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A New Hybrid Algorithm Based on the Firefly Algorithm and Cuckoo Search to Estimate Survival Analysis of Truncated Inverse Gompertz Distribution

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Abstract

This paper introduces the Right Truncated Inverse Gompertz Distribution (RTIGD) with two parameters δ and ϑ with some of its properties as; (Survival-Function, Hazard-Function, cumulative distribution function, probability density function, r-th moment, mean, variance, Moment Generating Function, Median, and Mode. In addition, we propose a new hybrid algorithm, namely the firefly Algorithm with Cuckoo Search Algorithm (FA_CSA) to estimate the parameters based on the survival functions of RTIGD. Simulation is utilized to compare the proposed algorithm with traditional methods, for example, the Maximum Likelihood Estimator and moment method and the standard algorithms Firefly Algorithm and Cuckoo Search. In most cases, the results demonstrate that the proposed algorithm (FA_CSA) provides an accurate estimation for the survival function since it has less mean squared error than the other estimation methods.

Keywords: Firefly Algorithm, Hybrid Algorithm, Cuckoo Search Algorithm, Right Truncated Inverse Gompertz distribution (RTIGD), Maximum Likelihood Estimation Method.

خوارزمية مهجنة جديدة تعتمد على خوارزمية الذباب الناري وخوارزمية بحث الوقواق لتقدير دالة البقاء لمعكوس توزيع جومبرتز المبتور

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الخلاصه

في هذا البحث تم تقديم معكوس توزيع جومبرتز المبتور من جهه اليمين مع معلمتين (δ, ϑ) ودراسة بعض خصائصه مثل (داله الكثافه الاحتماليه ،داله التوزيعة التراكمية ، دالة البقاء ، دالة الخطر ، العزوم، الدالة المولدة للعزوم، المتوسط ، الوسيط ، المنوال و التباين) لهذا التوزيع المبتور . بالاضافة الى ذلك , نقترح خوارزمية هجينة جديدة (خوارزمية الذباب الناري مع خوارزمية بحث الوقواق ((FA_CSA) لتقدير المعلمات بناءً على دالة البقاء للتوزيع لـ RTIGD . يتم استخدام المحاكاة لمقارنة الخوارزمية المقترحة مع الطرق التقليدية (طريقة الامكان الاعظم وطريقة العزوم) والخوارزميات (خوارزمية الذباب الناري و خوارزمية بحث الوقواق). توضح النتائج أن الخوارزمية المقترحة (FA_CSA) توفر تقديرًا دقيقًا لداله البقاء نظرًا لأنها تحتوي على اقل متوسط تربيعي الخطا من طرق التقدير الأخرى في معظم الحالات.

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1. Introduction

Recently, survival analysis has been executed in most science fields. The term survival analysis means an analysis that is performed to specify the probability of occurrence of the events associated with failure or death time after treatment [1].

For many years, statisticians have been interested in estimating survival functions [2-6]. In addition, statistical distributions have been long employed in survival analysis [7-9] .Recently, numerous researchers focused their consideration on suggestions for more flexible probability distributions, by using different approaches to describe a set of data. Distribution properties are very useful to illustrate the ability of that distribution. Without losing the generality, the Truncated Distributions (TD) are more realistic to describe phenomena. In statistics, TD refers to conditional distribution created by imposing limits on the domain of another probability distribution. The TD arises in the practical statistics in cases where the ability to record or even to know about the occurrences is limited to the values that lie above (below) a given threshold or in a specific range. Therefore, this paper introduces the truncated distribution and estimates the parameters depending on the survival function. However, the classical statistical methods are not appropriate for the survival function due to the nonlinearity of a model distribution makes the estimation of parameters more difficult and more challenging [10]. Thus, the research community aims to estimate the parameters by using meta-heuristic algorithms. Shin et al. [11] used a meta-heuristic algorithm presented by Genetic Algorithm (GA) to maximize the result of log-likelihood to estimate the parameters of the mixture's normal distribution. Yoon et al.[12] estimated the parameters for appropriate probability distribution by using a meta heuristic approach (harmony search algorithm). Ali [13] proposed the Jackknife algorithm which estimated parameters of the Gumbel distribution.

Even though meta-heuristic algorithms have been effectively used to solve complex problems, there is no standard algorithm that can be used for all problems depends on the no-free-lunch theorem [14]. Consequently, the contemporary concepts of hybrid algorithms facilitate the selection of an appropriate algorithm that aims to overcome the implied limitation of meta-heuristics in attempting to solve parameter estimation procedures. So, the aim of this is to introduce a hybrid algorithm to estimate the parameters of Right Truncated Inverse Gompertz Distribution based on survival functions.

The rest of the paper is organized as follows: Section 2 describes the Right Truncated Inverse Gompertz Distribution. Section 3 presents some properties of the Right Truncated Inverse Gompertz Distribution. Section 4 and section 5 introduce the Maximum Likelihood Estimation and Moment Estimation Methods, receptively. Section 6 describes the Meta-heuristic Algorithms. To demonstrate the effectiveness of the proposed Hybrid Meta-heuristic algorithm, numerical results and discussions are presented in Section 7. In section 8 conclusions are given.

2. Right Truncated Inverse Gompertz Distribution (RTIGD)

Gompertz presents a two parameters of the probability distribution [15] which is widely utilized in SA to represent behavioral science data and human mortality. It is an extension of the exponential distribution, and it has numerous utilize in real-life [16]. Eliwa et al.[17] introduced Inverse Gompertz Distribution (IGD) with different estimation methods and properties.

The random variable (X) is said to have IGD if it has the following the p.d.f :

$$f(x;\delta,\vartheta) = \frac{\delta}{x^2}\, e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}} \qquad ,x,\delta,\vartheta > 0\,. \tag{1}$$

Where δ and ϑ represent the shape and the scale parameters, respectively. The (CDF) of the Inverse Gompertz Distribution (IGD) is as follows:

$$F(x;\delta,\vartheta) = e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)} \tag{2}$$

The right-side truncation for the inverse Gompertz distribution is

$$f_{RTIGD}(x) = \frac{f(x;\delta,\vartheta)}{F(1;\delta,\vartheta)}\,,$$

Therefore, Right Truncated Inverse Gompertz Distribution (RTIGD) on [0,1] is given as follows:

$$f_{RTIGD}(x;\delta,\vartheta) = \frac{f(x;\delta,\vartheta)}{F(1;\delta,\vartheta)} = \frac{\frac{\delta}{x^2}\, e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}. \tag{3}$$

And the CDF, SF and HF for RTIGD are expressed respectively as follows:

$$F_{RTIGD}(x;\delta,\vartheta) = \frac{F(x;\delta,\vartheta)}{F(1;\delta,\vartheta)} = \frac{e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}\,, \tag{4}$$

$$S_{RTIGD}(x;\delta,\vartheta) = 1 - F_{RTIGD}(x;\delta,\vartheta) = 1 - \frac{e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} = \frac{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)} - e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} \tag{5}$$

And $H_{RTIGD}(x;\delta,\vartheta) = \frac{f_{RTIGD}(x;\delta,\vartheta)}{S_{RTIGD}(x;\delta,\vartheta)} = \dfrac{\dfrac{\frac{\delta}{x^2}\, e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}}{\dfrac{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)} - e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}} = \dfrac{\frac{\delta}{x^2}\, e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)} - e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}\,.$ (6)

Where x is a random variable value and $0 < x < 1$, ϑ is the scale parameter where ($\vartheta > 0$), andδ is the shape parameter where ($\delta > 0$).
Figures (1- 4) illustrate the (PDF, CDF, SF and HF) for the RTIGD of some cases of δ and ϑ

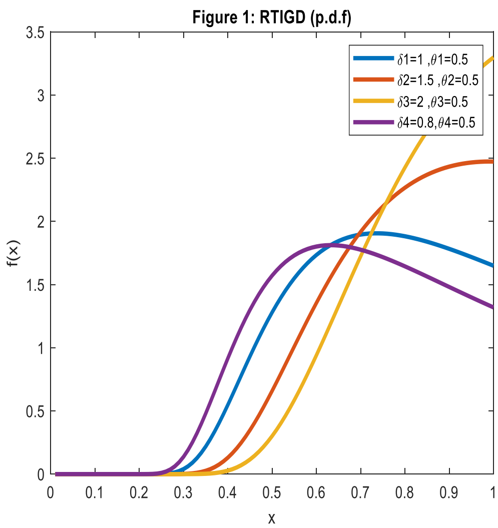

Figure 1: probability density function for RTIGD

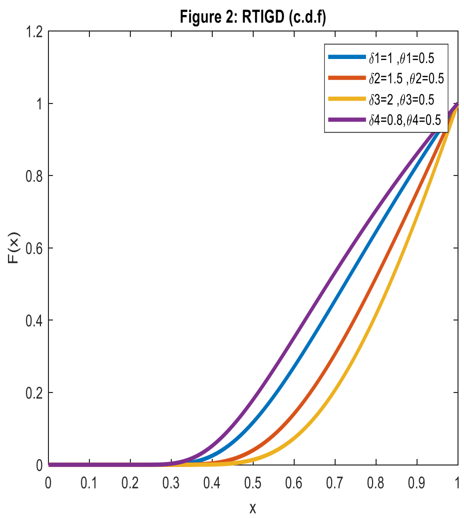

Figure 2: cumulative distribution function for RTIGD

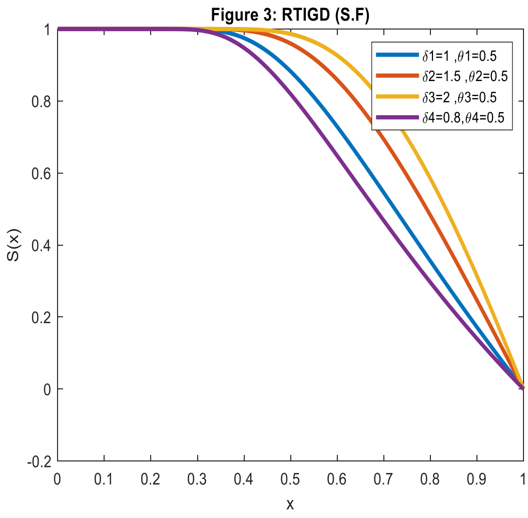

Figure 3: Survival function for the RTIGD

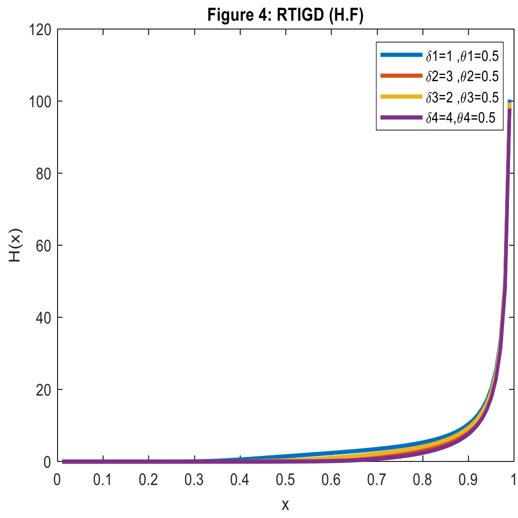

Figure 4: Hazard function for the RTIGD

3. Some properties of the Right Truncated Inverse Gompertz Distribution

In this section, some properties are given for RTIGD. However, some properties are complicated to be solved. For this reason, we use numerical analysis to find some of them. We also do some simplifications for the $E(x^r)$ by using the Binomial theorem and Tayler series $(a \mp x)^n = \sum_{j=0}^{n} \binom{n}{j} (\mp x)^j a^{n-j}$.

3.1 r-th Moment:

The r-th moment can be derived as follows:

$$E(x^r) = \int_0^1 x^r \, f_{RTIGD}(x) \, dx = \int_0^1 x^r \frac{\frac{\delta}{x^2} e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx = \int_0^1 x^r \frac{\frac{\delta}{x^2} e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)} \, e^{\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx$$

$$= \int_0^1 x^r \frac{\frac{\delta}{x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)\right)^k e^{\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx = \int_0^1 x^r \frac{\frac{\delta}{x^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\frac{\delta}{\vartheta}\right)^k \left(e^{\frac{\vartheta}{x}}-1\right)^k e^{\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx$$

$$= \int_0^1 x^r \frac{\frac{\delta}{x^2} \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{k!}\right) \left(\frac{\delta}{\vartheta}\right)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^j \left(e^{\frac{\vartheta}{x}}\right)^{k-j} e^{\frac{\vartheta}{x}}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx$$

$$= \int_0^1 (x^r) \frac{\frac{\delta}{x^2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^k \left(e^{\frac{\vartheta}{x}}\right)^{(k-j)+1}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx$$

$$= \int_0^1 (x^r) \frac{\frac{\delta}{x^2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^k \left(e^{\frac{\vartheta}{x}}\right)^{k-j+1}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx$$

$$= \int_0^1 x^r \frac{\frac{\delta}{x^2} \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^k e^{\frac{\vartheta}{x}(k-j+1)}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \, dx$$

$$= \int_0^1 x^r \frac{\frac{\delta}{x^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{\vartheta}{x}(k-j+1)\right)^m}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}\,dx$$

$$= \int_0^1 x^r \frac{\frac{\delta}{x^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{(\vartheta(k-j+1))^m}{x^m}\right)}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}\,dx$$

$$= \int_0^1 x^r \frac{\frac{\delta}{x^2}\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\frac{(\vartheta(k-j+1))^m}{x^m}\right)}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}\,dx$$

$$= \int_0^1 \varphi_{k,j,m}\ x^{r-m-2}\quad dx\,.$$

Where

$$\varphi_{k,j,m} = \frac{\delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{(-1)^{k+j}}{k!}\right)\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\vartheta((k-j)+1)\right)^m}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

$$= \int_0^1 \varphi_{k,j,m}\ x^{r-m-2}\quad dx = \frac{x^{r-m-1}}{r-m-1}\Big|_0^1 = \frac{\varphi_{k,j,m}}{r-m-1}$$

Therefore, $$E(x^r) = \frac{\delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{(-1)^{k+j}}{k!}\right)\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\vartheta((k-j)+1)\right)^m\left(\frac{1}{r-m-1}\right)}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

If r=1, then

$$E(x) = \frac{\delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\vartheta((k-j)+1)\right)^m\frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

When r=2, we get

$$E(x^2) = \frac{\delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\left(\vartheta((k-j)+1)\right)^m\frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

And if r=3, then

$$E(x^3) = \frac{\delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\left(\frac{(-1)^m}{m!}\right)\left(\left(\vartheta((k-j)+1)\right)^m\frac{1}{2-m}\right)}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

While , if r=4, then we have

$$E(x^4) = \frac{\delta\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{k+j}}{k!}\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^k\sum_{m=0}^{\infty}\left(\frac{(-1)^m}{m!}\right)\left(\left(\vartheta((k-j)+1)\right)^m\frac{1}{3-m}\right)}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}.$$

3.2 The Variance

The Variance (Var) of RTIGD can be found as follows:

$\sigma^2 = \mathrm{Var}(t) = E(t^2) - [E(t)]^2$

$$\mathrm{Var}(x) = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} - \left[\frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}\right]^2$$

3.3 The Moment Generating Function (MGF)

The MGF of RTIGD can be derived as follows:

$$\mathcal{M}_x(t) = E(e^{tx}) = \int_0^1 e^{tx}\, f_{RTIGD}(x)\, dx$$

$$= \int_0^1 e^{tx}\, \varphi_{k,j,m}\, (x^{-m-2})dx = \varphi_{k,j,m} \int_0^1 e^{tx}\, x^{-m-2}dx$$

$$= \varphi_{k,j,m} \int_0^1 \sum_{n=0}^{n} \frac{(t)^n}{n!}\, x^{n-m-2}dx$$

Where, $e^{tx} = \sum_{n=0}^{\infty} \left(\frac{(tx)^n}{n!}\right)$

$$= \varphi_{k,j,m} \left(\sum_{n=0}^{\infty} \left(\frac{(t)^n}{n!}\right)\right) \int_0^1 x^{n-m-2}\, dx = \varphi_{k,j,m} \left(\sum_{n=0}^{\infty} \left(\frac{(t)^n}{n!}\right)\right) \frac{x^{n-m-1}}{n-m-1}\Big|_0^1 = \varphi_{k,j,m} \left(\sum_{n=0}^{\infty} \frac{(t)^n}{n!}\right) \frac{1}{n-m-1}.$$

Therefore, $$\mathcal{M}_x(t) = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{k+j}}{k!} \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{m!} (\vartheta(k-j+1))^{m} \sum_{n=0}^{\infty} \frac{(t)^n}{n!} \frac{1}{n-m-1}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}$$

3.4 Median

The Median of RTIGD can be found as follows:

$$F_{RTIGD}(x) = \frac{1}{2}$$

$$\frac{e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} = \frac{1}{2} \Longrightarrow e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)} = \frac{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}{2}$$

$$\ln\left(e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}\right) = \ln\left(e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}\right) - \ln 2 \Longrightarrow \quad e^{\frac{\vartheta}{x}} - 1 = e^{\vartheta} - 1 + \frac{\vartheta}{\delta} ln\, 2$$

$$e^{\frac{\vartheta}{x}} = e^{\vartheta} + \frac{\vartheta}{\delta} ln\, 2 \;\Longrightarrow\; \ln e^{\frac{\vartheta}{x}} = \ln\left(e^{\vartheta} + \frac{\vartheta}{\delta} ln\, 2\right) \;\Longrightarrow\; \frac{\vartheta}{x} = ln\left(e^{\vartheta} + \frac{\vartheta}{\delta} ln\, 2\right)$$

$$x_{med} = \frac{\vartheta}{ln\left(e^{\vartheta} + \frac{\vartheta}{\delta} ln\, 2\right)}$$

3.5 Mode

The Mode of the RTIGD can be found as follows:

$$\frac{df_{RTIGD}(x)}{dx} = 0$$

$$\frac{df_{RTIGD}(x)}{dx} = \frac{\delta}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} \left(\frac{x^2 \left(e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}} \left(\frac{-\delta}{\vartheta}\, e^{\frac{\vartheta}{x}}\, \frac{-\vartheta}{x^2} - \frac{\vartheta}{x^2}\right)\right) - \left(2\,x\; e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}}\right)}{x^4}\right) = 0$$

$$\left(e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}} \left(\delta \ e^{\frac{\vartheta}{x}} \quad -\vartheta \ \right) \right) - \left(2\,x \ e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)+\frac{\vartheta}{x}} \right) = 0$$

$$\delta \ e^{\frac{\vartheta}{x}} \quad -\vartheta \ - \ 2\,x \ = 0 \ .$$

It can be obtained numerically, furthermore it always exists and unique.

4 -The Maximum Likelihood Estimation Method (MLE)

The likelihood function (L) is given by:

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$$\mathrm{L}\,(\mathrm{x}_1, \mathrm{x}_2\,, \ldots\ldots, \mathrm{x}_\mathrm{n}, \delta, \vartheta) = \prod_{i=1}^{n} f_{\mathrm{RTIGD}}(x_i) = \prod_{i=1}^{n}\left(\frac{\frac{\delta}{x_i^{2}}\,e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x_i}}-1\right)+\frac{\vartheta}{x_i}}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} \right)$$

Taking the natural logarithm to the two sides to get:

$$\mathrm{LnL} = \mathrm{n}\mathrm{Ln}\delta - 2\ \sum_{i=1}^{n}\mathrm{Ln}(x_i) + \vartheta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) - n\left(-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)\right)$$

$$\mathrm{LnL} = \mathrm{n}\mathrm{Ln}\delta - 2\ \sum_{i=1}^{n}\mathrm{Ln}(x_i) + \vartheta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) + n\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right) \qquad (7)$$

The partial derivative of equation (7) in terms of the unknown parameters (δ, ϑ)respectively:

$$\frac{\partial\,\mathrm{Ln}\,\mathrm{L}}{\partial\delta} = \frac{n}{\delta} - \frac{1}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) + n\frac{1}{\vartheta}\left(e^{\vartheta}-1\right)$$

$$\frac{n}{\delta} - \frac{1}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) + n\frac{1}{\vartheta}\left(e^{\vartheta}-1\right) = 0 \qquad (8)$$

$$\frac{\partial\,\mathrm{Ln}\,\mathrm{L}}{\partial\vartheta} = \sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta\vartheta\ \sum_{i=1}^{n}\frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \delta\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right)}{\vartheta^2} + \frac{\vartheta\,n\delta\,e^{\vartheta} - n\delta e^{\vartheta}}{\vartheta^2} + \frac{-n\delta}{\vartheta^2}$$

$$\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta\vartheta\ \sum_{i=1}^{n}\frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \delta\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right)}{\vartheta^2} + \frac{\vartheta\,n\delta\,e^{\vartheta} - n\delta e^{\vartheta}}{\vartheta^2} + \frac{-n\delta}{\vartheta^2} = 0$$

$$\vartheta^2\sum_{i=1}^{n}\frac{1}{x_i} - \delta\vartheta\ \sum_{i=1}^{n}\frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \delta\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) + \ n\delta\,e^{\vartheta}(\vartheta-1) - n\delta = 0. \qquad (9)$$

Since two non-linear equations are difficult to solve, so that the Newton-Raphson method is utilized to estimate the parameters δ *and* ϑ.

From the equations of (8) and (9), we get:

$$\begin{pmatrix}\hat{\delta}_{MLE}\\ \hat{\vartheta}_{MLE}\end{pmatrix} = \begin{pmatrix}\delta_0\\ \vartheta_0\end{pmatrix} - J^{-1}\ \begin{pmatrix}f(\delta,\vartheta)\\ g(\delta,\vartheta)\end{pmatrix}$$

$$let \ \ f(\delta,\vartheta) = \frac{n}{\delta} - \frac{1}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) + n\frac{1}{\vartheta}\left(e^{\vartheta}-1\right)$$

And,

$$g(\delta,\vartheta) = \vartheta^2\sum_{i=1}^{n}\frac{1}{x_i} - \delta\vartheta\ \sum_{i=1}^{n}\frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \delta\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right) + \ n\delta\,e^{\vartheta}(\vartheta-1) - n\delta$$

Then

$$\frac{\partial f(\delta,\vartheta)}{\partial\delta} = \frac{-n}{\delta^2} \qquad (10)$$

$$\frac{\partial f(\delta,\vartheta)}{\partial\vartheta} = \frac{\vartheta\ \sum_{i=1}^{n}\frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}}-1\right)}{\vartheta^2} + \frac{\vartheta\,n\,e^{\vartheta} - ne^{\vartheta}}{\vartheta^2} + \frac{-n}{\vartheta^2} \qquad (11)$$

And,

$$\frac{\partial g(\delta,\vartheta)}{\partial\delta} = \vartheta \sum_{i=1}^{n} \frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}} - 1\right) + ne^{\vartheta}(\vartheta - 1) - n \quad (12)$$

$$\frac{\partial g(\delta,\vartheta)}{\partial\vartheta} = 2\,\vartheta \sum_{i=1}^{n}\frac{1}{x_i} - \delta \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\left(\frac{\vartheta}{x_i}\ e^{\frac{\vartheta}{x_i}} + e^{\frac{\vartheta}{x_i}}\right) - \delta \sum_{i=1}^{n} e^{\frac{\vartheta}{x_i}} + n\delta\left(\vartheta\,(e^{\vartheta}) + e^{\vartheta}\right) - n\,\delta\left(e^{\vartheta}\right) \quad (13)$$

The Jacobean determate formulas are now as follows:

$$J = \begin{vmatrix} \frac{\partial f(\delta,\vartheta)}{\partial\delta} & \frac{\partial f(\delta,\vartheta)}{\partial\vartheta} \\ \frac{\partial g(\delta,\vartheta)}{\partial\delta} & \frac{\partial g(\delta,\vartheta)}{\partial\vartheta} \end{vmatrix}$$

As a result, the equation matrices are utilized to estimate the parameters for RTIGD on [0,1] using the Newton-Raphson method.

$$\begin{pmatrix}\hat{\delta}_{MLE}\\ \hat{\vartheta}_{MLE}\end{pmatrix} = \begin{pmatrix}\delta_0\\ \vartheta_0\end{pmatrix} - J^{-1}\left(\frac{\frac{n}{\delta} - \frac{1}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}} - 1\right) + n\frac{1}{\vartheta}\left(e^{\vartheta} - 1\right)}{\vartheta^2 \sum_{i=1}^{n}\frac{1}{x_i} - \delta\vartheta \sum_{i=1}^{n}\frac{1}{x_i}\ e^{\frac{\vartheta}{x_i}} - \delta\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}} - 1\right) + n\delta\ e^{\vartheta}(\vartheta - 1) - n\delta}\right)$$

Let $u_1 = \frac{\partial f(\delta,\vartheta)}{\partial\delta}$, $u_2 = \frac{\partial f(\delta,\vartheta)}{\partial\vartheta}$, $u_3 = f(\delta,\vartheta)$

$u_4 = \frac{\partial g(\delta,\vartheta)}{\partial\delta}$, $u_5 = \frac{\partial g(\delta,\vartheta)}{\partial\vartheta}$, $u_6 = g(\delta,\vartheta)$

Therefore,

$$\hat{\delta}_{MLE} = \delta_0 + h_1 \quad (14)$$

$$\hat{\vartheta}_{MLE} = \vartheta_0 + k_1 \quad (15)$$

Where:

$J^{-1} = \frac{1}{|J|}\ adj(J)$, $h_1 = \frac{u_4u_6 - u_3u_5}{u_1u_5 - u_2u_4}$, and $k_1 = \frac{-u_3 - u_1h_1}{u_4}$

which depend on (4-8) and (4-9)

So, the estimated survival function $\hat{S}_{RTIGD}(x)$ by MLE will be as follows:

$$\hat{S}_{\substack{MLE\\ RTIGD}}(x;\hat{\delta},\hat{\vartheta}) = \frac{e^{-\frac{\hat{\delta}_{MLE}}{\hat{\vartheta}_{MLE}}\left(e^{\hat{\vartheta}_{MLE}} - 1\right)} - e^{-\frac{\hat{\delta}_{MLE}}{\hat{\vartheta}_{MLE}}\left(e^{\frac{\hat{\vartheta}_{MLE}}{x}} - 1\right)}}{e^{-\frac{\hat{\delta}_{MLE}}{\hat{\vartheta}_{MLE}}\left(e^{\hat{\vartheta}_{MLE}} - 1\right)}} \quad (16)$$

5. Moments Estimation Method (MOM) :

The MOM was utilized to estimate the parameters of δ and ϑ for the Right Truncated Inverse Gompertz Distribution on [0,1]. the MOM can be get by equating sample moments to the population moments.

$E(x^k) = \frac{1}{n}\sum_{i=1}^{n} x_i^k$, where k=1, 2,…

The first and the second moments of the population and sample for two parameters of the RTIGT are given depending on the general form of x^r moment [18], respectively as follows:

$$E(x^r) = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right)\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty}\left(\frac{(-1)^m}{m!}\right)\left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{r-m-1}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}$$

when $r = 1$,

$$E(x) = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

Hence, when, $M_1 = E(x)$

$$\bar{x} = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

$$\bar{x} - \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} = 0 \qquad (17)$$

when $r = 2$ then,

$$E(x^2) = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

The second moment for the both population and sample is:

$M_2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right)$,

$M_2 = E(x^2)$

$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) = \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

$$\frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} = 0 \qquad (18)$$

The Newton-Raphson method is utilized to estimate the parameters of δ and ϑ. because two non-linear equations are difficult to solve.
From equations (17) and (18), we get:

$$\begin{pmatrix}\hat{\delta}_{MOM} \\ \hat{\vartheta}_{MOM}\end{pmatrix} = \begin{pmatrix}\delta_0 \\ \vartheta_0\end{pmatrix} - j^{-1} \begin{pmatrix}f(\delta,\vartheta) \\ g(\delta,\vartheta)\end{pmatrix}$$

let $f(\delta,\vartheta) = \bar{x} - \dfrac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$

And,

$$g(\delta,\vartheta) = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right) - \frac{\delta \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{(-1)^{k+j}}{k!}\right) \binom{k}{j} \left(\frac{\delta}{\vartheta}\right)^{k} \sum_{m=0}^{\infty} \left(\frac{(-1)^{m}}{m!}\right) \left(\vartheta\left((k-j)+1\right)\right)^{m} \frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}}$$

The Jacobean determate formulas are now as follows:

$$J = \begin{vmatrix} \frac{\partial f(\delta,\vartheta)}{\partial\delta} & \frac{\partial f(\delta,\vartheta)}{\partial\vartheta} \\ \frac{\partial g(\delta,\vartheta)}{\partial\delta} & \frac{\partial g(\delta,\vartheta)}{\partial\vartheta} \end{vmatrix}$$

As a result, the follows equation matrices are utilized to estimate the parameters for the RTIGD on [0,1] by using the Newton-Raphson method $\begin{pmatrix}\hat{\delta}_{MOM}\\ \hat{\vartheta}_{MOM}\end{pmatrix} = \begin{pmatrix}\delta_0\\ \vartheta_0\end{pmatrix} -$

$$J^{-1}\begin{pmatrix} \bar{x} - \dfrac{\delta \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{(-1)^{k+j}}{k!}\right)\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^{k}\sum_{m=0}^{\infty}\left(\frac{(-1)^{m}}{m!}\right)\left(\vartheta((k-j)+1)\right)^{m}\frac{1}{-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \\ \frac{1}{n}\left(\sum_{i=1}^{n}x_i^2\right) - \dfrac{\delta \sum_{k=0}^{\infty}\sum_{j=0}^{k}\left(\frac{(-1)^{k+j}}{k!}\right)\binom{k}{j}\left(\frac{\delta}{\vartheta}\right)^{k}\sum_{m=0}^{\infty}\left(\frac{(-1)^{m}}{m!}\right)\left(\vartheta((k-j)+1)\right)^{m}\frac{1}{1-m}}{e^{-\frac{\delta}{\vartheta}(e^{\vartheta}-1)}} \end{pmatrix}$$

Let $u_1 = \frac{\partial f(\delta,\vartheta)}{\partial \delta}, u_2 = \frac{\partial f(\delta,\vartheta)}{\partial \vartheta}, u_3 = f(\delta,\vartheta)$

$u_4 = \frac{\partial g(\delta,\vartheta)}{\partial \delta}, u_5 = \frac{\partial g(\delta,\vartheta)}{\partial \vartheta}, u_6 = g(\delta,\vartheta)$

Therefore,

$$\hat{\delta}_{MOM} = \delta_0 + h_1, \quad (19)$$

$$\hat{\vartheta}_{MOM} = \vartheta_0 + k_1, \quad (20)$$

Where,

$J^{-1} = \frac{1}{|J|}\, adj(J),\ h_1 = \frac{u_4 u_6 - u_3 u_5}{u_1 u_5 - u_2 u_4}$, and $k_1 = \frac{-u_3 - u_1 h_1}{u_4}$

That depend on (19) and (20)

So, the estimated survival function $\hat{S}_{RTIGD}(x)$ by MOM will be given as follows:

$$\hat{S}_{\substack{MOM\\RTIGD}}(x;\hat{\delta},\hat{\vartheta}) = \frac{e^{-\frac{\hat{\delta}_{MOM}}{\hat{\vartheta}_{MOM}}\left(e^{\hat{\vartheta}_{MOM}}-1\right)} - e^{-\frac{\hat{\delta}_{MOM}}{\hat{\vartheta}_{MOM}}\left(e^{\frac{\hat{\vartheta}_{MOM}}{x}}-1\right)}}{e^{-\frac{\hat{\delta}_{MOM}}{\hat{\vartheta}_{MOM}}\left(e^{\hat{\vartheta}_{MOM}}-1\right)}}, \quad (21)$$

6. The Meta-Heuristic Algorithms.

Recently, the traditional mathematical method can occasionally fail to solve and address the parameter of the model estimation[19-21]. In the real world, the Meta Heuristics algorithm (MA) provides a near-optimal solution[22]. These algorithms have grown and used in popularity due to their important properties and benefits such as ease of implementation and flexibility [23].

6.1. Firefly Algorithm (FA)

The FA was developed by Xin_She Yang in 2008 [24] by modeling the brightness of the fireflies and their behavior in nature. The majority of the fireflies produce brief, rhythmic, and distinctively patterned lights. The primary function of these lights is to draw in hunters. Each firefly's attractiveness varies based on how much light it can be produced. In fact, the lighter side of each pair of fireflies attracts the one with the less light. [25, 26]

The firefly algorithm (FA) steps :

Step 1: Generate values of parameters FA (the parameter for randomization (α), firefly attractiveness (β_0) , population size $\mathbb{N}$, media light absorption coefficient γ and a maximum number of generations.

Step 2: Generate a random solution set X_i.

Step 3: Evaluate the light intensity (I), at (X_i) from the likelihood function (objective function) of all solutions in the population.

$$f = n\mathrm{Ln}\delta - 2\sum_{i=1}^{n}\mathrm{Ln}(x_i) + \vartheta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}} - 1\right) + n\frac{\delta}{\vartheta}(e^{\vartheta} - 1)$$

Step 4: Update each solution with the position update equation

$$x_i^{t+1} = x_i^t + \beta_0 e^{-\gamma r_{ij}^2}\left(x_j^t - x_i^t\right) + \alpha\left(\text{rand} - \frac{1}{2}\right);$$

$$r_{ij} = \| x_i - x_j \| = \sqrt{\sum_{k=1}^{d}(x_{i,k} - x_{j,k})^2}\ ,\ \gamma \in [0,\infty),\ \ \alpha \in [0,1]$$

Step 5: Perform Greedy selection.
Step 6: Terminate if a termination criterion is fulfilled otherwise go back to step 3.

6.2. Cuckoo Search Algorithm (CSA)

Cuckoo search (CS) was created in 2009 by Suash Deb and Xin-She Yang. The CSA is a population based meta-heuristic algorithm inspired by the reproductive behaviors of the cuckoo bird [27].Cuckoo birds may remove other eggs from communal nests where they lay their eggs in order to increase the probability of hatch [28, 29]. Because their eggs hatch before the host bird's eggs, the cuckoos lay their eggs in a nest where the host bird had just laid its own eggs. As soon as the eggs hatch, the cuckoo chick begins to push the host eggs outside of the nest in order to acquire a larger portion of the food provided by its host bird. Flights on Levy Recent studies, including those by [30].

The steps of CSA are as follows:
Step 1: Generate values of parameters CSA ($p_a \in [0,1]$), maximum the number of generations and population size $\mathbb{N}$
Step 2: Generate a random solution set X_i.
Step 3: Evaluate the objective function from the likelihood function.

$$f = n\text{Ln}\delta - 2\sum_{i=1}^{n}\text{Ln}(x_i) + \vartheta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}} - 1\right) + n\frac{\delta}{\vartheta}(e^{\vartheta} - 1)$$

Step 4: A new solution is randomly generated by using a Levy flight as follows.

$$x_i^{t+1} = x_i^t + \rho \oplus Levy(\lambda)$$

where (ρ) is the step size, $\oplus$ denotes entry-wise multiplication, and Levy(λ) is the Levy distribution.
Step 5 : if (f ($x_i^{(t+1)}$) > f($x_i^{(t)}$))
Then simply replace the old solution with the new solution $x_i^{(t+1)}$.
Step 6: A Part (p_a) of solutions is randomly chosen and replaced with new solutions generated by using local random paths.as follows $x_i^{t+1} = x_i^t + \omega\left(x_j^t - x_k^t\right)$, where ($x_j^t$) and ($x_k^t$) are two various solutions at random chosen and (ω)is a random number.
Step 7: a ranking of the solutions based on (f), The best solution has been chosen. The iteration count goes up.
Step 8: The procedure is repeated until the termination conditions are met.
Step 9: Produce the best solution discovered.

7. The Proposed Hybrid Meta-Heuristic Approach

There are different types of algorithms that are used for optimization each one has a different weaknesses and strengths. Some of them work well with specific problems, while others may not. Not only do they differently perform on different problem classes, but also they differently behave on distinct problem instances. According to Wolpert and Macready (1997)'s the NFL theorem in heuristic search, there is no method that performs well on all problems [14]. As a result, we consider two critical components of modern meta-heuristics: exploration and exploitation. Thus, a new hybrid algorithm namely FA_CSA combing the firefly Algorithm with Cuckoo Search Algorithm to estimate the parameters (δ, ϑ) of the Right Truncated Inverse Gompertz distribution based on SF. Since the diversity is very effective if the algorithm converges slowly to solutions that hop around some potentially optimal solutions.
The steps of FA_CSA are as follows:

Step 1: Generate values of parameters FA and CSA (the randomization parameter α, firefly attractivenessβ_0, population size $\mathbb{N}$ media light absorption coefficient γ and a maximum number of generations.
Step 2: Generate a random solution set X_i.
Step 3: Evaluate the objective function by minimizing the log-likelihood of all solutions in the population.

$$f = nLn\delta - 2\sum_{i=1}^{n} Ln(x_i) + \vartheta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{\delta}{\vartheta}\sum_{i=1}^{n}\left(e^{\frac{\vartheta}{x_i}} - 1\right) + n\frac{\delta}{\vartheta}\left(e^{\vartheta} - 1\right)$$

Step 4: Light intensity (I) is determined by fitness function at (X_i). Step 5: Update each solution with position using a Levy flight for CSA and Update Equation as follows. $x_i^{t+1} = x_i^t + \beta_0 e^{-\gamma r_{ij}^2}\left(x_j^t - x_i^t\right) + \alpha\left(rand - \frac{1}{2}\right)\left(\rho \oplus Levy(\lambda)\right)$

where (ρ) is the step size, $\oplus$ denotes entry-wise multiplication, and $Levy(\lambda)$ is the Levy distribution.
Step 6: Perform Greedy selection
Step 7: Terminate if a termination criterion is fulfilled otherwise go back to (step 3).

8. Simulation study

The proposed method's estimation performance is validated using simulation. Furthermore, 1000 replications were used to generate each simulation condition. To investigate the effective size of the samples, mutable sizes are tested: 15, 30, 60, 90,120 ,160, and 200. The steps of simulation based on Mean Squared Errors criteria were introduced as follows.
Step 1: Initialize all the parameters of FA, CSA and FA_CSA Methods.
Step 2: Generate random samples as $(u_1, u_2, \dots, u_n)$, which can be defined by continuous uniform distribution of interval (0,1). After that, using (CDF) to transform it to random samples that is followed the RTIGD:

$$F_{RTIGD}(x;\delta,\vartheta) = \frac{e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}}, \quad u_i = \frac{e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)}}{e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}} \quad \Rightarrow e^{-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}}-1\right)} = u_i\, e^{-\frac{\delta}{\vartheta}\left(e^{\vartheta}-1\right)}$$

$$-\frac{\delta}{\vartheta}\left(e^{\frac{\vartheta}{x}} - 1\right) = \ln u_i + \left(-\frac{\delta}{\vartheta}\left(e^{\vartheta} - 1\right)\right) \quad \Rightarrow e^{\frac{\vartheta}{x}} - 1 = -\frac{\vartheta}{\delta}\, ln\, u_i + \left(e^{\vartheta} - 1\right)$$

$$e^{\frac{\vartheta}{x}} = -\frac{\vartheta}{\delta}\, ln\, u_i + e^{\vartheta} \quad \Rightarrow e^{\frac{\vartheta}{x}} = e^{\vartheta} - \frac{\vartheta}{\delta}\, ln\, u_i \quad \Rightarrow \frac{\vartheta}{x} = \ln\left(e^{\vartheta} - \frac{\vartheta}{\delta}\, ln\, u_i\right)$$

$$x_i = \frac{\vartheta}{ln\left(e^{\vartheta} - \frac{\vartheta}{\delta}\, ln\, u_i\right)}$$

Then, define a vector which is used for all required parameters; such as Ҳ = [δ, ϑ], and generates N solutions for Ҳ
Step 3: Compute the S from equation (5).
Step 4: Then, Compute $(\hat{S})$ based on (MLE) and (MOM) by using equations (16) and (21)
Step 5: Calculate the best solution $(\hat{S})$ from FA, CSA, FA_CSA methods.
Step 6: Based on (L=1000) trials, MSE will be calculated as follows;

$$\mathrm{MSE} = \left(\frac{1}{\mathrm{L}}\sum_{\mathrm{i}=1}^{\mathrm{L}}\left(\hat{\mathrm{S}}_{\mathrm{i}} - \mathrm{S}\right)^2\right).$$

8. Numerical Results

To determine the best method of the proposed estimation method (FA_CSA) by estimating the survival function based on the scale and shape parameters (δ, ϑ) of RTIGD, seven sample sets (15, 30, 60, 90, 120, 160, 200) are used. Tables (1)-(4) illustrate the results of the simulation of the proposed estimation by utilizing the survival function and MSE of all the estimation and the classical methods, meta-heuristic algorithm (FA_CSA, CSA, FA, MLE, MOM) depend on the survival analyses and MSE. The sets of parameters (δ_1,ϑ_1), (δ_2,ϑ_2), (δ_3,ϑ_3), and (δ_4,ϑ_4)

that are used in the fourth table are (2,1), (3,2), (2,2), and (1,2), respectively. These tables showed that the Hybrid Algorithm (FA_CSA) provided less Mean Square Error in all cases for the survival function. This implies that the (FA_CSA) method was the best of the other estimators. In addition, in Table 1, we notice if the two parameters are different i.e. the scale parameter is less than the shape parameter (first case), then the CSA algorithm showed that it has less MSE, than FA algorithm, MLE, and MOM. In this paper, when n= 160 and the survival function value is equal to 0.24853619, the classical methods MLE and MOM showed that MSE is stronger than the CS algorithm FA. While in Table 1, we notice that the best algorithm comes after the FA_CSA algorithm. The CSA algorithm showed that it has less MSE than MLE, then MOM, and then FA in all the different samples that are taken in this paper. In Table 3, when the parameters of shape and scale are equal and their value is equal to 2, we note that the classical methods (MLE, MOM) provided reasonably suitable solutions, showing that MSE is less than CSA and FA in all the different samples that are chosen except the case when n = 160 and the value of the survival function is (0.70066124), we note that CSA and then FA have MSE less than MLE and MOM, While in Table 4, when the shape parameter is less than the scale parameter, we note that the classical methods (MLE, MOM) provided reasonably suitable solutions, showing that MSE is less than CSA, and FA when n = 15,30,60,200, but in the case of n = 90,120, it was shown that CSA possesses Less MSE, so it is better than FA, MLE, and MOM, but when n=160, we notice that CSA is better than MLE, MOM, and FA.

Table 1: MSE values of $\hat{s}$ when $\delta_1 = 2$ and $\vartheta_1 = 1$

N	S_{RTIGD}	MLE	MOM	FA	CSA	FA_CSA	Best
15	0.51399103	0.00022452	0.00026418	0.00021073	0.000109201	**0.000058592**	**FA_CSA**
30	0.54621339	0.00025122	0.00029835	0.00020593	0.000205111	**0.000129229**	**FA_CSA**
60	0.93782066	0.00067983	0.00087951	0.000003867	0.000003861	**0.0000037455**	**FA_CSA**
90	0.80864843	0.00053203	0.00065391	0.000036615	0.0000366153	**0.0000294678**	**FA_CSA**
120	0.50921327	0.00021976	0.00025929	0.00024086	0.000218407	**0.000124167**	**FA_CSA**
160	0.24853619	0.000053238	0.00006177	0.00056469	0.0003744895	**0.0000302333**	**FA_CSA**
200	0.80622061	0.000018751	0.000405884	0.00003755	0.000037551	**0.0000173341**	**FA_CSA**

Table 2: MSE values of $\hat{s}$ when $\delta_2 = 3$ and $\vartheta_2 = 2$

n	S_{RTIGD}	MLE	MOM	FA	CSA	FA_CSA	Best
15	0.22568305	0.000049232	0.0000509328	0.0005995666	0.0000358966	**0.00001411691**	**FA_CSA**
30	0.52969078	0.000269217	0.0002805723	0.0002211907	0.0000659206	**0.00006301401**	**FA_CSA**
60	0.32883322	0.000104295	0.0001081313	0.0004504643	0.0000449796	**0.00003519722**	**FA_CSA**
90	0.21891907	0.000046323	0.0000479256	0.0006100871	0.0000302368	**0.00001403676**	**FA_CSA**
120	0.46764682	0.000210211	0.0002186941	0.000283399	0.0000490429	**0.00004534779**	**FA_CSA**
160	0.23787034	0.000054672	0.0000565823	0.0005808381	0.0000182799	**0.00001793788**	**FA_CSA**
200	0.74849087	0.000532149	0.0005602385	0.0003596656	0.0001046597	**0.00008154242**	**FA_CSA**

Table 3: MSE values of $\hat{s}$ when $\delta_3 = 2$ and $\vartheta_3 = 2$

n	S_{RTIGD}	MLE	MOM	FA	CS	FA_CSA	Best
15	0.38501026	0.00014005	0.0001482329	0.000378212	0.0001415739	**0.0000077046**	**FA_CSA**
30	0.35738965	0.00012090	0.0001277273	0.000412948	0.0001555131	**0.0000212501**	**FA_CSA**
60	0.10866648	0.00001124	0.0000118084	0.000794475	0.0000335452	**0.0000020396**	**FA_CSA**
90	0.41725363	0.00016441	0.0001741005	0.000339593	0.0001481170	**0.0000245383**	**FA_CSA**
120	0.22350023	0.00004743	0.0000499523	0.000602952	0.0000875128	**0.0000015754**	**FA_CSA**
160	0.70066124	0.00045711	0.0004909262	0.000089601	0.0000517870	**0.0000292561**	**FA_CSA**
200	0.58990013	0.00032628	0.0003479821	0.000168181	0.0001136663	**0.0000104644**	**FA_CSA**

Table 4: MSE values of $\hat{s}$ when $\delta_4 = 1$ and $\vartheta_4 = 2$

n	S_{RTIGD}	MLE	MOM	FA	CSA	FA_CSA	Best
15	0.45161268	0.00018208	0.000203954	0.000300728	0.000296004	**0.000037627**	**FA_CSA**
30	0.34015476	0.00010384	0.000115705	0.000435396	0.000337102	**0.000001822**	**FA_CSA**
60	0.14215532	0.00001825	0.0000202081	0.000735897	0.000226647	**0.000000498**	**FA_CSA**
90	0.54327752	0.00026195	0.0002951504	0.000208595	0.000192656	**0.000002786**	**FA_CSA**
120	0.90183611	0.00068672	0.0008133084	0.000009636	0.000009635	**0.000001397**	**FA_CSA**
160	0.19990599	0.00003604	0.0000399624	0.000427033	0.000015499	**0.000003792**	**FA_CSA**
200	0.33880791	0.00010295	0.0001147907	0.000351734	0.000032424	**0.000002115**	**FA_CSA**

8. **Conclusions**

In this paper, we propose three methods (CSA), (FA) and (FA_CSA) to estimate survival functions based on the parameters (δ, ϑ) of RTIGD. Simulation is utilized to compare the suggested method with a classical method which includes (MLE and MOM). The results show that in the first, second, third and fourth cases, it was found that FA_CSA algorithm is better than the other algorithms in all the different sample sizes that are randomly selected in this paper, where the proposed algorithms are strengthened by taking the strengths of each proposed algorithm in this paper and merging them with Some in order to give better and stronger results, as the results proved that the new algorithms are the best because they have less MSE.

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