DOI: 10.24996/ijs.2023.64.11.30

ISSN: 0067-2904

Determination of Spacewise− Dependent Heat Source Term in Pseudoparabolic Equation from Overdetermination Conditions

Sayl Gani[1,2*], M. S. Hussein[3]

[1] Department of Mathematics, College of Education for Pure Sciences Ibn-AL-Haithem, University of Baghdad, Baghdad, Iraq

[2] Department of Computer Techniques Engineering, Imam Al-Kadhum College, Baghdad, Iraq

[3] Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received: 22/12/2022 Accepted: 10/2/2023 Published: 30/11/2023

Abstract:

This paper examines the finding of spacewise dependent heat source function in pseudoparabolic equation with initial and homogeneous Dirichlet boundary conditions, as well as the final time value / integral specification as additional conditions that ensure the uniqueness solvability of the inverse problem. However, the problem remains ill-posed because tiny perturbations in input data cause huge errors in outputs. Thus, we employ Tikhonov's regularization method to restore this instability. In order to choose the best regularization parameter, we employ L-curve method. On the other hand, the direct (forward) problem is solved by a finite difference scheme while the inverse one is reformulated as an optimization problem. The later problem is accomplished by employing *lsqnolin* subroutine from MATLAB. Two test examples are presented to show the efficiency and accuracy of the employed method by including many noises level and various regularization parameters.

Keywords: Pseudo-parabolic equation, Inverse problem, von Neumann stability analysis, Finite difference method, Tikhonov regularization method.

تحديد المصدر الحراري المعتمد على الفضاء في معادلة شبه قطع مكافئ من شروط إضافية

سيل غني[1و2*]، محمد صباح حسين[3]

[1] قسم هندسة تقنيات الحاسوب، كلية الامام الكاظم (ع) للعلوم الإسلامية الجامعة، بغداد، العراق.

[2] قسم الرياضيات، كلية التربية للعلوم الصرفة/ ابن الهيثم، جامعة، بغداد، العراق.

[3] قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق.

الخلاصة

هذا البحث يبحث في ايجاد المصدر الحراري المعتمد على الفضاء في معادلة شبه قطع مكافئ مع شروط حدودية من نوع Dirichlet المتجانسة، بالإضافة إلى القيمة الزمنية النهائية / شرط التكامل كشرط إضافي. التي تضمن وحدانية و وجود الحل للمسألة العكسية. ومع ذلك، لا تزال المسألة عليلة الوضع لأن الاضطرابات الصغيرة في بيانات الإدخال تسبب أخطاء جسيمة في المخرجات. وبالتالي، فإننا نستخدم طريقة تنظيم تيخونوف لاستعادة الاستقرارية. من أجل اختيار أفضل معلمة تنظيم، نستخدم طريقة L– curve. من ناحية أخرى، تم حل المسألة المباشرة (الأمامية) من خلال مخطط الفروق المحدودة بينما تتم إعادة صياغة المسألة

*Email: saylgani@alkadhum-col.edu.iq

العكسية كمسألة امثليه. تم إنجاز المسألة الأخيرة من خلال استخدام روتين lsqnonlin الفرعي من MATLAB. تم عرض مثالين للاختبار كفاءة ودقة الطريقة المستخدمة من خلال تضمين العديد من مستويات الضوضاء ومختلف معاملات التنظيم.

1. Introduction

The pseudoparabolic equations of higher order play an important role in the Mathematical modeling of moisture transfer, fluid filtration and heat propagation [1], [2]. The pseudo-parabolic inverse problems are of great interest, which arise in the modeling of numerous phenomena like the wave processes, chemical, engineering, diffusion, plasma physics and heat conduction [3]. Also, pseudoparabolic inverse problems have many applications in real life phenomena like the theory of small oscillation of a rotating fluid [4] and infiltration of homogeneous fluids in strata [5].

A. I. Ismailov in [6] theoretically studied the two-dimensional pseudoparabolic inverse problem with the additional integral conditions. Sh. Lyubonova and A. Tani in [7] discussed the stabilization of multi-dimensional pseudoparabolic inverse problem with the coefficient of Piezo conductivity also they discussed the regularity of the solution. In [8], the authors analysed the existence and uniqueness of the solution of the third order pseudoparabolic inverse problem with periodic and integral conditions. Abylkairov and Khompysh [9] studied the existence and uniqueness of solution for right side of pseudoparabolic inverse problem which is described the motion of the Kelvin-Voight fluids. Antotsev et. al [10] proved the unique solvability for pseudoparabolic inverse problem with a P-Laplacian and under nonlocal integral over-determination condition by using the Galerkin method. For the other related work of pseudoparabolic inverse problems, see [11], [12], [13], [14], [15], [16].

The authors in [17] solved the pseudoparabolic problem to identify the space dependent forcing term. The authors in [18] presented the fourth order pseudoparabolic inverse problem to determine the unknown coefficient. In [19], the authors presented the inverse problem for pseudoparabolic equation with periodic boundary conditions, and it has been numerically solved to identify the space dependent heat source. Whilst, the authors in [20] considered the pseudoparabolic inverse problem with integral overdetermination condition to reconstruction the unknown time coefficient.

The other related work is found on the pseudoparabolic inverse problems. Irem and Timar in [21] solved the quasilinear pseudoparabolic equation with an unknown coefficient under periodic boundary conditions and overdetermination data to determine the coefficient and source term. While in [22] the multi-dimensional pseudoparabolic problem by using the meshless radial basis function method is solved. In [23], the authors employed the Cubic B-spline collection method to reconstruct time-dependent coefficients of the fourth order pseudoparabolic inverse problem subject to the additional nonlocal data.

In this paper, we investigate the third order psudoparabolic inverse problem to reconstruct the space dependent heat source coefficient with Dirichlet boundary conditions and two different types of additional data. To discretize the direct problem numerically, we use the finite difference method then for obtaining the solution for inverse problems, we minimised the least square's objective functional in a suitable norm. The novelty of this work occurs in firstly solving 3rd order pseudoparabolic inverse coefficient problem numerically via finite difference scheme combined with the Tikhonov regularization technique. Secondly, employing two parameters selection strategy, L- curve and minimum rmse curve in order to obtain an accurate and stable solution.

The article consists of eight sections: Section 2 is devoted to the mathematical formulation of inverse problem. In section 3, FDM is used to discretize the direct problem. In Section 4, the stability analysis is covered. Test examples for the direct problem are in Section 5. Whereas, Section 6 presents the minimization of the function-based numerical technique. In Section 7, the numerical results of the inverse problem I and II are described, respectively. Section 8 is the conclusions that are highlighted.

2. Mathematical Formulation

Consider the source determination problem for pseudoparabolic equation of the form

$$\frac{\partial v(x,\tau)}{\partial \tau}-\frac{\partial^3 v(x,\tau)}{\partial x^2 \partial \tau}-\frac{\partial^2 v(x,\tau)}{\partial x^2}=f(x) \quad (1)$$

Where $D_T := \{0 < x < l < \infty, 0 < \tau < T < \infty\}$ is the solution domain, under the initial condition

$$v(x,0)=\eta(x), \ \ 0 \le x \le l. \quad (2)$$

The homogeneous Dirichlet boundary condition

$$v(0,\tau)=v(l,\tau)=0, \quad 0 \le \tau \le T, \quad (3)$$

and the final time specification condition

$$v(x,T)=h(x), \quad 0 \le x \le l, \quad (4)$$

or integral mass additional condition

$$\int_0^T v(x,\tau)d\tau = e(x), \quad 0 \le x \le l. \quad (5)$$

We call equations (1) - (4) the inverse problem I (IP- I) and equations (1) - (3) and (5) the inverse problem II (IP- II). The unique solvability of above problems has been established in [24] and reads as follows:

Definition: The pair of functions give $(v(x,\tau), f(x))$ is called a classical solution to the IP-I or IP-II, if $v(x,\tau) \in C^{2,1}_{x,\tau}(D_T)$ and $f(x) \in C(0,l)$ and satisfies the equations (1)- (4) or (1)-(3) and (5).

Theorem 1. Consider the listed below conditions [24]:
$A1$: $\eta(x), h(x), e(x) \in C^3[0,l]$
$A2$: $\eta(0)=\eta(l)=0, \quad \eta''(0)=\eta''(l)=0$
$A3$: $h(0)=h(l)=0, \ \ h''(0)=h''(l)=0$
$A4$: $e(0)=e(l)=0, \ \ e''(0)=e''(l)=0$
If the conditions (A1)- (A3) hold, then IP-I has a unique solution and if the conditions (A1), (A2) and (A3) holds, then IP-II has a unique solution.

3. Discretization of direct problem

We present Eqs. (1) - (4) which is the direct problem when $f(x)$ and $\eta(x)$ are known and the temperature $v(x,\tau)$ is to be determined together with the required outputs $h(x)$or $e(x)$. Rewriting Eq. (1) by a form of (FDM) as follows [25], [26], [27]: Denote for $v(x_i,\tau_j)=v_{i,j}, and\ f(x_i)=f_i$ where space node $x_i = i\Delta x$, and time node $\tau_j = j\Delta\tau$, $\Delta x = \frac{1}{M_x}\ and\ \Delta\tau = \frac{T}{N_\tau}\ for\ i=0,1,\dots,M_x\ \ where\ \ M_x, N_\tau$ are positive integers and based on the finite difference method, Eq.(1) can be rewritten as:

$$\frac{v_{i,j+1}-v_{i,j}}{\Delta\tau}=\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{(\Delta x)^2}+\frac{1}{\Delta\tau}\left(\frac{v_{i+1,j+1}-2v_{i,j+1}+v_{i-1,j+1}}{(\Delta x)^2}\right)-\frac{1}{\Delta\tau}\left(\frac{v_{i+1,j}-2v_{i,j}+v_{i-1,j}}{(\Delta x)^2}\right)+f_{i,j}, \quad (6)$$

where,

$$v(x,0)=\eta(x_i),\quad i=0,1,\dots,M_x$$
$$v(0,\tau_j)=v(1,\tau_j)=0,\ \ j=0,1,\dots,N_\tau$$

rearranging Eq. (6), we get:

$$-\alpha v_{i-1,j+1}+(1+2\alpha)v_{i,j+1}-\alpha v_{i+1,j+1}=\gamma v_{i-1,j}+(1-2\gamma)v_{i,j}+\gamma v_{i+1,j}+\Delta\tau f_{i,j}, \quad (7)$$

where,

$$\alpha=\frac{1}{(\Delta x)^2},\quad \gamma=\frac{\Delta\tau}{(\Delta x)^2}-\frac{1}{(\Delta x)^2} \quad (8)$$

Applying boundary condition in Eq. (6) and then Eq. (1) can be rewritten as a system of linear algebraic equations $(M_x-1)\times(M_x-1)$ at each time τ_{j+1} for j = 0, …, N_τ.

$$DV^{j+1}=EV^j+b$$

where

$$V^{j+1}=\left(v_{1,j+1},v_{2,j+1},\dots,v_{M_x-1,j+1}\right)\ and\ V^j=\left(v_{1,j},v_{2,j},\dots,v_{M_x-1,j}\right).$$

The last equation is solved for the unknown vector V^{j+1} using the Gauss elimination method to solve the system to march the time level from j to j + 1. Where D and E are tridiagonal matrices that satisfy the condition of diagonal dominated entries.

$$\boldsymbol{D}=\begin{cases}1+2\alpha, & \text{(maindiogonal element)}\\ -\alpha, & \text{(upper and lower diagonal element)}\\ 0 & o.w\end{cases}$$

$$\boldsymbol{E}=\begin{cases}1-2\gamma, & \text{(maindiogonal element)}\\ -\gamma, & \text{(upper and lower diagonal element)}\\ 0 & o.w\end{cases}$$

$$b_1=\Delta\tau\left(f_{1,j}\right)+\alpha v_{0,j+1},\qquad j=0,1,\dots,N_\tau$$
$$b_i=\Delta\tau\left(f_{i,j}\right),\qquad i=2,\dots,M_x,\qquad j=0,1,\dots,N_\tau$$
$$b_{M_{x-1}}=\Delta\tau\left(f_{M_x-1,j}\right)+\alpha v_{M_x,j+1},\quad j=0,1,\dots,N_\tau$$

4. Stability analysis

We presented the Von Neumann stability analysis [28], [29] direct problem. We are taking $f_i=0$ in Eq. (7) then we get:

$$-\alpha v_{i-1,j+1}+(1+2\alpha)v_{i,j+1}-\alpha v_{i+1,j+1}=\gamma v_{i-1,j}+(1-2\gamma)v_{i,j}+\gamma v_{i+1,j}, \quad (9)$$

where,

$$\alpha=\frac{1}{(\Delta x)^2},\ \ \gamma=\frac{\Delta\tau}{(\Delta x)^2}-\frac{1}{(\Delta x)^2}. \quad (10)$$

now, we present the error as follows:

$$\varepsilon_{i,j}=V_{i,j}-v_{i,j}, \quad (11)$$

where $v_{i,j}$ is the numerical solution and $V_{i,j}$ is the exact solution and both $V_{i,j}$ and $v_{i,j}$ verifying Eq. (9). For the linear partial differential equations, the variation of error can express as a finite Fourier series in the interval [l, T] as:

$$\varepsilon(x,\tau)=\sum_{8=1}^{M_x=N_\tau}S_s e^{w\theta_s x}, \quad (12)$$

where $\theta_s = \frac{\pi s}{1}$; $s = 1,2,\dots,M_x$ and $w = \sqrt{-1}\,.\,e^{w\theta_s x}$ is called the Euler formula and S is a function of time. The term $v_{i,j} = S^j e^{wi\theta}$as the proposed solutions at x_i applying this data into Eq. (9) to find S as follows:

$$-\alpha S^{j+1}e^{w\theta(i-1)} + (1+2\alpha)S^{j+1}e^{wi\theta} - \alpha S^{j+1}e^{w\theta(i+1)}$$
$$= \gamma S^j e^{w\theta(i-1)} + (1-2\gamma)S^j e^{w\theta(i)}$$
$$+ \gamma S^j e^{w\theta(i+1)}, \quad (13)$$

simplifying above equation, we get:

$$\big((1+2\alpha) - 2\alpha\cos\theta\big)S = (1-2\gamma) + 2\gamma\cos\theta$$

which can be written as,

$$S = \frac{(1-2\gamma) + 2\gamma\, cos\,\theta}{(1+2\alpha) - 2\alpha\, cos\,\theta}$$

for $\gamma < 0$ and $\alpha > 0$ hence $\gamma < \alpha$ from this we get $|S| < 1$, then the scheme is unconditionally stable.

5. Example for direct problem

Consider the direct problem in Eqs. (1)-(3) with $T = l = 1$:

$$v(x,0) = \eta(x) = \sin(\pi x) - \sin(2\pi x), \quad x \in [0,1]$$

The analytical solution for the given data is:

$$v(x,\tau) = \sin(\pi x) - e^{-\frac{4\pi^2}{1+4\pi^2}\tau}\sin(2\pi x), \quad (x,\tau) \in D_T,$$

and heat source is:

$$f(x) = \pi^2\sin(\pi x), \quad x \in [0,1],$$

Figure 1 explains the exact and numerical solution for $v(x,\tau)$, and the absolute error. From this figure, one can notice an excellent matching with the error magnitude of order $O(10^{-3})$. Whilst, Figure 2 explains the comparison between the numerical and the exact solution for desired outputs $h(x)$ and $e(x)$. Also, accurate solutions are obtained.

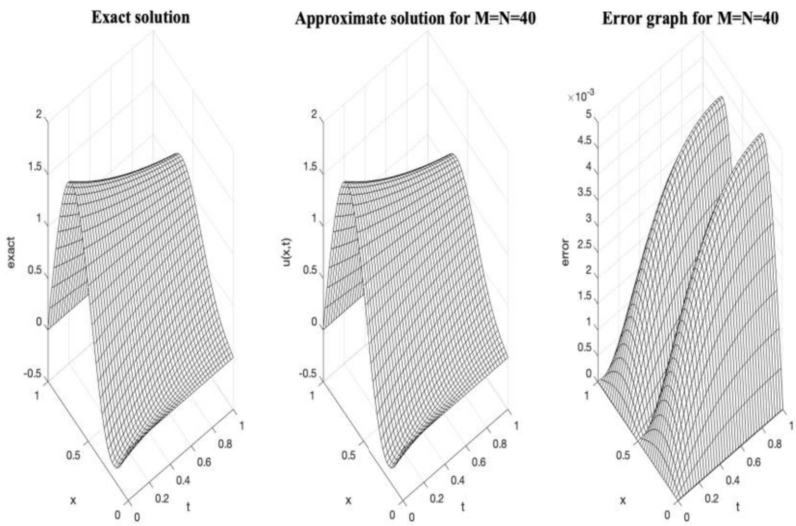
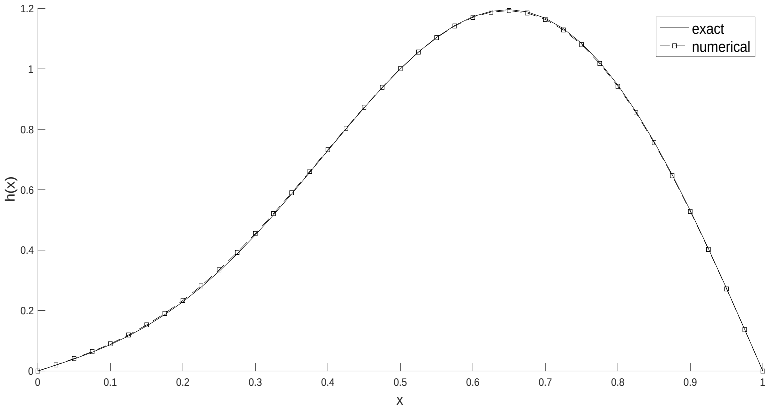
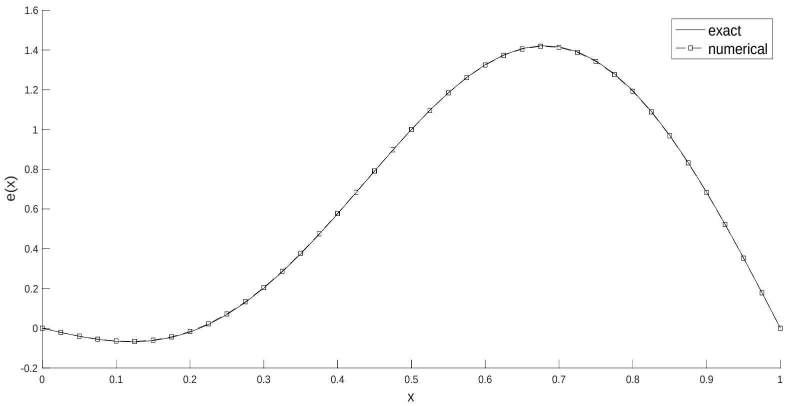


Figure 1: Exact (left), numerical (middle) solutions and absolute error (right) for direct problem (1)- (3) with mesh size $M_x = N_\tau = 40$.



(a)



(b)

Figure 2: Numerical and Exact solutions for (a) $h(x)$, (b) $e(x)$ when $M_x = N_\tau = 40$.

6. Inverse problem

This section is devoted to solving the inverse problems (1)- (4) and (1)- (3), (5), that is when $f(x)$ is unknown, in addition to heat distribution $v(x,T)$ which satisfy Eqs. (1)- (4) for IP-I and Eqs. (1)- (3) and (5) for IP-II. The numerical technique used is to recast the problems to an optimization problem by imposing the extra measurement (4) or (5) in a suitable norm and applying Tikhonov regularization to maintain stability. The cost functional
For IP − I is;

$$K_I(f) = \|v(x,T) - h(x)\|^2 + \beta\|f(x)\|^2, \tag{14}$$

For IP-II is:

$$K_{II}(f) = \left\|\int_0^T v(x,\tau)dx - e(x)\right\|^2 + \beta\|f(x)\|^2, \tag{15}$$

where, $\beta \geq 0$ the regularization parameter should be selected according to some techniques like L-curve [30], Morozov's discrepancy principle [31], or trial and error [32]. The discretized form of (14) and (15) are

$$K_I\left(\underline{f}\right) = \sum_{i=1}^{M_x}\left(v(x_i,T) - h(x_i)\right)^2 + \beta\sum_{i=1}^{M_x} f_i^2, \tag{16}$$

$$K_{II}\left(\underline{f}\right) = \sum_{i=1}^{M_x}\left(\int_0^T v(x_i,\tau)dx - e(x_i)\right)^2 + \beta\sum_{i=1}^{M_x} f_i^2, \tag{17}$$

The objective functions (16) and (17) are minimized by subroutine *lsqnonlin* from MATLAB optimization toolbox. This routine tries to solve nonlinear least- squares curve fitting problem starting from the initial guess f_0 for an unknown source term. Further, one can impose an upper and lower bound for f which are taken to be $[-10^2, 10^2]$, respectively. Also, in this routine, we did not need to provide the gradient of K_I, K_{II} that can be approximated internally by finite - differences. In order to solve this optimization problem, the Trust-Region-Reflective (TRR) algorithm was applied which mainly depends on the interior-reflective - Newton method [33- 36].

The following parameters are essential to start the minimization process and to terminate the minimization process when of the following prescribed parameters are achieved:

- Allowed number of iterations = 6000.
- Specified solution and objective function Tolerance = 10^{-15}.

The IP-I given by (1)-(4) and IP-II given by (1)-(3) and (5) are solved subject to noisy measurement and the exact data (4) or (5). The noise contaminated is simulated as [37- 41]:

$$h^{\epsilon}(x_i) = h(x_i) + \epsilon_i, \quad i = 1,2,\dots,M_x, \tag{18}$$

For IP-I

$$e^{\epsilon}(x_i) = e(x_i) + \epsilon_i, \quad i = 1,2,\dots,M_x, \tag{19}$$

for IP-II

where ϵ represents a Gaussian random vector with mean equal to zero and standard deviation μ is given by:

$$\mu = p \times \max_{x\in[0,l]} |h(x)|, \tag{20}$$

$$\mu = p \times \max_{x\in[0,l]} |e(x)|, \tag{21}$$

where p represents the percentage of noise. We use the normrnd built in function to generate the random variables $\epsilon = (\epsilon_i)$ $i = 1,2,\dots,M_x$ as follows:

$$\epsilon = normrnd(0, \mu, M_x)$$

7. Results and discussion

We introduce some test examples for each inverse problem to explain the stability and accuracy of the computational procedure based on the finite difference method combined with the minimization of the Tikhonov function (16) and (17).

To assess the reconstruction accuracy of the heat source, we use the root mean square error *rmse* which is given by the next expression

$$rmse(f) = \sqrt[2]{\frac{1}{M_x}\sum_{i=1}^{M_x}\left(f_i - f_{exact}(x_i)\right)^2},$$

7.1 Numerical results for IP -I

Assume the IP-I with l = T = 1 and input data:

$$v(x,0) = \eta(x) = \sin(\pi x) - \sin(2\pi x), \quad x \in [0,1],$$

$$v(x,T) = h(x) = \sin(\pi x) - e^{-\frac{4\pi^2}{1+4\pi^2}}\sin(2\pi x),$$

where the analytical solution are:

$$v(x,\tau) = sin(\pi x) - e^{-\frac{4\pi^2}{1+4\pi^2}\tau}\, sin(2\pi x),$$

$$f(x) = \pi^2 \sin(\pi x),$$

that can be checked by direct substitution.

Firstly, we examine the IP-I with various mesh sizes such as $M_x = N_\tau \in \{20, 40, 60\}$ in the absence of noise and regularization. From Figure 3 and Table 1, it can be realized that the best choice for M_x, N_τ is 60 which presents the lowest $rmse(f)$.

One can notice that all the conditions for IP −I are satisfied and hence the unique solvability of the solution is guaranteed. Initially, we try to retrieve the function $f(\mathrm{x})$ and $v(x,\tau)$for noise free case $(\mathrm{p}=0)$ then for $\mathrm{p} \in \{0.05\%, 0.5\%\}$ noisy data. The objective function (16) is plotted as a function of the number of iterations in Figure 4. The fast convergence that can be seen in Figure 4, which reaches a very low value of $O(10^{-14})$.

Table 1: Numerical information for IP-I.

$M_x = N_\tau$	No. of iteration	Obj. fun. Value at final iteration	$rmse$	Time consumed in second
20	11	1.42644E-17	0.3980	5.4
40	16	2.33092E-17	0.2003	15.6
60	15	2.71453E-14	0.1337	34.6

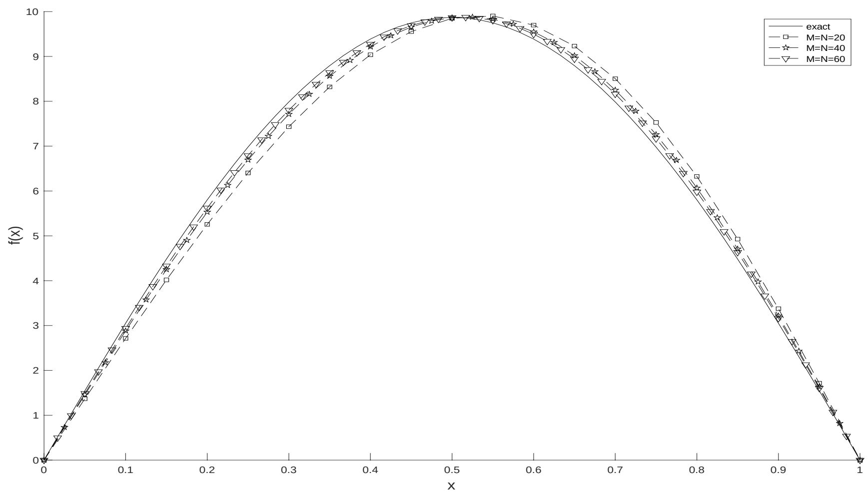


Figure 3: Numerical and exact solution for heat source $f(x)$ when $M_x = N_\tau \in \{20, 40, 60\}$.

Now, for evaluating the stability of the approximate solution with respect to (20), the noise was added $p \in \{0.05\%, 0.5\%\}$ to the additional data $h(x)$ as in (25). Figure 7 presents the identification of the estimated $f(x)$. For this case, the obtained results were inaccurate and unstable when the regularization parameter $\beta = 0$, see Figure 5. Hence, the Tikhonov regularization scheme was applied by adding the regularization parameter $\beta = \{10^{-4}, 10^{-5}, 10^{-6}\}$ (for both cases of noise data) to restore the stability. Figures 6.a and 7.a show that the objective function (16) decreases rapidly and reach a stationary value of $O(10^{-3})$. A set of values of regularization parameter $\beta = \{10^{-4}, 10^{-5}, 10^{-6}\}$ was applied for $p = 0.05\%$ and $\mathrm{p} = 0.5\%$ noise seen in Figure 6.b and Figure 7.b respectively, these figures show that the approximate solution of heat source $f(x)$ is stable and reasonable. Exact and numerical temperature $v(x,\tau)$ with regularization data for both cases $p \in \{0.05\%, 0.5\%\}$ are plotted in Figure 12. Table 2 depicts other details about $rmse(f)$, the number of iterations and computational time.

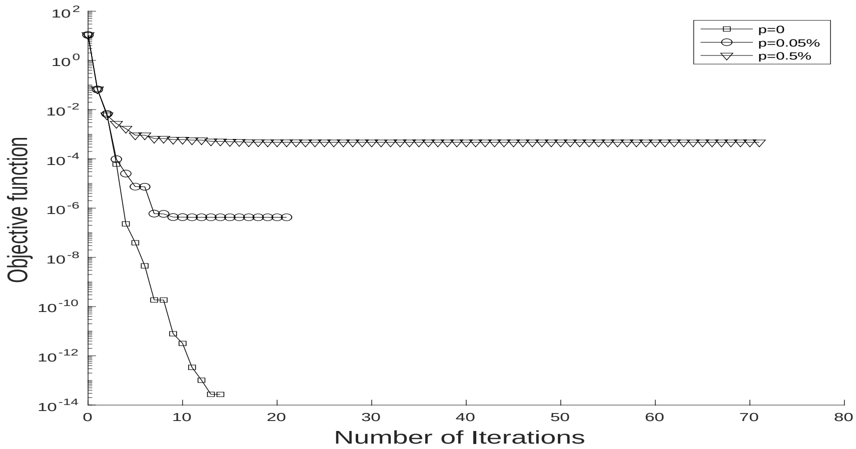


Figure 4: The unregularized objective function (16), with $p = \{0, 0.05\%, 0.5\%\}$ noise.

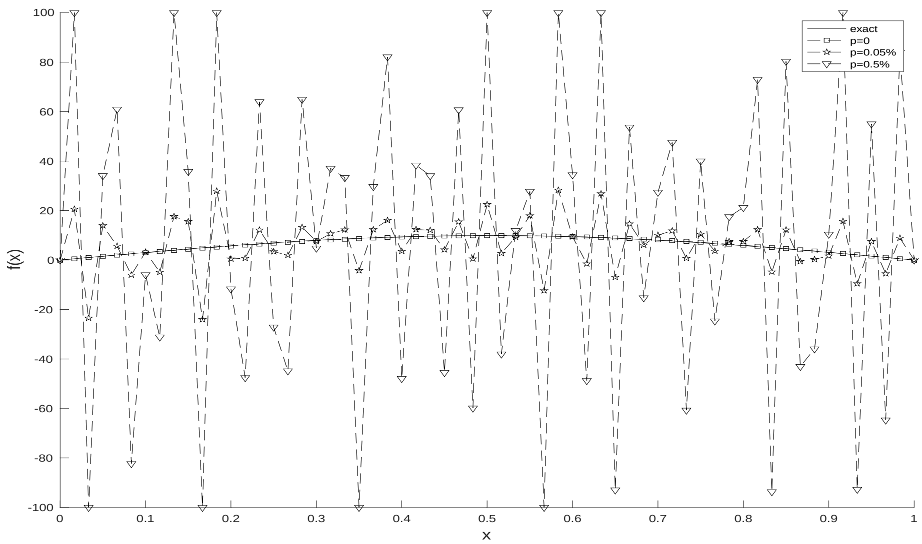
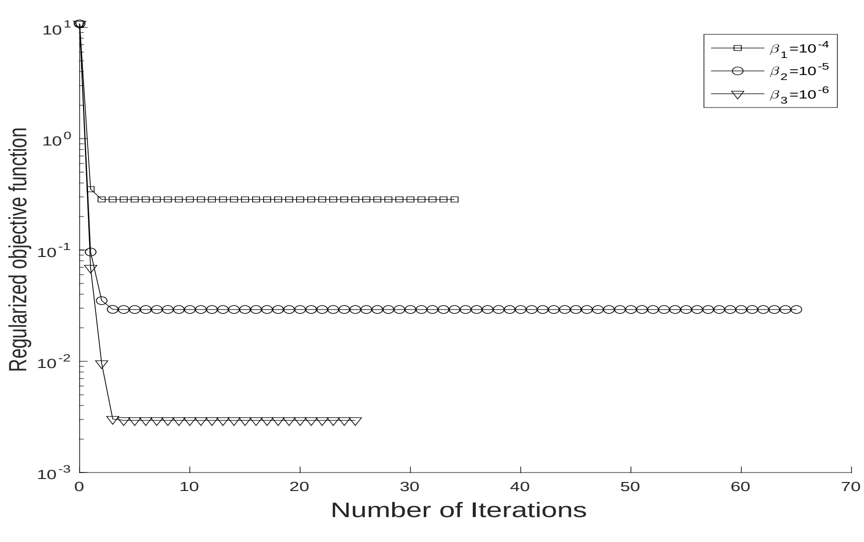
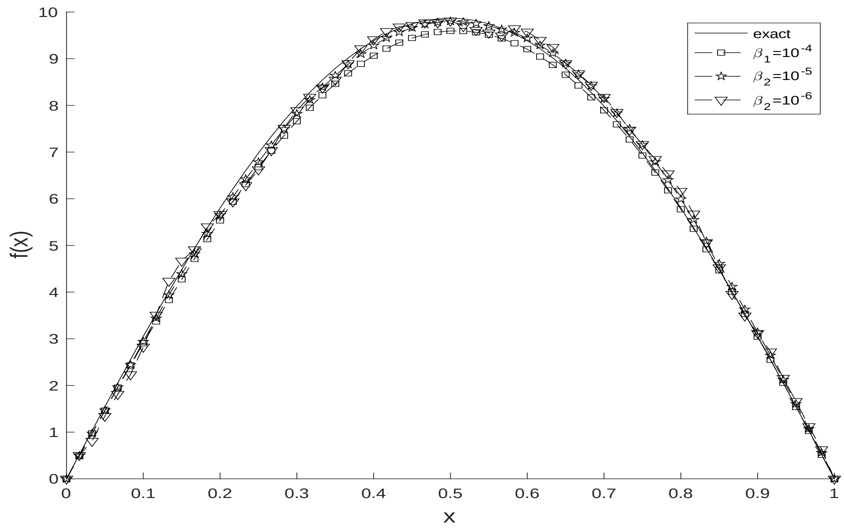


Figure 5: Numerical reconstructions and exact solution for $f(x)$, with various noise level $p = \{0, 0.05\%, 0.5\%\}$, without regularization applied for IP-I.

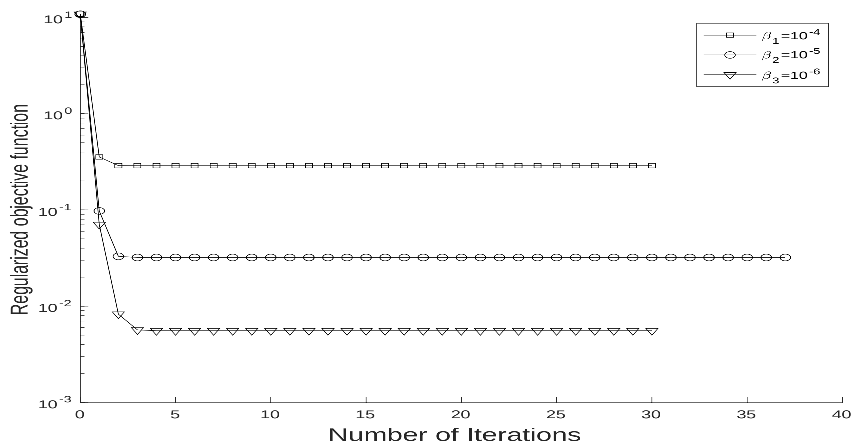


(a)

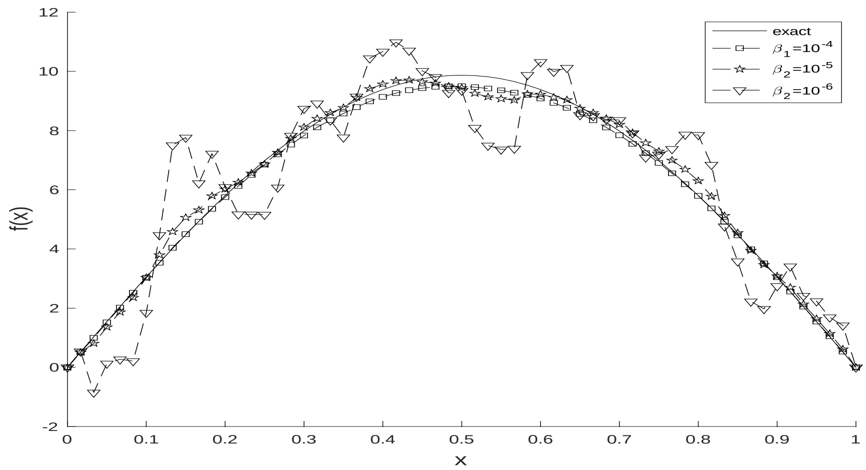


(b)

Figure 6: (a) The objective function (16), (b) Numerical reconstructions and exact solution for $f(x)$, with regularization parameter $\beta = \{10^{-4}, 10^{-5}, 10^{-6}\}$ and $p = 0.05\%$ noise.



(a)



(b)

Figure 7: (a) The objective function (16), (b) Numerical reconstructions and exact solution for $f(x)$, with regularization parameter $\beta = \{10^{-4}, 10^{-5}, 10^{-6}\}$ and $p = 0.5\%$ noise.

The L-curve plot is a powerful tool to choose the suitable regularization parameter for the given data in the regularization method. The L- curve is based on a log -log plot of the residual norm $\|v(x,\tau)-h(x)\|$versus the corresponding regularized solution norm $\|f_\beta\|$. It is an appropriate graphical tool for displaying trade-off between the fit of the given data and the size of a regularization solution [42], [43].

Figures 8 and 9 show that the best value of regularization parameters are $\beta = \{10^{-4}, 10^{-5}, 10^{-6}\}$ which are located near the corner of L- shape curve.

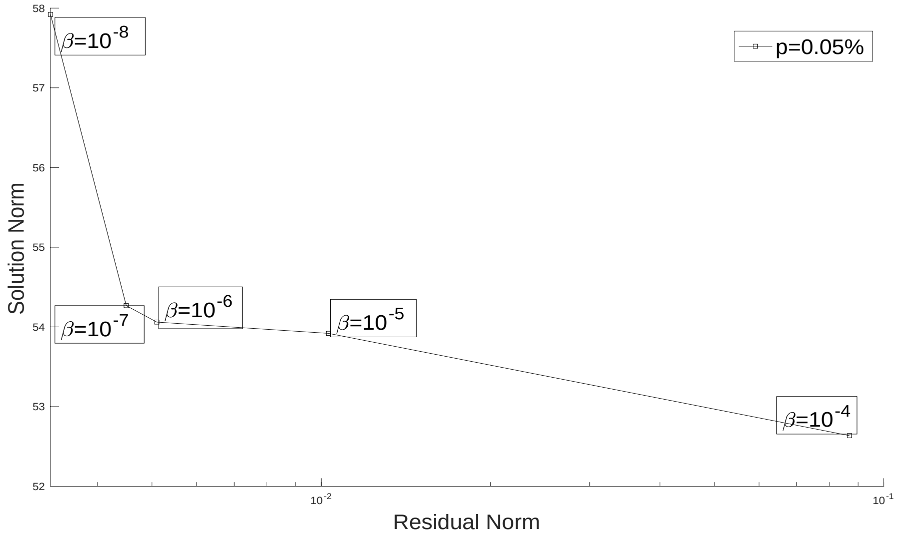


Figure 8: The L-curve plot with various regularization when *p* = 0.05% noise.

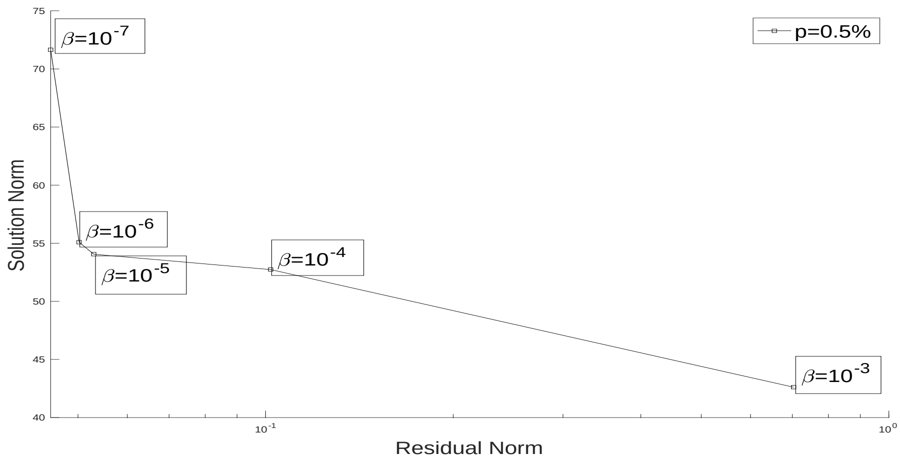


Figure 9: The L-curve plot with various regularization when p = 0.5% noise.

In fact, L-curve consists of two parts: the vertical part for small values of β such as $\{10^{-8}, 10^{-7}, 10^{-6}\}$. In this part, the solution norm starts to increase. Then horizontal part for large values of β such as $\{10^{-4}, 10^{-3}, \ldots, 1\}$. Which is this part as β increases the residual decreases. Therefore, the best selection for β can be found in corner of the L- curve, as clearly visual in Figures 8 and 9. Also, Figures 10 and 11 present the graph of $rmse(f)$ versus the regularization parameter β varies from 10^{-8} to 10^{-2}. From these figures, it can deduce that the best value for β meets the minimum point of the curve which is associated with appropriate β, see Table 2, for more numerical information obtained from an iterative process of minimization subroutine *lsqnonlin*.

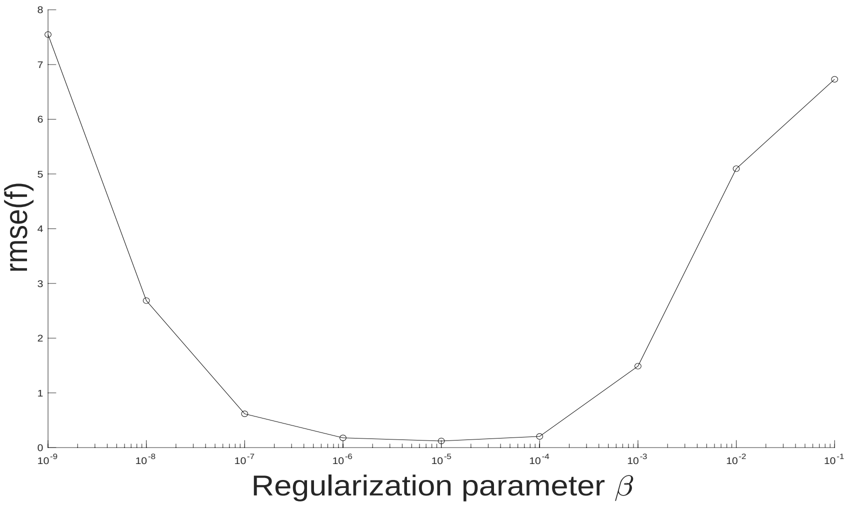


Figure 10: The $rmse(f)$ plot with various regularization when $p = 0.05\%$ noise.

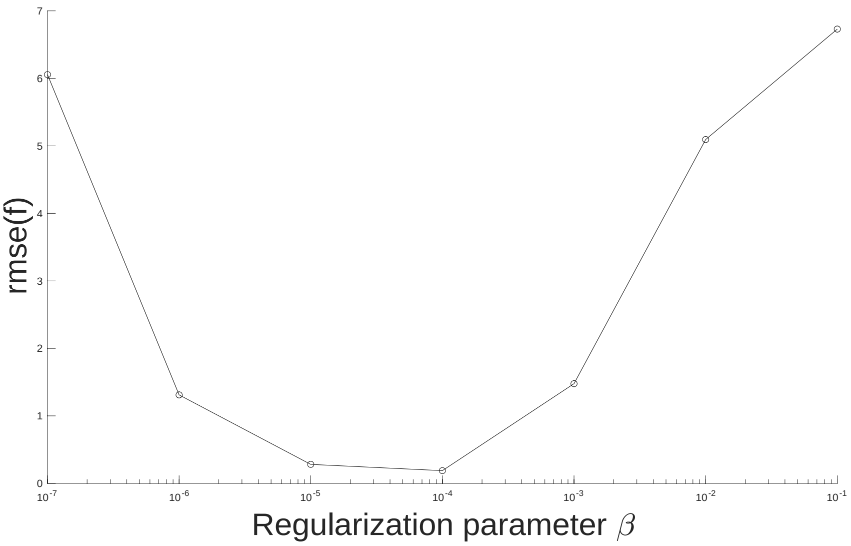


Figure 11: The $rmse(f)$ plot with various regularization when $p = 0.5\%$ noise.

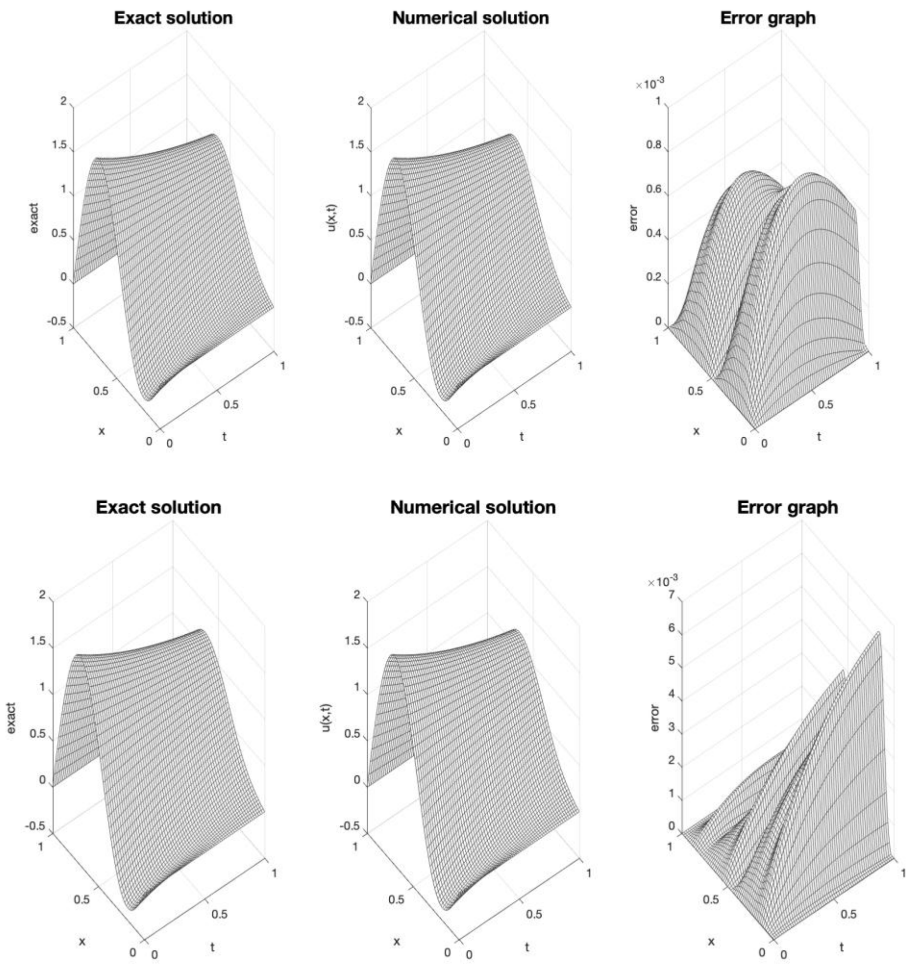


Figure 12: Exact and numerical temperature $v(x,\tau)$ with (a) $p = 0.05\%$ and $\beta = 10^{-6}$, (b) $p = 0.5\%$ noise and $\beta = 10^{-6}$.

Table 2: Numerical information for IP-I with noise

$P = 0.05\%$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
No. of iteration	34	65	25
Objective function (16) at final iteration	0.2846	0.0291	0.0029
$rmse(f)$	0.2036	0.1209	0.1777
Computational time	93.1	128.6	57.6
$P = 0.5\%$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
No. of iteration	31	38	31
Objective function (16) at final iteration	0.2885	0.0320	0.0056
$rmse(f)$	0.1888	0.2823	1.3114
Computational time	69.8	75.0	64.8

7.2 Numerical results of IP-II

We examine the IP −II which described by (1)- (3) and (5) with the following data when T = 1:

$$v(x,0) = \eta(x) = \sin(\pi x) - \sin(2\pi x), \quad x \in [0,1].$$

$$e(x) = \sin(\pi x) - \frac{\left(1 - e^{-\frac{4\pi^2}{1+4\pi^2}}\right)(1+4\pi^2)\sin(2\pi x)}{4\pi^2}, \quad x \in [0,1],$$

with unknown heat source and solution

$$f(x) = \pi^2 \sin(\pi x)$$

$$v(x,\tau) = \sin(\pi x) - e^{-\frac{4\pi^2}{1+4\pi^2}\tau} \sin(2\pi x),$$

The conditions in Theorem 1 are satisfied and hence, the solution of IP − II exists and is unique. Firstly, starting with case $p = 0$ i.e, without noise. Figure 13, represents the objective minimization in this case can be seen a rapid convergence to reach a low stationary value $O(10^{-15})$ in 16 iterations only. For evaluating the stability of numerical results, adding a small percentage $p = \{0.05\%, 0.5\%\}$ of noise. Figure 14 shows that the inaccurate and unstable results of heat source $f(x)$ are obtained in this case. The Tikhonov regularization method employs to obtain stable reconstruction for $f(x)$. L-curve method and minimum $rmse$ curve (as explained in previous example of IP-I) are applied to identify the appropriate regularization parameter, these are shown in Figures 17- 20.

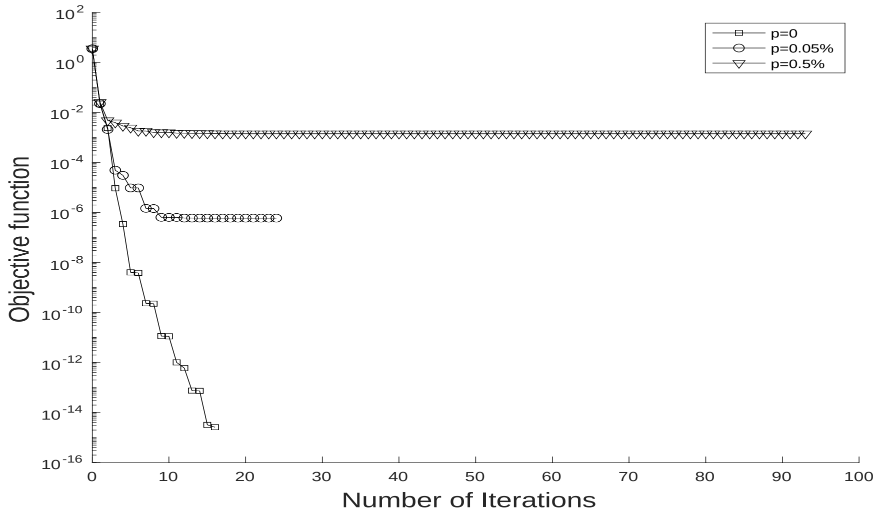


Figure 13: Objective function (17) with various noise and no regularization.

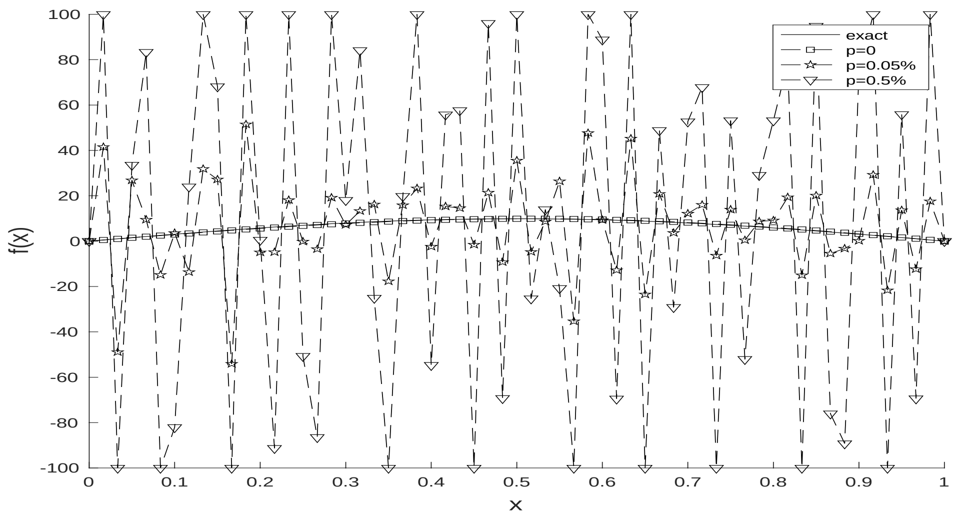
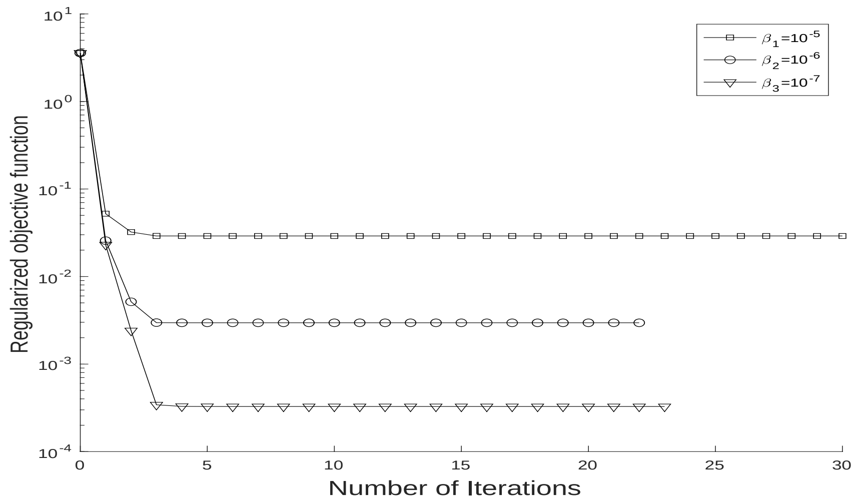
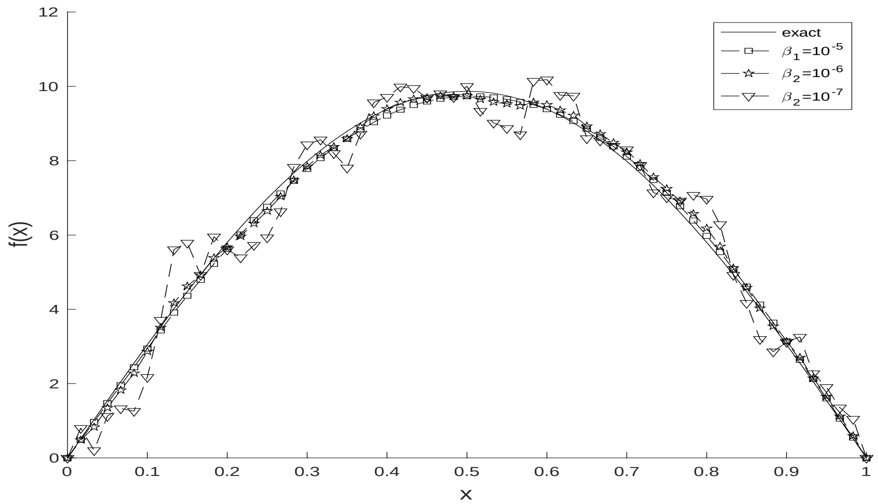


Figure 14: Numerical solution and Exact solution for $f(x)$ with no regularization.

Regularization parameters $\beta = \{10^{-7}, 10^{-6}, 10^{-5}\}$ was chosen for the noise data $p = 0.05\%$. Figure 17.a shows that the objective function (17) decreases rapidly in a small number of iterations. The Tikhonov approach with selected parameters gives a reasonable and stable approximate solution of heat source $f(x)$(see Figure 17.b). When $p = 0.05\%$, deduce the regularization parameters $\beta = \{10^{-6}, 10^{-5}, 10^{-4}\}$, one can observe that these choices of β give the stable and accurate approximate solution for heat source $f(x)$ (see Figure 16). It can be seen in Table 3 that the numerically observing results become more accurate and stable when the percentage of noise p decreases from 0.5% to 0.05%. In Figures 17- 20, one can notice that $\beta = 10^{-5}$ represents the optimal value based on L-curve criteria or $rmse$-graph. The numerical result of $v(x,\tau)$ are presented in Figure 21, this figure shows the analytical (31) and numerical solution for temperature $v(x,\tau)$ together with the absolute error graph.

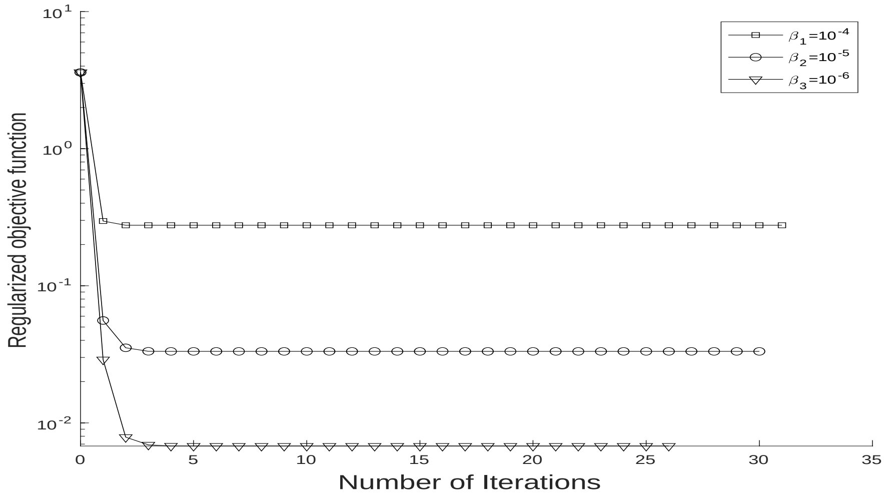


(a)

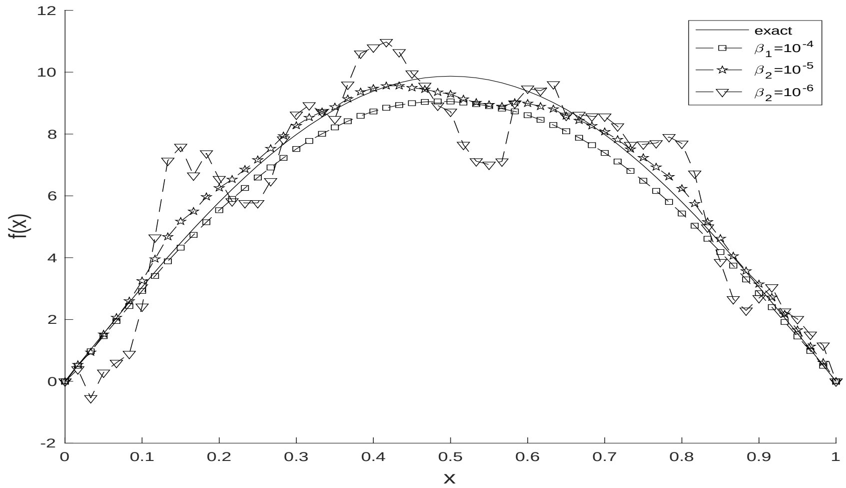


(b)

Figure 15: (a) The objective function (17), (b) Numerical reconstructions and exact solution for $f(x)$, with regularization parameter $\beta = \{10^{-5}, 10^{-6}, 10^{-7}\}$ and p $= 0.05\%$ noise applied for
IP-II.



(a)



(b)

Figure 16: (a) The objective function (17), (b) Numerical reconstructions and exact solution for $f(x)$, with regularization parameter $\beta = \{10^{-4}, 10^{-5}, 10^{-6}\}$ and p $=$ 0.5% noise.

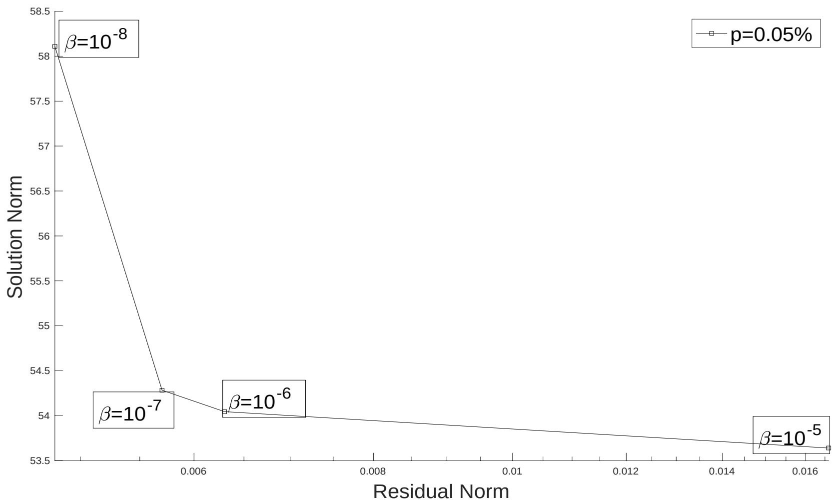


Figure 17: The L-curve plot with various regularization parameter when $p = 0.05\%$ noise.

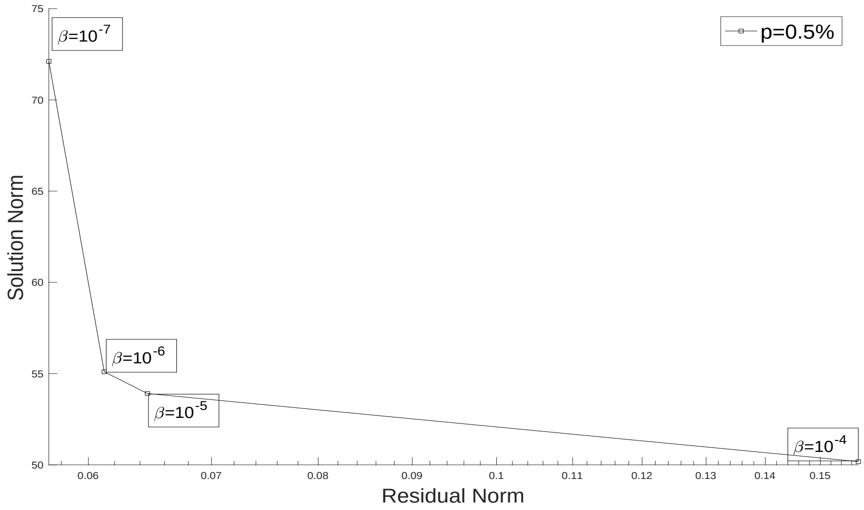


Figure 18: The L-curve plot with various regularization parameter when $p = 0.5\%$ noise.

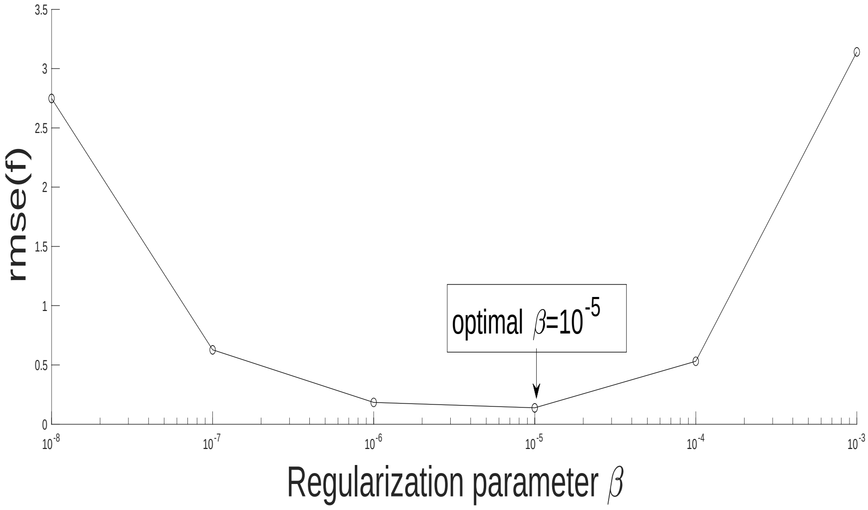


Figure 19: The $rmse(f)$ plot with various regularization parameter when p = 0.05% noise.

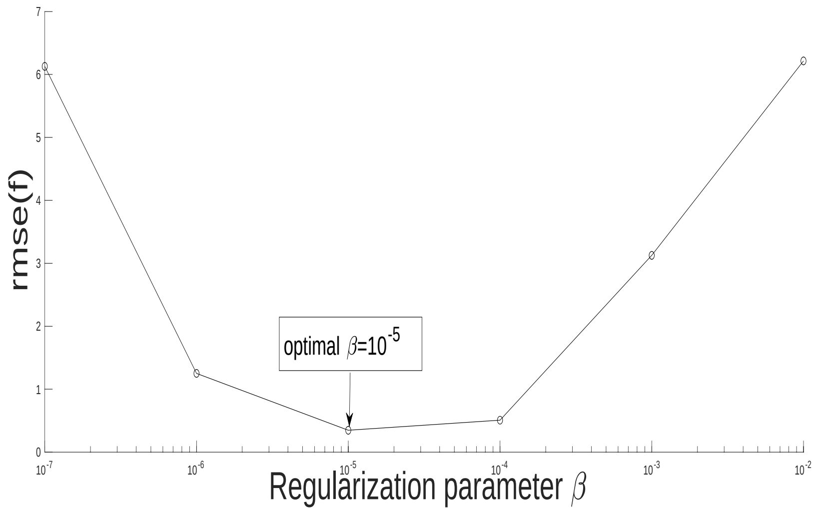


Figure 20: The $rmse(f)$ plot with various regularization parameter when $p = 0.5\%$ noise.

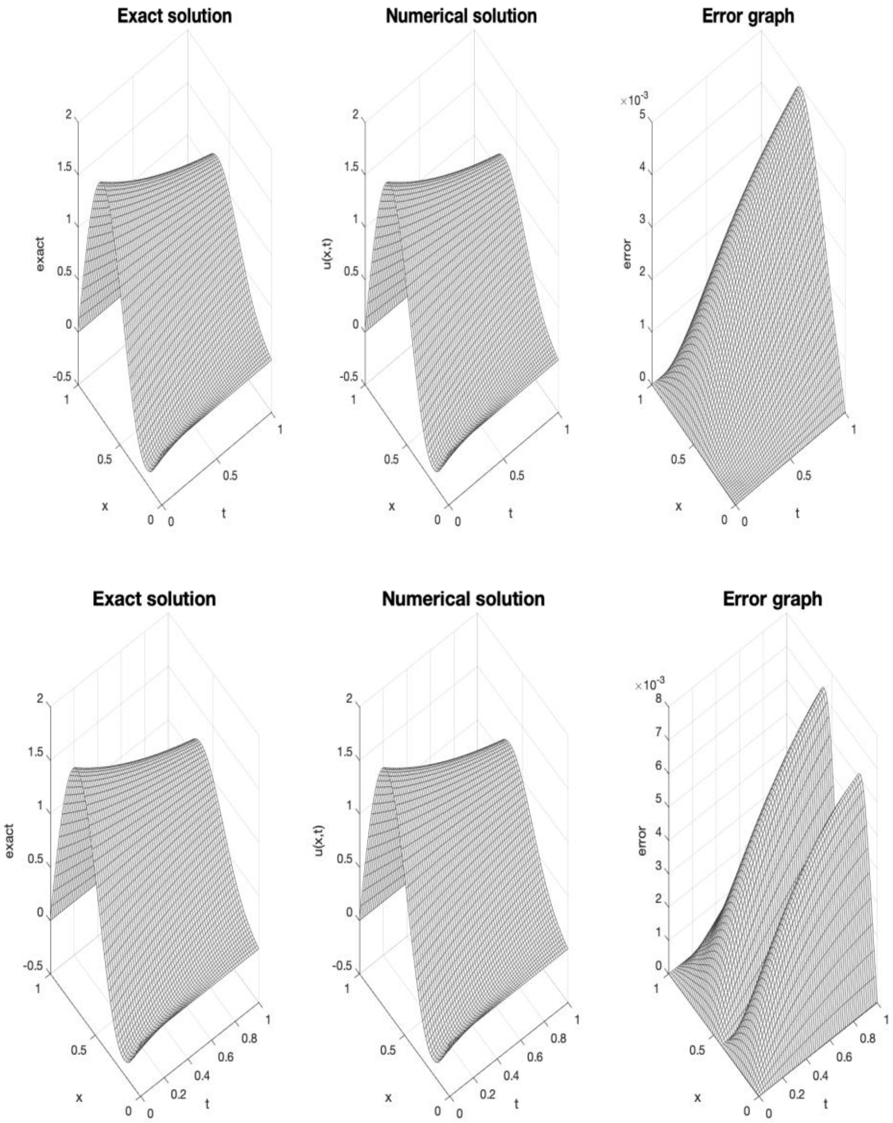


Figure 21: The exact temperature and numerical temperature $v(x,\tau)$, with (a) $p = 0.05\%$ noise and regularization parameter $\beta = 10^{-5}$, (b) $p = 0.5\%$ noise and $\beta = 10^{-5}$.

Table 3: Numerical information of IP-II with noise and regularization

$P = 0.05\%$	$\beta = 10^{-5}$	$\beta = 10^{-6}$	$\beta = 10^{-7}$
No. of iteration	31	23	24
Objective function (16) at final iteration	0.0290	0.0030	3.2E-4
$rmse(f)$	0.1378	0.1836	0.6273
Computational time	64.6	49.8	50.1
$P = 0.5\%$	$\beta = 10^{-4}$	$\beta = 10^{-5}$	$\beta = 10^{-6}$
No. of iteration	32	31	27
Objective function (16) at final iteration	0.2765	0.0332	0.0068
$rmse(f)$	0.5084	0.3476	1.2510
Computational time	64.9	76.0	55.7

Finally, we notice that from Tables 2 and 3, the IP-I is better than IP-II in $rmse(f)$ but in time consumed IP-II is better than IP-I. Mathematically, this case is because IP-I is a particular case from IP-II in terms of approximating integral data as the average of temperature distribution over a time interval. However, from a practical point of view, the second problem is more applicable than the first one. Because the average temperature distribution is much easier to capture than the final time data.

8- Conclusions

The pseudoparabolic inverse problems from a class of third order are presented to recover the space dependent source $f(x)$ numerically. The initial and Dirichlet boundary conditions with overdetermination conditions are used for unique recovery. The inverse problems investigate under temperature distribution at final time condition and mass/energy specification (integral type) data. FDM method based on the Cranck-Nicholson scheme utilised to discrete the direct problem. Also, the von Neumann technique was used to study the stability and convergence of the proposed numerical direct algorithm. The inverse problems were reformulated as a nonlinear optimization problem and solved numerically by *lsqnonlin* iterative routine from MATLAB. We test the source reconstruction for both exact and noisy data to evaluate the stability of the approximated solution. To stabilize the ill-posed problem under investigation, we apply Tikhonov's regularization method. Which is based on converting ill-posed problem to a family of well-posed problems related to the regularization parameters. Choosing regularization parameter β that balances accuracy and stability is the major challenge when solving the ill-posed problem with regularization. In this research, we employ L -curve criteria and minimum $rmse$ curve to find the best/ optimal regularization parameters which give the stable and accurate solution. Finally, a couple of numerical test examples are given, and the accuracy of results is presented by figures and tables which confirm the stability of our results.

References

[1] A. Asanov and E.R. Atamanov." *Nonclassical and inverse problems for pseudoparabolic equations*". In Nonclassical and Inverse Problems for Pseudoparabolic Equations. De Gruyter, 2014.

[2] D. Colton. "Pseudoparabolic equations in one space variable". *Journal of Differential Equations*, vol.12, no.3 , pp.559–565, 1972.

[3] K. Khompysh. *Inverse problem for 1d pseudo-parabolic equation*. Vol. 216, pp. 382–387. Springer New York LLC, 2017.

[4] S. L. Sobolev. *On a new problem of mathematical physics*. In Selected Works of SL Sobolev, pp. 279–332. Springer, 2006.

[5] G. I. Barenblatt, I. P. Zheltov, and I.N. Kochina. "Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]". *Journal of applied mathematics and mechanics*, vol.24, no.5, pp.1286–1303, 1960.

[6] A.I. Ismailov. "Inverse boundary value problem for two-dimensional pseudo parabolic equation of third order with addi-tional integral condition". *Caspian Journal of Applied Mathematics, Ecology and Economics*, vol. 1 , pp. 43–54, 2020.

[7] A. S. Lyubanova and A. Tani. "An inverse problem for pseudoparabolic equation of filtration: the stabilization". *Applicable Analysis*, vol.93, no.3, pp.573–585, 2013.

[8] Y.T. Mehraliyev and G.K. Shafiyeva. "Inverse boundary value problem for the pseudoparabolic equation of the third order with periodic and integral conditions". *Applied Mathematical Sciences*, vol.8, no.23, pp.1145–1155, 2014.

[9] U.U. Abylkairov and Kh. Khompysh. "An inverse problem of identifying the coefficient in kelvin-voight equations". *Appl Math Sci*, vol.9, no.101-104, pp. 5079–5088, 2015.

[10] S. N. Antontsev, S. E. Aitzhanov, and G. R. Ashurova. "An inverse problem for the pseudoparabolic equation with p-laplacian". *Evolution Equations and Control Theory*, vol.11, no.2, pp.399, 2022.

[11] X. Gao. "The solution of the inverse problem for a one-dimensional pseudo-parabolic equation". *In WORLD SCIENCE: PROBLEMS AND INNOVATIONS*, pp.17–22, 2022.

[12] L.D. Long and R. Saadati. "Regularization of inverse initial problem for conformable pseudoparabolic equation with inhomogeneous term". *Journal of Function Spaces*, vol.2022, 2022.

[13] A. Sh. Lyubanova and A. V. Velisevich. "Inverse problems for the stationary and pseudoparabolic equations of diffusion". *Applicable Analysis*, vol.98, pp.1997–2010, 8 2019.

[14] T. B. Ngoc, Y. Zhou, D. O'Regan, and N. H. Tuan. "On a terminal value problem for pseudoparabolic equations involving riemann–liouville fractional derivatives". *Applied Mathematics Letters*, vol.106:106373, 2020.

[15] A. Sh. Lyubanova "The regularity of the solutions of inverse problems for the pseudoparabolic equation". *Journal of Siberian Federal University. Mathematics and Physics*, vol.14, no.4, pp.414–424, 2021.

[16] S.N. Shergin, E. I. Safonov, and S. G. Pyatkov. "On some inverse coefficient problems with the pointwise overdetermination for mathematical models of filtration". *Math-Net.Ru All Russian mathematical portal*, vol.12, no.1, pp.82–95, 2019.

[17] D. Serikbaev and N. Tokmagambetov. "An inverse problem for the pseudo-parabolic equation for a sturm-liouville operator". *News of the National Academy of sciences of the Republic of Kazakhstan*., vol.4, pp. 122–128, 2019.

[18] A.T. Ramazanova, Y.T. Mehraliyev, and S.I. Allahverdieva. "On an inverse boundary value problem with non-local integral terms condition for the pseudo-parabolic equation of the fourth order". *Differential equations and their applications in mathematical modeling, Saransk*, July, pp. 9–12, 2019.

[19] M. J. Huntul. "Space-dependent heat source determination problem with nonlocal periodic boundary conditions". *Results in Applied Mathematics*, vol.12:100223, 2021.

[20] M. J. Huntul. "Identifying an unknown heat source term in the third-order pseudo-parabolic equation from nonlocal integral observation". *International Communications in Heat and Mass Transfer*, vol.128, no.11, 2021.

[21] I. BAGLAN and T. CANEL. "An inverse coefficient problem for quasilinear pseudo- parabolic of heat conduction of poly (methyl methacrylate) (pmma)". *Turkish Journal of Science*, vol.5, no.3, pp.199–207, 2020.

[22] N. H. Can, Y. Zhou, N. H. Tuan, and T. N. Thach. "Regularized solution approximation of a fractional pseudo-parabolic problem with a nonlinear source term and random data". *Chaos, Solitons & Fractals*, vol. 136:109847, 2020.

[23] M. J. Huntul, M. Tamsir, and N. Dhiman. *An inverse problem of identifying the timedependent potential in a fourth-order pseudo-parabolic equation from additional condition.Numerical Methods for Partial Differential Equations*, 2021.

[24] Kh. Kenzhebai. "An inverse problem of recovering the right hand side of 1D peseudoparabolic equation". *Journal of Mathematics, Mechanics and Computer Science*, vol.3, pp. 28–37,2021.

[25] M. J Huntul and M. S Hussein. "Simultaneous identification of thermal conductivity and heat source in the heat equation". *Iraqi Journal of Science*, pp.1968–1978, 2021.

[26] M. S. Hussein and D. Lesnic. "Determination of the time-dependent thermal conductivity in the heat equation with spacewise dependent heat capacity". *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics*), vol.9045, pp.217–224, 2015.

[27] M. S. Hussein and D. Lesnic. "Simultaneous determination of time-dependent coefficients and heat source". *International Journal for Computational Methods in Engineering Science and Mechanics*, vol.17, pp.401–411, 2016.

[28] H. N. aglar, S. H. aglar, and E. H. Twizell. *The numerical solution of fifth-order boundary value problems with sixth-degree b-spline functions*, 1999.

[29] N. Dhiman and M. Tamsir. "A collocation technique based on modified form of trigonometric cubic b-spline basis functions for fisher's reaction-diffusion equation. Multidiscipline Modeling in Materials and Structures", vol.14, pp.923–939, 10 2018.
[30] P. C. Hansen. "Analysis of discrete ill-posed problems by means of the l-curve". *SIAM review*, vol.34, no.4, pp.561–580, 1992.
[31] V. A. Morozov. "*On the solution of functional equations by the method of regularization*". In Doklady Akademii Nauk, vol. 167, pp. 510–512. Russian Academy of Sciences, 1966.
[32] M. S. Hussein and D. Lesnic. "Simultaneous determination of time and space-dependent coefficients in a parabolic equation". *Communications in Nonlinear Science and Numerical Simulation*, vol.33, pp.194–217, 2016.
[33] T. F. Coleman and Y. Li. "On the convergence of interior-reflective newton methods for nonlinear minimization subject to bounds". *Mathematical programming*, vol.67,no.1, pp.189–224, 1994.
[34] T. F Coleman and Y. Li. "An interior trust region approach for nonlinear minimization subject to bounds". *SIAM Journal on optimization*, vol.6, no.2, pp.418–445, 1996.
[35] F. Anwer and MS Hussein. "Retrieval of timewise coefficients in the heat equation from nonlocal overdetermination conditions". *Iraqi Journal of Science*, pp. 1184–1199, 2022.
[36] Q. W. Ibraheem, M. S. Hussein, "Determination of time-dependent coefficient in time fractional heat equation". *Partial Differential Equations in Applied Mathematics*, vol.100492, 2023.
[37] M. Qassim, M. S. Hussein, Numerical Solution to Recover Time-dependent Coefficient and Free Boundary from Nonlocal and Stefan Type Overdetermination Conditions in Heat Equation. *Iraqi Journal of Science*, pp.950-960, 2021.
[38] M. S. Hussein, D. Lesnic, and Ivanchov, M.I. Identification of a heterogeneous orthotropic conductivity in a rectangular domain. *International Journal of Novel Ideas: Mathematics*, vol.1. 2017.
[39] M. S. Hussein, D. Lesnic, Kamynin, V.L., Kostin, A.B. Direct and inverse source problems for degenerate parabolic equations. *Journal of Inverse and Ill-Posed Problems*, vol.28,no.3, pp. 425-448. 2020.
[40] M. S. Hussein, N. Kinash, D. Lesnic, Ivanchov, M.I. Retrieving the time-dependent thermal conductivity of an orthotropic rectangular conductor. *Applicable Analysis*, vol.96, no.15, pp. 2604-2618. 2017.
[41] M. S. Hussein, D. Lesnic, B. T. Johansson, and A. Hazanee. "Identification of a multidimensional space-dependent heat source from boundary data". *Applied Mathematical Modelling*, vol.54, pp.202–220, 2018.
[42] P. C. Hansen. *The l-curve and its use in the numerical treatment of inverse problems*. 1999.
[43] P. C. Hansen and D. P. O'Leary. "The use of the l-curve in the regularization of discrete ill-posed problems". *SIAM journal on scientific computing*, vol.14, no.6, pp.1487–1503, 1993.