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Closed (St-Closed) Compressible Modules and Closed (St-Closed) Retractable Modules

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Abstract:
Let R be a commutative ring with 1 and M be a left unitary R-module. In this paper, we give a generalization for the notions of compressible (retractable) module. As well as, we study closed (St-closed) compressible and closed (St-closed) retractable. Furthermore, some of their advantages, properties, categorizations and instances have been given. Finally, we study the relation between them.

Keywords: st-closed compressible, closed compressible, st-closed retractable, closed retractable.

مقاسات قابلة للانضغاط مغلقة ومغلقة من النمط ***St***– ومقاسات قابلة للانسحاب مغلقة ومغلقة من النمط ***St***–

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الخلاصة:
لتكن R حلقة إبدالية ذات عنصر محايد وليكن M مقاساً أيسرَ ذات عنصر احادي. بحثنا هذا هو تعميم لمفهوم المقاسات القابلة للضغط (قابلة للانسحاب) .كذلك سندرس المقاسات القابلة للانضغاط مغلقة ومغلقة من النمط –St ومقاسات قابلة للانسحاب مغلقة ومغلقة من النمط –St بالاضافة الى ذلك سنعطي في هذا البحث بعض الخواص والصفات والامثلة. واخيرا سندرس العلاقة بينهم.

1. Introduction

Let M be a left unitary module over a commutative ring R with 1. A non-zero submodule U is termed a closed submodule of M (briefly$U \leq_c M$), if K has no proper essential extension in M, i.e., if there exists a submodule U of M such that $U \leq_c K \leq M$, then $U = K$,[1]. A non-zero submodule U is termed an essential submodule U of M if $K \cap L \neq 0$ for all non-zero submodule L of M, [11].

A non-zero submodule U of an R-module Y is termed a semi-essential submodule U briefly $U \leq_{s.e} Y$ if $N \cap P \neq 0$ for all non-zero prime submodule P of Y,[27].

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A submodule P of M is said to be prime if P is a proper submodule U in M and whenever $rx \in P$ for all $r \in R, x \in M$ up to either $x \in P$ or $r \in [P:M]$, where $[P:M] = \{r \in R: rM \subseteq P\}$, [2,12]. A submodule U of an R-module Y is termed St-closed (briefly $N \leq_{Stc} M$), if N has no proper semi-essential extension in M, i.e., if there exist a submodule K of M such that $N \leq_{Stc} K \leq M$, then $N = K$,[3]. If M can be embedded in all non-zero sub-module, then M is termed compressible,[4]. If $Hom_R(M, U) \neq 0$, for each non-zero submodule U in R-module Y, then M is termed retractable [5] The researchers generalized compressible,[19] and retractable, [20].

In this paper, we will introduce a new concept of generalizations of compressible and retractable modules. We shall introduce and study the notions of closed (St-closed) compressible and St-closed (closed) retractable modules as a generalization of compressible module and retractable module respectively and give some of their properties, instance and give some of their advantages.

2. Closed Compressible Modules:

As a generalization of compressible module we will introduce the concept of closed compressible module, by giving some of basic properties, instances and characterization of this concept.

Definition 2.1: An R-module Y is said to be closed compressible if Y can be embedded in all non-zero closed submodule U of Y. Equivalently, Y is closed compressible if it can be found $f: Y \longrightarrow U$ whenever $0 \neq U \leq_c Y$ and f is an monomorphism. A ring R is a closed compressible if R is a closed compressible as R-module.

Examples and Remarks 2.2:

(1) All compressible R-module is closed compressible, the converse is not true, for instances Z-module Z_4 is closed compressible (because Z_4 is the only closed submodule U of Z_4), hence is not compressible.

(2) Because Z_6 as Z-module is not possible to be embedded in $(\bar{3})$, then Z_6 is not closed compressible (where $(\bar{3})$ is closed submodule in Z_6) .

(3) Z as Z-module is closed compressible (since Z is compressible), so is not simple.

(4) Q as Z-module is closed compressible (because Q is the only closed submodule of Q).

(5) $\frac{Z}{6Z} \simeq Z_6$ is not closed compressible module, however Z as Z-module is closed compressible module. In general, a homomorphic image of a closed compressible module need not be closed compressible.

Proposition 2.3: All closed submodule U of a closed compressible is a closed compressible.

Proof: Suppose that L be a closed compressible with $0 \neq K \leq_c Y$. Let $0 \neq L \leq_c K$ implies that $L \leq_c Y$,[1]. Because Y is a closed compressible module, so found a monomorphism $f: Y \longrightarrow L$ and $i: K \longrightarrow Y$ is the inclusion non-zero homomorphism, then $f \circ i: K \longrightarrow L$ is a monomorphism. Therefore, K is a closed compressible.

Remark 2.4: The direct sum of a closed compressible module need not be closed compressible. Consider the following instances let $Z_6 = Z_2 \oplus Z_3$ as Z-module. Z_2 and Z_3 is closed compressible (Since Z_2 and Z_3 are simple module), however Z_6 is not closed compressible see Remark and Examples 2.2 point (2).

Proposition 2.5: Let Y be a semi simple R-module, then the following statements are equivalent:
(1)Y is a compressible module;
(2)Y is a closed compressible module.
Proof: (1) $\Longrightarrow$ (2) It is clear, see Remarks and Examples 2.2, (1).
(2)$\Longrightarrow$ (1) Let $0 \neq U \leq Y$ and Y is a semi simple, then U is closed submodule of Y. However, Y is a closed compressible, thus there exists monomorphism $f: Y \longrightarrow U$. Therefore, Y can be embedded in U.

Proposition 2.6: A direct summand of a closed compressible module Y is moreover closed compressible module.
Proof: Since all direct summand of an R-module Y is a closed submodule of Y, hence by Proposition 2.3, we have done.

Corollary 2.7: For a semi simple R-module Y. All non-zero sub-module U of closed compressible module Y is closed compressible.
Proof: Let $0 \neq N \leq Y$ and Y is a closed compressible R-module, then N is direct summand of Y, hence N is closed compressible by Proposition 2.6.

Remark 2.8: If an R-module Y is semi simple, then Y is not necessarily closed compressible.
Proof: See Remarks and Examples 2.2, (2).

Definition 2.9:[**1**] An R-module Y is termed semi simple if all submodule U of Y is closed. For instance Z_6 as Z-module is semi simple closed.

3. Closed Retractable Modules:

An R-module Y is termed a retractable if $Hom(M.N) \neq 0$ for all non-zero submodule N of M. [20].

Definition 3.1: An R-module M is termed a closed retractable if $Hom_R(M.N) \neq 0$ for all non-zero a closed submodule of M. Equivalently, M is a closed retractable if it can be found non-zero homomorphism $f: M \longrightarrow N$ whenever $0 \neq N \leq_c M$. A ring R is a closed retractable if R is a closed retractable as R-module.

Examples and Remarks 3.2:
1. All retractable R-module Y is a closed retractable R-module. Conversely is not true for instance Q as Z-module is a closed retractable, however not retractable,[8].
2. Q as Z-module is a closed retractable (because Q is the only non-zero closed submodule of Q).
3. Z_n as Z-module is a closed retractable for all $n \in Z^+$ (Since it is a retractable), [8].
4. All a closed compressible R-module is a closed retractable. Conversely is not true for instances Z-module Z_6 is a closed retractable, on the other hand not a closed compressible.
5. Z as Z-module is a closed retractable (since it is a retractable), [8].
6. Every closed simple R-module is closed retractable, however not in opposition for instance Z-module Z is closed retractable, on the other hand not simple. We say that an R-module M is a closed simple if (0) and M are the only closed submodules of M.
7. All commutative ring with 1 is a closed retractable (by Remarks and Examples 1.2.2), [8].

Remark 3.3: $R\oplus Y$ is a closed retractable (since $R\oplus Y$ is retractable), [8].

Proposition 3.4: A closed submodule of a closed retractable is a closed retractable.
Proof: The proof is similar to the proof of Proposition 2.3.

Proposition 3.5: Let M be a semi simple R-module, and then M is a closed retractable module if and only if M is retractable module.
Proof: Let $0 \neq N \leq M$ and M be a semi-simple R-module with M is a closed retractable, thus there exists a non-zero homomorphism $\psi: M \longrightarrow N$, by assumption $\psi \neq 0$ Therefore, M is a retractable. Conversely see (Remarks and Examples 3.2) point (6).

4. St-Colsed Compressible Module:

Recall that a submodule U of an R-module Y is termed St-closed (briefly$U \leq_{Stc} Y$), if U has no proper semi-essential extension in Y, i.e., if there exists submodule U of Y such that $U \leq_{Stc} K \leq Y$, then $U = K$, [16].

As a generalization of compressible module we will introduce the concept of St-closed compressible module; with giving some of basic properties, instances and characterization of this concept.

Definition 4.1: An R-module Y is termed St-closed compressible if Y can be embedded in all non-zero St-closed submodule U of M. Equivalently, M is an St-closed compressible if it can be found a monomorphism $f: M \longrightarrow U$ whenever $0 \neq U \leq_{Stc} M.$ R is a St-closed compressible ring if R is an St-closed compressible as R-module.
Remarks and Examples 4.2:

(1)All compressible R-module is St-closed compressible, but the converse is not true for instance Z_4 as Z-module is St-closed compressible (because Z_4 is the only St-closed submodule of Z_4). However not compressible.
(2)Z as Z-module is St-closed compressible (since Z is compressible).
(3)Q as Z-module is St-closed compressible (because Q is the only St-closed submodule of Q.
(4)All St-closed simple R-module is St-closed compressible, but the converse is not true for instance as Z-module is St-closed compressible, which is not simple St-closed. We say that an R-module M is simple St-closed if (0) and M are the only St-closed submodules of M.
(5)Z_6 as Z-module is not St-closed compressible (since Z_6 is impossible to be embedded in $(\bar{3})$, where $(\bar{3})$ is St-closed submodule in Z_6.
(6)A direct sum of an St-closed compressible module need not be St-closed compressible. For instances, let $Z_6 = Z_2 \oplus Z_3$ as Z-module. Z_2 and Z_3 is St-closed compressible (since Z_2 and Z_3 are St-closed simple module), however Z_6 is not closed compressible see Remarks and Examples 4.2, (5).
(7)All closed compressible R-module Y is St-closed compressible. Conversely, is not true for instances Q as Z-module.

Proof: Assume that an R-module Y is a closed compressible. We have to show that Y is an St-closed compressible module. Let $0 \neq U \leq_{Stc} Y$, then $U \leq_c Y$,[3], As Y is a closed compressible, then Y can be embedded in St-closed submodule U of Y. Therefore, Y is an St-closed compressible module.

An R-module Y is termed to be fully prime if all proper submodule U of Y is a prime submodule U, [21]

Proposition 4.3: Let Y be an R-module . If Y is fully prime, then Y is St-closed compressible if and only if Y is closed compressible.
Proof: Let $0 \neq U \leq_c Y$, then $U \leq_{Stc} Y$ by [3], so Y is St-closed compressible. Thus found a monomorphism $f: Y \longrightarrow U$. Therefore, Y is a closed compressible. To proof the other direction, see Remarks and Examples 4.2, (1).

Recall that an R-module M is called chained if for each submodules A and B of M either A ≤ B or B ≤ A [25].

Proposition 4.4: An St-closed submodule of an St-closed compressible chained R-module M is an St-closed compressible.
Proof: Let $0 \neq B \leq_{Stc} M$ and assume that M is St-closed compressible module. Let $0 \neq A \leq_{Stc} B$, Since M is chained module then $A \leq_{Stc} M$, [3] . However M is an St-closed compressible module, then found a monomorphism $h: M \longrightarrow A$ and $i: B \longrightarrow M$ is the inclusion non-zero homomorphism, then $h \circ i: B \longrightarrow A$ is a monomorphism. Therefore, B can be embedded in A.

Recall that an R-module M is called a fully essential if every submodule in M is essential,[2].

Proposition 4.5: Let Y be an R-module in which all semi-essential extension of any submodule of M is fully essential. Then Y is an St-closed compressible if and only if M is a closed compressible.
Proof: Let $0 \neq B$ be a closed submodule of an R-module Y, by assumption $\leq_{Stc} Y$, [3]. However, Y is an St-closed compressible, then there exists a monomorphism $h: Y \longrightarrow B$. Therefore, Y is a closed compressible. The converse is clear by Remarks and Examples 4.2, (1).

Recall that an R-module M is called semi-essentially compressible if M can be embedded in every of it is a non-zero submodule of M. Equivalently, M is compressible if there exists a non-zero monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \leq_{s.e} M$, [13].

Proposition 4.6: Suppose Y is an St-closed compressible R-module in which all submodule U of Y is semi-essentially compressible, then M is a compressible module.
proof: Let $0 \neq C \leq M$, then found an St-closed submodule H in M such that $C \leq_{sem} H$, [3]. However, M is an St-closed compressible R-module, then if can be found monomorphism $\psi: M \longrightarrow H$ and $\phi: H \longrightarrow C$ is a monomorphism (since all submodule U of M is semi-essentially compressible), thus we have a composition $\phi \circ \psi: M \longrightarrow C$ which is monomorphism. Therefore, M is a compressible module.

Corollary 4.7: Let M be an St-closed compressible R-module and M is semi-essentially compressible, then M is a compressible module.

5. St-Closed Retractable Modules:
Definition 5.1: An R-module Y is termed St-closed retractable if $Hom_R(M.U) \neq 0$, for all non-zero St-closed submodule U of M. Equivalently, M is an St-closed retractable if found a non-zero homomorphism $f: M \longrightarrow U$ whenever $0 \neq U \leq_{Stc} M$. A ring R is an St-closed retractable if R is an St-closed retractable as R-module.

Remarks and Examples 5.2:
1. Q as Z-module is an St-closed retractable (because Q is the only St-closed submodule of Q).
2. Every retractable R-module st-closed retractable.
3. Z_n as Z-module is an st-closed retractable for all $n \in Z^+$(because it is retractable),[8].

4. All an St-closed compressible R-module is an St- closed retractable. Conversely is not true for instances Z_6 as Z-module is an St-closed retractable, however not St-closed compressible.
5. Z as Z-module is an St-closed retractable (Since it is a retractable),[8].
6. All St-closed simple R-module is St-closed retractable, however not conversely for instances Z as Z-module is St-closed retractable, however not simple. We say that an R-module M is called an st-closed simple if (0) and Y are the only St-closed submodule of Y.
7. All commutative ring with 1 is an St-closed retractable (By remarks and examples (1.2.2), [8].
8. All retractable R-module Y is an St-closed retractable R-module.
9. All closed retractable R-module Y is an St-closed retractable R-module, however not conversely.
10. $Z_P\infty$ as Z-module is not St-closed retractable(Since it is not a retractable) ,[8].

Remark (5.3): For any R-module Y , $R\oplus Y$ is an St-closed retractable (Since $R\oplus M$ is retractable), [8].

Recall that an R-module M is called multiplication module, if every submodule N of M is of the form IM for some ideal I of R [26].

Recall that an R-module M is called a faithful R-module if $ann_R(M) = 0$,[1]

Proposition(5.4): Let Y be a faithful multiplication R-module. Then Y is an St-closed retractable.

Proof: Let $0 \neq U \leq_{Stc} Y$. Since Y is a multiplication R-module, then $U = IY$ for some non-zero ideal I of a ring R and since R is an St-closed retractable ring by remarks and examples (5.2) point(7) implies that $Hom_R(R, I) \neq 0$. Let $\phi: R \longrightarrow I$ be a non-zero homomorphism. Put $\phi(1) = x$ for some xI, then $x \neq o$. Define $f: Y \longrightarrow U$ by $f(m) = xm$ for all $m \in Y$. Clearly f is a well-defined monomorphism. Moreover $f \neq 0$. Since Y is a faithful. Therefore $Hom_R(Y.U) \neq 0$ and hence Y is St-closed retractable.

In above proposition the condition Y is a multiplication is necessary, for instances $Z_P\infty$ as Z-module is not St-closed retractable and it is not multiplication.

Corollary (5.5): All faithful cyclic R-module is an St-closed retractable.

Proposition (5.6): Let $M = M_1 \oplus M_2$ be a fully prime R-module such that $ann_R(M_1) + ann_R(M_2) = R$. Then M is an St-closed retractable if and only if M_1 and M_2 are St-closed retractable.

proof: Let $0 \neq A \leq_{Stc} M_1$ and $0 \neq B \leq_{Stc} M_2$, then $(0.0) \neq A\oplus B \leq_{Stc} M$,[3, Theorem(1.20)].Since M is an St-closed retractable, then there exists a non-zero homomorphism $\phi: M \longrightarrow A\oplus B$ and if it can be found $j: M_1 \longrightarrow M$ is the inclusion non-zero homomorphism. Then there exists a non-zero homomorphism $\rho: A\oplus B \longrightarrow A$, so $\rho \circ \phi \circ j: M_1 \longrightarrow A$. Thus M_1 is an St-closed retractable. Conversely let$(0.0) \neq A\oplus B \leq_{Stc} M$, then $0 \neq A \leq_{Stc} M_1$,[3,Theorem (1.20)]. Suppose that M_1 is an St-closed retractable, then if it can be found $: M_1 \longrightarrow A$, $j: A \longrightarrow A\oplus B$ and $\rho: M \longrightarrow M_1$ .Thus we have a composition $j \circ g \circ \rho: M \longrightarrow A$, then M is an St-closed retractable.

Corollary 5.7: Let $\{Y_i\}_{i=1}^{n}$ be a finite family of St-closed retractable fully prime R-module such that $\sum_{i=1}^{n} ann_R(Y_i) = R$. Then $\oplus_{i=1}^{n} Y_i$ is also St-closed retractable.

6. Closed Prime Modules:

An R-module Y is termed prime module if $ann_R(Y) = ann_R(U)$, for all non-zero sub-modules U of Y, [23]

Definition 6.1: Let M be an R-module, M is termed a closed prime module if $ann_R(M) = ann_R(U)$, for all non-zero closed sub-modules U of M.

Lemma 6.2: Let M be a closed prime R-module, then $ann_R(U)$ is a prime ideal of R for each non-zero closed submodule U of M.

Proof: let $a.b \in R$ such that $a.b \in ann_R(U)$, then $abU = 0$. Suppose $bU \neq 0$. However, $0 \neq U \leq_c M$ and $bU \subseteq U$, then $bU \leq_c M$ [1], since M is a closed prime module and $a \in ann_R(bU)$ implies that $a \in ann_R(M)$. But $ann_R(M) = ann_R(U)$, so $ann_R(U)$ is a prime ideal of R.

Remarks and Examples 6.3:

1. All prime R-module is a closed prime module. The converse is not true in general for instances Z_8 as Z-module is a closed prime $module$ because Z_8 is the only non-zero closed submodule of Z_8 , thus is not prime.
2. Z_6 as Z-module is not closed prime $module$. (Because $ann_R(Z_6) \neq ann_R(\langle\bar{2}\rangle)$; $\langle\bar{2}\rangle \leq_c Z_6$).
3. Z and Q as Z −module are closed prime modules, because Z as Z-module is the only closed submodule in Z. Similarly for Q .
4. All simple R-module is a closed prime module, however the converse is not true, because Z as Z-module is a closed prime module so its not simple.
5. $\frac{Z}{6Z} \simeq Z_6$ is not closed prime module, however Z as Z-module is closed prime module, see point (3). In general, a homomorphic image of a closed prime module need not be closed prime.
6. The direct sum of a closed prime module need not be closed prime. Consider the following instances let $Z_6 \simeq Z_3 \oplus Z_2$as Z −module.Z_3 and Z_2 are closed prime submodules see (4) , however Z_6 is not closed prime see (2) .

Proposition 6.4: A closed submodule U of a closed prime module is also closed prime module.

Proof: Let $0 \neq K \leq_c Y$ and Y be a closed prime module. Let $0 \neq L \leq_c K \leq_c Y$, then by [1] $L \leq_c Y$. Because M is closed prime module, then $ann_R(L) = ann_R(Y) = ann_R(K)$. Therefore K is a closed prime module.

Proposition 6.5: If Y is a closed compressible R-module, then Y is closed prime module.

Proof: Suppose $0 \neq U \leq_c Y$. We have to show that $ann_R(Y) = ann_R(U)$. Let $r \in ann_R(U)$, then $rU = 0$, as Y is a closed compressible R-module, $f: Y \longrightarrow U$ is a monomorphism , then $f(rY) = rf(Y) \subseteq rU = 0$, so $rY = 0$, thus $r \in ann_R(Y)$. Therefore, $ann_R(Y) = ann_R(U)$ and it is a closed prime module.

Proposition 6.6: Let Y be an R-module. Y is closed compressible module if and only if there exists a monomorphism $\varphi \in End_R(Y)$ such as $Im\varphi \subseteq U$ for each a non-zero closed submodule U of Y.

Proof: Suppose that Y is a closed compressible $module$. Let $0 \neq U \leq_c Y$, then $f: Y \longrightarrow U$ is a monomorphism , so found $\varphi = i \circ f \in End_R(Y)$ where $i: U \longrightarrow Y$ is the inclusion monomorphism and $Im\varphi = i \circ f(Y) = f(Y) \subseteq U$. Conversely let $0 \neq U \leq_c Y$, by assumption found a monomorphism $\varphi \in End_R(Y)$ and $\varphi(Y) \subseteq U$. Thus $\varphi: Y \longrightarrow U$ is a monomorphism. Therefore, Y is a closed compressible module.

Conclusions:

In this work, the class of compressible and retractable modules have been generalized to new concepts called closed (St-closed) compressible modules and closed (St-closed) retractable modules. Several characteristics of this type of modules have been studied. In addition, we see relations between closed (St-closed) compressible modules, closed (St-closed) retractable modules and other related modules as retractable module.

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