)} dx\, \mathrm{d}t\right)^{1/q(x)} \cdot \left(\iint_{\mathcal{D}_t} (\varrho^n |\nabla u_k^m|)^{p(x)} dx\, \mathrm{d}t\right)^{1/p(x)}$$

where $m = \gamma(p(x)-1)/p(x),\ n = \gamma/p(x),\ q(x) = p(x)/p(x) - 1.$

Due to $\varrho^\gamma \left|\nabla u\right|^{p(x)}, \varrho^\gamma \left|\nabla v\right|^{p(x)} \in L^1(\mathcal{D}_t)$, then

$$\iint_{\mathcal{D}_t} \varrho^\gamma \left|\nabla v^m\right|^{p(x)} dx\, dt + \iint_{\mathcal{D}_t} \varrho^\gamma \left|\nabla v^m\right|^{p(x)-1} |\nabla u_k^m| dx\, dt \leqslant c.$$

Let $k \to 0$ in (61). It converges to 0. Thus,

$$1/(m+1)\iint_{\mathcal{D}_t}\psi_t u^{m+1}dxdt-\iint_{\mathcal{D}_t}u^m\xi\nabla\psi dxdt$$
$$-\iint_{\mathcal{D}_t}\psi\,\xi\cdot\nabla v^m dxdt-\iint_{\mathcal{D}_t}\psi\varrho^\gamma|\nabla v^m|^{p(x)-2}\nabla v^m\cdot\nabla(u^m-v^m)dxdt$$
$$+\iint_{\mathcal{D}_t}\left[\alpha(u^m)\int_{\mathcal{D}}\Lambda(y)|\nabla u^m(s,t)|\,ds\right]u^m\psi dxdt\geq 0 \qquad (62)$$

Now, we pick $\varphi=\psi\,u^m$ in (55),

$$1/(m+1)\iint_{\mathcal{D}_t}\psi_t u^{m+1}dxdt-\iint_{\mathcal{D}_t}\xi\cdot\nabla\psi u^m dxdt$$
$$+\iint_{\mathcal{D}_t}\left[\alpha(u^m)\int_{\mathcal{D}}\Lambda(s)|\nabla u^m(s,t)|\,ds\right]\psi u^m dxdt=\iint_{\mathcal{D}_t}\xi\,\psi\cdot\nabla u^m dxdt$$

From this form and (62), we obtain

$$\iint_{\mathcal{D}_t}\psi\left(\xi-\varrho^\gamma|\nabla v^m|^{p(x)-2}\nabla v^m\right)\cdot\nabla(u^m-v^m)dxdt\geq 0 \qquad (63)$$

Let $v^m=u^m-\lambda\varphi,\lambda\geq 0,\varphi\in C_0^\infty(\mathcal{D}_t)$. Then

$$\iint_{\mathcal{D}_t}\psi\left(\xi_i-\varrho^\gamma|\nabla(u^m-\lambda\varphi)|^{p(x)-2}(u^m-\lambda\varphi)_{x_i}\right)dxdt\geq 0$$

Let$\lambda\to 0$. We have

$$\iint_{\mathcal{D}_t}\psi\left(\xi_i-\varrho^\gamma|\nabla u^m|^{p(x)-2}u^m_{x_i}\right)dxdt\geq 0,\forall\varphi\in C_0^\infty(\mathcal{D}_t)$$

Furthermore, let's choose $\lambda\leq 0$, to obtain

$$\iint_{\mathcal{D}_t}\psi\left(\xi_i-\varrho^\gamma|\nabla u^m|^{p(x)-2}u^m_{x_t}\right)dxdt\leq 0\,,\forall\varphi\in C_0^\infty(\mathcal{D}_t).$$

Therefore, if pick ψ satisfies $supp\ \psi\supset supp\ \varphi$, and on $supp\varphi,\ \psi=1$, then we can get (56).

Proof of Theorem 4.2. Consider the following rescaling function

$$v(x,t)=u_{kr}(x,t),\qquad r\in(0,1)$$

then we get

$$v(x,t)=ru_k\left(x,r^{m(p(x)-1)-1}t\right),$$

which is the solution to the Dirichlet problem

$$v_t(x,t)=div\left(\varrho^\gamma(x)|\nabla v^m|^{p(x)-2}\nabla v^m\right)+r^{m[p(x)-2]}\alpha(r^{-m}v^m)\int_{\mathcal{D}}\Lambda(s)|\nabla v^m|\ ds \quad (64)$$
$$v=ru_k,\ (x,t)\in\mathcal{D}\times\{0\} \qquad (65)$$
$$v=0,\quad (x,t)\in\partial\mathcal{D}\times(0,\infty) \qquad (66)$$

Noticing that $\alpha(r^{-m}v^m)\geq\alpha(v^m)$, $p(x)-2<0$, and $r^{m(p(x)-2)}>1$, $0<r<1$

$$v_t(x,t)\geq div\left(\varrho^\gamma|\nabla v^m|^{p(x)-2}\nabla v^m\right)+\alpha(v^m)\int_{\mathcal{D}}\Lambda(s)\,|\nabla v^m|\,ds.$$

Using an argument similar to that in [24] , we can prove this $u_k\geq u_{kr}$.
As a result of

$$\left[u_k\left(x,r^{m(p(x)-1)-1}t\right)-u_k(x,t)\right]/\left[\left(r^{m(p(x)-1)-1}-1\right)t\right]\ \geq$$
$$\left[(r-1)/\left(1-r^{m(p(x)-1)-1}\right)t\right]u_k\left(x,r^{m(p(x)-1)-1}t\right)$$

as $r\to 1$, then

$$-\,u_k/(m(p(x)-1)-1)t\leq u_{kt} \qquad (67)$$

Proof of Theorem 4.3. For a small positive constant $\lambda>0$, let

$$\mathcal{D}_\lambda=\{x\in\mathcal{D}:\varrho(x)=\mathrm{dist}(x,\partial\mathcal{D})>\lambda\}$$

and let

$$\phi(x)=\begin{cases}1, & \text{if } x\in\mathcal{D}_{2\lambda},\\ \frac{1}{\lambda}(\varrho(x)-\lambda), & x\in\mathcal{D}_\lambda\setminus\mathcal{D}_{2\lambda}\\ 0, & \text{if } x\in\mathcal{D}\setminus\mathcal{D}_\lambda\end{cases} \qquad (68)$$

for any given positive integer$N$, let $\tau_N(z)$ be an odd function, and

$$\tau_N(z) = \begin{cases} 1, & z > 1/N, \\ N^2z^2\mathrm{e}^{1-N^2z^2}, & 0 \le z \leqslant 1/N \end{cases} \tag{69}$$

clearly,

$$\lim_{N\to 0} \tau_N(z) = sgn(z), z \in (-\infty, +\infty), \tag{70}$$

and

$$0 \le \tau'_N(z) \le c/z,\ 0 < z < 1/N \tag{71}$$

where c is independent of N. From a process of limit, we can take $\tau_N\big(\phi(u^m - v^m)\big)$ as the test function, so

$$\int_{\mathcal{D}} \tau_N\big(\phi(u^m - v^m)\big)\ \partial(u^m - v^m)/\partial t \quad dx +$$
$$\int_{\mathcal{D}} \varrho^\gamma(x)\big(|\nabla u^m|^{p(x)-2}\nabla u^m - |\nabla v^m|^{p(x)-2}\nabla v^m\big)\cdot \phi\nabla(u^m - v^m)\tau'_N dx$$
$$+ \int_{\mathcal{D}} \varrho^\gamma(x)\big(|\nabla u^m|^{p(x)-2}\nabla u^m - |\nabla v^m|^{p(x)-2}\nabla v^m\big)\cdot \nabla\phi(u^m - v^m)\tau'_N dx \tag{72}$$
$$+\alpha(u^m)\int_{\mathcal{D}} \Lambda(s)|\nabla u^m(s,t)|ds \cdot \tau_N\big(\phi(u^m - v^m)\big)dx = 0$$

Thus

$$\lim_{N\to\infty}\lim_{\lambda\to 0}\ \int_{\mathcal{D}} \tau_N\big(\phi(u^m - v^m)\big)\partial(u^m - v^m)/\partial t \quad dx = \frac{d}{dt}\|u^m - v^m\|_1, \tag{73}$$

$$\int_{\mathcal{D}} \varrho^\gamma(x)\big(|\nabla u^m|^{p(x)-2}\nabla u^m - |\nabla v^m|^{p(x)-2}\nabla v^m\big)\cdot \nabla(u^m - v^m)\tau'_N\phi(x)dx \ge 0. \tag{74}$$

By the L'Hospital rule,

$$\lim_{\lambda\to 0}\Big[\int_{\mathcal{D}_\lambda\setminus\mathcal{D}_{2\lambda}} \tau'_N\big(\phi(u^m - v^m)\big)(u^m - v^m)dx\Big]/\lambda = \lim_{\lambda\to 0}\Big[\int_\lambda^{2\lambda}\int_{\varrho=\xi}\tau'_N\big(\phi(u^m - v^m)\big)(u^m - v^m)d\Gamma d\zeta\Big]/\lambda$$
$$= \lim_{\lambda\to 0}\ \int_{\varrho=2\lambda}\tau'_N(u^m - v^m)(u^m - v^m)d\Gamma = \int_{\partial\mathcal{D}}\tau'_N\big(2(u^m - v^m)\big)(u^m - v^m)d\Gamma$$
$$= \int_{\Gamma'_{p(\mathrm{x})}}\tau'_N(u^m - v^m)(u^m - v^m)d\Gamma \tag{75}$$

Since assuming that

$\varrho(x)|\nabla u|^{p(x)} < \infty, \varrho(x)|\nabla v|^{p(x)} < \infty$, we have

$$\lim_{\lambda\to 0}\left|\int_{\mathcal{D}} \varrho^\gamma(x)\big(|\nabla u^m|^{p(x)-2}\nabla u^m - |\nabla v^m|^{p(x)-2}\nabla v^m\big)\cdot \nabla\phi(u^m - v^m)\tau'_N\big(\phi(u^m - v^m)\big)dx\right|$$
$$= \lim_{\lambda\to 0}\left|\int_{\mathcal{D}_\lambda\setminus\mathcal{D}_{2\lambda}} \varrho^\gamma(x)\big(|\nabla u^m|^{p(x)-2}\nabla u^m - |\nabla v^m|^{p(x)-2}\nabla v^m\big)\cdot\nabla\phi\right| u^m - v^m\Big|\ \ \tau'_N\big(\phi(u^m - v^m)\big)dx\Big| \tag{76}$$
$$\le c\lim_{\lambda\to 0}\ \Big[\int_{\mathcal{D}_\lambda\setminus\mathcal{D}_{2\lambda}}\tau'_N\big(\phi(u^m - v^m)\big)(u^m - v^m)dx\Big]/\lambda = \int_{\Gamma'_{p(\mathrm{x})}}\tau'_N(u^m - v^m)(u^m - v^m)d\Gamma,$$

and

$$\Big|\int_{\mathcal{D}}\tau_N(v^m)\Big[\alpha(u^m)\int_{\mathcal{D}}\Lambda(s)|\nabla u^m|ds - \alpha(v^m)\int_{\mathcal{D}}\Lambda(s)|\nabla v^m|ds\Big]dx\Big|$$
$$\le \Big|\int_{\mathcal{D}}\Lambda(s)|\nabla u|ds\int_{\mathcal{D}}[\alpha(u^m) - \alpha(v^m)]dx\Big| + c\Big|\int_{\mathcal{D}}\alpha(v^m)(u^m - v^m)|v^m(t)|\Big|dx$$
$$\le c\|u^m - v^m\|_1\|\nabla u^m\|_1 + c\|v^m\|^2{}_1.$$

Now, let $\lambda \to 0$, and $N \to \infty$ in (72). Then

$$\frac{d}{dt}\|u^m - v^m\|_1 \leqslant c\lim_{N\to\infty} sup\int_{\Gamma'_p}\tau_N(u^m - v^m)|u^m - v^m|d\Gamma +$$
$$c\ \|u^m - v^m\|_1\|\nabla u^m\|_1 + c\|v^m\|_1^2.$$

It implies that

$$\int_{\mathcal{D}}|u^m(x,t) - v^m(x,t)|dx \leqslant \int_{\mathcal{D}}|u_0^m - u_0^m|dx + c\lim_{N\to\infty} sup\ \int_{\Gamma'_p}\tau_N(u^m - v^m)\,|u^m - v^m|d\Gamma$$
$$+ c\ \|u^m - v^m\|_1\|\nabla u^m\|_1 + c\|v^m\|_1^2,\ \forall t \in [0,T).$$

Therefore, Theorem 4.3. is proved.

6 . Conclusion

The nonlinear degenerate singular parabolic equations that describes the nonlinear $p(x)$-Laplacian equation is illustrated. The gradient term has a significant role and influences the solution's qualitative behavior. The goal of this paper is to prove the positive existence and uniqueness of the local weak solution of a nonlinear $p(x)$-Laplacian equation in Theorem 4.1, 4.2. Also, we proved the stability of the solution under some restrictions in Theorem 4.3. Also, by estimating the regularization issue and employing the Moser iterative approaches, the locally uniform properties of the solution for the gradients can be determined. Furthermore, uniformly bounded properties and the $L^{p(x)}$-norm to gradient estimations are required for specific properties of the local solutions.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] R. Bassem and N. Al-Jawary, "Fixed Point Theory for Study the Controllability of Boundary Control Problems in Reflexive Banach Spaces," *Iraqi Journal of Science,* pp. 222-232, 2022.

[2] H. A. Aal-Rkhais and R. H. Qasim, "The Development of interfaces in a Parabolic p-Laplacian type diffusion equation with weak convection," in *Journal of Physics: Conference Series*, 2021, vol. 1963, no. 1: IOP Publishing, p. 012105.

[3] X. Fan and D. Zhao, "On the spaces Lp (x)(Ω) and Wm, p (x)(Ω)," *Journal of mathematical analysis and applications,* vol. 263, no. 2, pp. 424-446, 2001.

[4] J. A. A. Al-Hawasy and N. F. Mansour, "The Galerkin-Implicit Method for Solving Nonlinear Variable Coefficients Hyperbolic Boundary Value Problem," *Iraqi Journal of Science,* pp. 3997-4005, 2021.

[5] S. N. Antontsev and S. I. Shmarev, "A model porous medium equation with variable exponent of nonlinearity: existence, uniqueness and localization properties of solutions," *Nonlinear Analysis: Theory, Methods & Applications,* vol. 60, no. 3, pp. 515-545, 2005.

[6] C. Chen, "On global attractor for m-Laplacian parabolic equation with local and nonlocal nonlinearity," *Journal of mathematical analysis and applications,* vol. 337, no. 1, pp. 318-332, 2008.

[7] R. A. Hameed, M. A. Rasheed, O. E. Achieng, and W. M. Taha, "On the existence of periodic solutions to a p-Laplacian Allen-Cahn equation with Neumann boundary condition," in *AIP Conference Proceedings*, 2022, vol. 2398, no. 1: AIP Publishing.

[8] R. A. Hameed, F. N. Ghaffoori, H. S. Mustafa, W. M. Taha, and M. A. Rasheed, "Using Leray-Schauder topological degree to solve a linear diffusion parabolic equation with periodic initial conditions," *International Journal of Nonlinear Analysis and Applications,* vol. 13, no. 1, pp. 1629-1635, 2022.

[9] L. H. Ali and J. A. Al-Hawasy, "Boundary Optimal Control for Triple Nonlinear Hyperbolic Boundary Value Problem with State Constraints," *Iraqi Journal of Science,* pp. 2009-2021, 2021.

[10] C. Chen and R. Wang, "Global Existence and L^∞ Estimates of Solution for Doubly Degenerate Parabolic Equation," *ACTA MATHEMATICA SINICA-CHINESE EDITION-*, vol. 44, no. 6, pp. 1089-1098, 2001.

[11] E. DiBenedetto, *Degenerate parabolic equations*. Springer Science & Business Media, 1993.

[12] J. W. Cholewa and T. Dlotko, *Global attractors in abstract parabolic problems*. Cambridge University Press, 2000.

[13] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*. Springer Science & Business Media, 2012.

[14] P. Lei, Y. Li, and P. Lin, "Null controllability for a semilinear parabolic equation with gradient quadratic growth," *Nonlinear Analysis: Theory, Methods & Applications,* vol. 68, no. 1, pp. 73-82, 2008.
[15] J. Lions, "Quelques problémes de la théorie des équations non linéaires d'évolution," in *Problems in Non-Linear Analysis*: Springer, 1971, pp. 189-342.
[16] J. J. Manfredi and V. Vespri, "Large time behavior of solutions to a class of doubly nonlinear parabolic equations," 1994.
[17] J. Ildefonso Diaz, "New applications of monotonicity methods to a class of non-monotone parabolic quasilinear sub-homogeneous problems," *arXiv e-prints,* p. arXiv: 1908.11171, 2019.
[18] T. Nanbu, "On solutions of some nonlinear degenerate parabolic equations," *Nonlinear Analysis: Theory, Methods & Applications,* vol. 47, no. 3, pp. 1681-1692, 2001.
[19] M. Nakao and C. Chen, "Global existence and gradient estimates for the quasilinear parabolic equations of m-Laplacian type with a nonlinear convection term," *Journal of Differential Equations,* vol. 162, no. 1, pp. 224-250, 2000.
[20] H. Wang and C. Chen, "L∞ $ L^{\infty} $ decay estimates of solutions of nonlinear parabolic equation," *Boundary Value Problems,* vol. 2021, no. 1, pp. 1-11, 2021.
[21] G. Kristensson, *Second order differential equations: special functions and their classification*. Springer Science & Business Media, 2010.
[22] H. Zhan, "The asymptotic behavior of a doubly nonlinear parabolic equation with a absorption term related to the gradient," *WSEAS Transactions on Mathematics,* vol. 10, no. 7, pp. 229-238, 2011.
[23] S. Uchida, "Solvability of Doubly Nonlinear Parabolic Equation with $ p $-Laplacian," *arXiv preprint arXiv:2010.10020,* 2020.
[24] H. A. Aal-Rkhais, A. H. Kamil, and K. F. Al Oweidi, "The Approximation of Weighted Hölder Functions by Fourier-Jacobi Polynomials to the Singular Sturm-Liouville Operator," *Baghdad Science Journal,* vol. 19, no. 6, pp. 1387-1387, 2022.
[25] K. F. Al Oweidi *et al.*, "Partial differential equations of entropy analysis on ternary hybridity nanofluid flow model via rotating disk with hall current and electromagnetic radiative influences," *Scientific Reports,* vol. 12, no. 1, p. 20692, 2022.
[26] K. F. Al Oweidi *et al.*, "Partial differential equations modeling of thermal transportation in Casson nanofluid flow with arrhenius activation energy and irreversibility processes," *Scientific Reports,* vol. 12, no. 1, p. 20597, 2022.