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Existence and Uniqueness Theorem of Fuzzy Stochastic Ordinary Differential Equations

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Abstract
A fuzzy valued diffusion term, which in a fuzzy stochastic differential equation refers to one-dimensional Brownian motion, is defined by the meaning of the stochastic integral of a fuzzy process. In this paper, the existence and uniqueness theorem of fuzzy stochastic ordinary differential equations, based on the mean square convergence of the mathematical induction approximations to the associated stochastic integral equation, are stated and demonstrated.

Keywords: Existence and uniqueness theorem, Fuzzy differential equations, Stochastic differential equations, Brownian motion.

مبرهنة الوجود والوحدانية للمعادلات التفاضلية التصادفية الضبابية الاعتيادية

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الخلاصة
تتكون المعادلة التفاضلية التصادفية الضبابية من مصطلح انتشار ذو قيمة غامضة والذي يتم تعريفة من خلال معنى التكامل العشوائي لعملية ضبابية فيما يتعلق بالحركة البراونية احادية البعد. في هذا البحث، يتم ذكر مبرهنة الوجود والوحدانية للمعادلات التفاضلية العشوائية الضبابية الاعتيادية واثبات انها تستند على متوسط التقارب للمعادلة التكاملية الضبابية.

1. Introduction

In recent years, the theory of Fuzzy Stochastic Ordinary Differential Equations (FSODE's) has been extensively developed in conjunction with fuzzy valued mappings [1,2]. The FSODEs are employed in several real-world systems, such as economics, and consisting phenomenas elated to fuzziness and types of uncertainty [2].

There are a variety of papers that are concerned with FSODE's [3]. An explanation of the fuzzy stochastic Itô integral was provided by Kim in [3,4-7]. The authors drove up the fuzzy

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Itô stochastic integral using the Brownian motion and fuzzy non-anticipating stochastic processes in order to create a fuzzy random variable, the procedure entails embedding a crisp Itô stochastic integral into fuzzy space. In this work, the existence and uniqueness theorem of the solution to such system are stated and proved [7].

Also, it is notable that fuzzy set theory is a generalization of abstract set theory and it has a wide scope of applications than abstract or crisp set theory in solving problems that involve some degree of subjective evaluation [8]. In order to recall fuzzy sets, let X be a space of objects (called the *universal set*) and x be the generic element of X, a classical (nonfuzzy or crisp) set A, $A \subseteq X$, is defined as a collection of elements or objects $x \in X$, such that each element x can either belong or can not to the set A. By defining a characteristic (or membership) function for each element $x \in X$, one can represent a classical set A by a set of order pairs (x,0) or (x,1), which indicates that $x \notin A$ or $x \in A$, respectively. A fuzzy set $\tilde{A}$ expresses the degree to which an element belongs to a set [9,10].
Hence, for simplicity, the membership function of a fuzzy set $\tilde{A}$ is allowed to have values between 0 and 1, which reflects the degree of the membership of an element in $\tilde{A}$. In mathematical symbols, the membership function is given by $\mu_{\tilde{A}}: X \longrightarrow [0,1]$, and the fuzzy subset $\tilde{A}$ of X is defined as a set of ordered pairs [11-13]:
$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X,\ \mu_{\tilde{A}}(x) \in [0,1]\}$.
Now, the general governing model of FSODEs is given in the following:

$$d\tilde{x}_t(t, W_t) = f(t, \tilde{x}_t(t, W_t))dt + g(t, \tilde{x}_t(t, W_t))\, dW_t,\ t \in I \subset \mathbb{R}, \tag{1}$$

with the initial condition:

$$\tilde{x}_t(t_0, W_t) = \tilde{x}_{t_0}, \tag{2}$$

where $f: I \times \mathcal{F}_C(\mathbb{R}) \longrightarrow \mathcal{F}_C(\mathbb{R})$, $g: I \times \mathcal{F}_C(\mathbb{R}) \longrightarrow \mathcal{F}_C(\mathbb{R})$, where $\mathcal{F}_C(\mathbb{R})$ is the family of all fuzzy subset of X for which their level sets are non-empty closed convex subset of the reals $\mathbb{R}$, and $W_t, \forall\, t \geq 0$ are one-dimensional Brownian motion[14,15].

The SFODE given in Eq.(1) may be written in an equivalent form as fuzzy integral equation:

$$\tilde{x}_t(t, W_t) = \tilde{x}_{t_0} + \int_0^t f(s, \tilde{x}_s(s, W_s))ds + \int_0^t g(s, \tilde{x}_s(s, W_s))dW_s. \tag{3}$$

However, the second integral that is given in Eq.(3) cannot be defined in the usual integral meaning, where W_t is the Brownian motion or Wiener process. Then by using the concept of α-levels or α-cuts of fuzzy sets and letting $\tilde{x}_t(W_t) = [\underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)]$, $\alpha \in [0,1]$, where $\underline{x}_t^\alpha$ and $\bar{x}_t^\alpha$ are the lower and upper bound solutions constituting the fuzzy function $\tilde{x}_t$, as a solution to Eq.(1) or Eq.(3), and hence Eq.(1) may be written in terms of its lower and upper solutions as follows:

$$d\underline{x}_t^\alpha(W_t) = f_1\left(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\right)dt + g_1\left(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\right)dW_t, \tag{4}$$

$$d\bar{x}_t^\alpha(W_t) = f_2\left(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\right)dt + g_2\left(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\right)dW_t, \tag{5}$$

with the initial conditions , respectively:

$$\left.\begin{aligned} \underline{x}_{t_0}^\alpha &= \underline{x}_0^\alpha, \\ \bar{x}_{t_0}^\alpha &= \bar{x}_0^\alpha. \end{aligned}\right\} \tag{6}$$

Also, the fuzzy integral equations that related to Eqs.(4) and (5) are given by:

$$\underline{x}_t^\alpha(W_t) = \underline{x}_{t_0}^\alpha + \int_{t_0}^t f_1\left(s, \underline{x}_s^\alpha(W_s), \bar{x}_s^\alpha(W_s)\right) ds + \int_{t_0}^t g_1\left(s, \underline{x}_s^\alpha(W_s), \bar{x}_s^\alpha(W_s)\right) dW_s, \tag{7}$$

$$\bar{x}_t^\alpha(W_t) = \bar{x}_{t_0}^\alpha + \int_{t_0}^t f_2\left(s, \underline{x}_s^\alpha(W_s), \bar{x}_s^\alpha(W_s)\right) ds + \int_{t_0}^t g_2\left(s, \underline{x}_s^\alpha(W_s), \bar{x}_s^\alpha(W_s)\right) dW_s. \tag{8}$$

2. Basic concepts

In this section, some basic concepts related to this work will be introduced. We start by recalling the concept of the probability space $(\Omega, \mathcal{F}, p)$, which comprises the sample space Ω, a σ-algebra $\mathcal{F}$ of subsets of Ω (called events) and a probability measure p of $\mathcal{F}$.

Definition 1 [16,17]. A collection of random variables $x_t(\omega)$ (or briefly x_t) on probability space $(\Omega, \mathcal{F}, p)$ is known as a stochastic process, which assumes real values and p-measurable as a function of $\omega \in \Omega$ for every fixed $t \in [t_0,T] \subset [0,\infty)$. The time is considered to be the parameter t, and $x_t(.)$ represents a random variable on the above probability space Ω, while $x_.(\omega)$ is a known trajectory or sample path of the stochastic process.

Definition 2 [18]. A stochastic process $W_t, t \in [0, \infty)$, is said to be a Wiener process or a Brownian motion if:
1. $p(\{\omega \in \Omega | W_0(\omega) = 0\}) = 1$.
2. For $0 < t_0 < t_1 < \cdots < t_n$, the increments $W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent.
3. For $h > 0$ and an arbitrary t, $W_{t+h} - W_t$ has a Gaussian distribution with mean 0 and variance h.
Sequence of stochastic process may have different meaning of convergence, as it is seen in the next three definitions:

Definition 3 [18,19]. A sequence of random variables $\{x_n(\omega)\}\}$, n = 1, 2, … is said to be converge with probability one (denoted by p-w.p.1 or w.p.1) to $x(\omega)$ if $p(\{\omega \in \Omega: \lim_{n\to\infty} x_n(\omega) = x(\omega)\}) = 1$.
Another name for this type of convergence is called almost sure (denoted by a.s.) convergence.

Definition 4 [20]. A sequence of random variables $\{x_n(\omega)\}\}$, n = 1, 2, … such that $E(x_n^2) < \infty$, for all $n \in \mathbb{N}$, is said to be converge in the mean square to $x(\omega)$ if $\lim_{n\to\infty} E(|x_n - x|^2) = 0$, $\forall\ \omega \in \Omega$, where E stands for the mathematical expectation.

Definition 5 [21]. A sequence of random variables $\{x_n(\omega)\}\}$, n = 1, 2, … is said to be converge in probability or stochastically to $x(\omega)$, if:
$lim_{n\to\infty} p(\{\omega \in \Omega \mid x_n(\omega) - x(\omega)) \geq \varepsilon\} = 0, \forall\ \varepsilon > 0$.
Several results are needed for this work,we will start with the next two theorems:

Theorem 1 [22]. If f is step function in $\mu^2_{W_t}[\alpha, \beta]$, where $\mu^2_{W_t}[\alpha, \beta]$ is the set of measurable functions, which are square integrable over $[0, T]$ and W_t is a Brownian motion, then:
$E\int_\alpha^B f(t)\, dW_t = 0,$
$E\left|\int_\alpha^\beta f(t)\, dW_t\right|^2 = E\int_\alpha^\beta f^2 dt.$

Theorem 2 [22]. Let $f \in \mu^2_{W_t}[\alpha, \beta]$, then:
$E\left\{Sup_{0\leq t\leq T}\left|\int_0^T f(s)\, dW_s\right|^2\right\} \leqslant 4E\left|\int_0^T f(s)\, dW_s\right|^2$
$= 4E\int_0^T |f(s)|^2\, ds.$

3. Existence and uniqueness theorem of stochastic differential equations

Consider the FSODE of the lower-case solution given by Eq. (4), which is:
$d\underline{x}_t^\alpha(W_t) = f_1\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)dt + g_1\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)dW_t$,
with initial conditions $\underline{x}_{t_0}^\alpha = \underline{x}_0^\alpha, \bar{x}_{t_0}^\alpha = \bar{x}_0^\alpha$.

Hence, to find the equivalent stochastic integral equation, integrate both sides of Eq.(4) and using the initial condition, we get:
$\int_0^t d\underline{x}_s^\alpha(W_s)\, ds = \int_0^t f\big(s, \underline{x}_0^\alpha(W_s), \bar{x}_s^\alpha(W_s)\big)\, ds + \int_0^t g\big(s, \underline{x}_0^\alpha(W_s), \bar{x}_s^\alpha(W_s)\big)\, dW_s$.
Therefore:

$$\underline{x}_t^\alpha(W_t) = \underline{x}_0^\alpha + \int_0^t f\big(s, \underline{x}_s^\alpha(W_s), \bar{x}_s^\alpha(W_s)\big)\, ds + \int_0^t g\big(s, \underline{x}_s^\alpha(W_s), \bar{x}_s^\alpha(W_s)\big)\, dW_s,$$

and then an iterated sequence of solutions of the resulting integral equation may be evaluated as follows:

$$\left.\begin{aligned}
\underline{x}_{1_t}^\alpha(W_t) &= \underline{x}_0^\alpha + \int_0^t f\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)ds + \int_0^t g\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)dW_s,\\
\underline{x}_{2_t}^\alpha(W_t) &= \underline{x}_0^\alpha + \int_0^t f\big(s, \underline{x}_{1_s}^\alpha(W_s), \bar{x}_{1_s}^\alpha(W_s)\big)ds + \int_0^t g\big(s, \underline{x}_{1_s}^\alpha(W_s), \bar{x}_{1_s}^\alpha(W_s)\big)dW_s,\\
&\vdots\\
\underline{x}_{m+1_t}^\alpha(W_t) &= \underline{x}_0^\alpha + \int_0^t f\big(s, \underline{x}_{m_s}^\alpha(W_s), \bar{x}_{m_s}^\alpha(W_s)\big)ds + \int_0^t g\big(s, \underline{x}_{m_s}^\alpha(W_s), \bar{x}_{m_s}^\alpha(W_s)\big)dW_s.
\end{aligned}\right\} \quad (9)$$

Theorem 3 (Existence Theorem). Suppose $f\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)$, $g\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)$ are measurable functions in $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, which satisfies:

$$\left.\begin{aligned}
\big|f\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big) - f\big(t, \underline{x}_{n_t}^\alpha(W_t), \bar{x}_{n_t}^\alpha(W_t)\big)\big| &\leqslant K_*\big(\big|\underline{x}_t^\alpha(W_t) - \underline{x}_{n_t}^\alpha(W_t)\big| + \big|\bar{x}_t^\alpha(W_t) - \bar{x}_{n_t}^\alpha(W_t)\big|\big),\\
\big|g\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big) - g\big(t, \underline{x}_{n_t}^\alpha(W_t), \bar{x}_{n_t}^\alpha(W_t)\big)\big| &\leqslant K_*\big(\big|\underline{x}_t^\alpha(W_t) - \underline{x}_{n_t}^\alpha(W_t)\big| + \big|\bar{x}_t^\alpha(W_t) - \bar{x}_{n_t}^\alpha(W_t)\big|\big),
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
\big|f\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)\big| &\leqslant K\big(1 + \big|\underline{x}_t^\alpha(W_t)\big| + |\bar{x}_t^\alpha(W_t)|\big),\\
\big|g\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)\big| &\leqslant K\big(1 + \big|\underline{x}_t^\alpha(W_t)\big| + |\bar{x}_t^\alpha(W_t)|\big),
\end{aligned}\right\} \quad (11)$$

where K and K_* are the Lipschitz constants. Let $\underline{x}_0^\alpha$ be any n-dimensional random vector independent of $f\big(t, \underline{x}_t^\alpha(W_t), \bar{x}_t^\alpha(W_t)\big)$, $0 \le t \le T$, such that $E\big|\underline{x}_0^\alpha\big| < \infty$. Then there exists a solution of Eqs. (4) and (6) in $\mu_\omega^2[0, T]$, where $\mu_\omega^2[0, T]$ is the space of measurable functions which are square integrable over $[0, T]$.

Proof. Since the iterated sequence of solutions of the integral equation may be given as:

$$\underline{x}_{m+1_t}^\alpha(W_t) = \underline{x}_0^\alpha + \int_0^t f\big(s, \underline{x}_{m_s}^\alpha(W_s), \bar{x}_{m_s}^\alpha(W_s)\big)\, ds + \int_0^t g\big(s, \underline{x}_{m_s}^\alpha(W_s), \bar{x}_{m_s}^\alpha(W_s)\big)\, dW_s, \quad (12)$$

for all $m = 0,1$ ....
The proof will proceed by induction on the sequence of solutions $\underline{x}_{t_m}^\alpha \in \mu_\omega^2[0, T]$.
If $m = 0$, then:
$\big|\underline{x}_{1_t}^\alpha(W_t) - \underline{x}_{0_t}^\alpha(W_s)\big| =$
$\Big|\underline{x}_0^\alpha + \int_0^t f\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)ds + \int_0^t g\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)dW_s - \underline{x}_0^\alpha\Big|$
$= \Big|\int_0^t f\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)ds + \int_0^t g\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)dW_s\Big|$
$\le \Big|\int_0^t f\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)ds\Big| + \Big|\int_0^t g\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)dW_s\Big|$.
Taking the expectation on both sides and using Eq.(11), give:
$E\big|\underline{x}_{1_t}^\alpha(W_t) - \underline{x}_{0_t}^\alpha(W_s)\big| \le$
$E\Big|\int_0^t f\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)ds\Big| + E\Big|\int_0^t g\big(s, \underline{x}_{0_s}^\alpha(W_s), \bar{x}_{0_s}^\alpha(W_s)\big)dW_s\Big|$.
Using the following inequality $(a + b)^2 \leqslant 2a^2 + 2b^2$, where $a, b \in \mathbb{R}$, then:

$E\left|\underline{x}_{1_t}^{\alpha}(W_t)-\underline{x}_{0_t}^{\alpha}(W_s)\right|^2 \leq$
$2E\left|\int_0^t f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)ds\right|^2+2E\left|\int_0^t g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)dW_s\right|^2.$
The Cauchy-Schwarz inequality implies that:
$\left|\int_0^t f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)ds\right|^2 \leqslant t\int_0^t \left|f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right|^2 ds$, for any $t>0$.
Now, from Theorem 1, we have:
$E\left|\underline{x}_{1_t}^{\alpha}(W_t)-\underline{x}_{0_t}^{\alpha}(W_s)\right|^2 \leq 2Et\int_0^t\left|f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right|^2 ds+$
$2E\int_0^t\left|g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right|^2 ds$
$\leq (2K^2t+2K^2)\left(1+E\left|\underline{x}_{0_s}^{\alpha}(W_s)\right|^2+E\left|\bar{x}_{0_s}^{\alpha}(W_s)\right|^2\right)t$
$=Mt=\frac{Mt}{1!},$
where $M=(2K^2t+2K^2)\left(1+E\left|\underline{x}_{0_s}^{\alpha}(W_s)\right|^2+E\left|\bar{x}_{0_s}^{\alpha}(W_s)\right|^2\right)$.
If $m=1$, then:
$\left|\underline{x}_{2_t}^{\alpha}(W_t)-\underline{x}_{1_t}^{\alpha}(W_s)\right|=$
$\left|\underline{x}_0^{\alpha}+\int_0^t f\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)ds+\int_0^t g\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)dW_s-\right.$
$\left.\left[\underline{x}_0^{\alpha}+\int_0^t f\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)ds+\int_0^t g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)dW_s\right]\right|$
$=$
$\left|\int_0^t f\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)ds-\int_0^t f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)ds+\right.$
$\left.\int_0^t g\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)dW_s-\int_0^t g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)dW_s\right|$
$\leq$
$\left|\int_0^t f\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)ds-\int_0^t f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)ds\right|+$
$\left|\int_0^t g\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)dW_s-\int_0^t g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)dW_s\right|.$
Taking the expectation on both sides and using Eq.(10), yields to:
$E\left|\underline{x}_{2_s}^{\alpha}(W_t)-\underline{x}_{1_s}^{\alpha}(W_t)\right|^2 \leq 2E\left|\int_0^t\left[f\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)-f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right]ds\right|^2$
$+2E\left|\int_0^t\left[g\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)-g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right]ds\right|^2$
$\leq$
$2E\int_0^t\left|f\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)-f\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right|^2 ds+$
$2E\int_0^t\left|g\left(s,\underline{x}_{1_s}^{\alpha}(W_s),\bar{x}_{1_s}^{\alpha}(W_s)\right)-g\left(s,\underline{x}_{0_s}^{\alpha}(W_s),\bar{x}_{0_s}^{\alpha}(W_s)\right)\right|^2 ds$
$\leq 2K_*^2 t\int_0^t E\left(\left|\underline{x}_{1_s}^{\alpha}(W_s)-\underline{x}_{0_s}^{\alpha}(W_s)\right|^2+\left|\underline{x}_{1_s}^{\alpha}(W_s)-\bar{x}_{0_s}^{\alpha}(W_s)\right|^2\right)ds+2K_*^2 t\int_0^t E\left(\left|\underline{x}_{1_s}^{\alpha}(W_s)-\right.\right.$
$\left.\left.\underline{x}_{0_s}^{\alpha}(W_s)\right|^2+\left|\underline{x}_{1_s}^{\alpha}(W_s)-\bar{x}_{0_s}^{\alpha}(W_s)\right|^2\right)ds$
$\leq (2K_*^2+2K_*^2)E\left[\left|\underline{x}_{1_s}^{\alpha}(W_s)-\underline{x}_{0_s}^{\alpha}(W_s)\right|^2+\left|\underline{x}_{1_s}^{\alpha}(W_s)-\bar{x}_{0_s}^{\alpha}(W_s)\right|^2\right]t \leq \frac{(Mt)^2}{2!}.$
So on, one may proceed similarly until $m=k$, then we have:
$E\left|\underline{x}_{k+1_t}^{\alpha}(W_t)-\underline{x}_{k_t}^{\alpha}(W_t)\right|^2 \leq$
$2E\left|\int_0^t\left[f\left(s,\underline{x}_{k_s}^{\alpha}(W_s),\bar{x}_{k_s}^{\alpha}(W_s)\right)-f\left(s,\underline{x}_{(k-1)_s}^{\alpha}(W_s),\bar{x}_{(k-1)_s}^{\alpha}(W_s)\right)\right]ds\right|^2+$
$$2E\left|\int_0^t\left[g\left(s,\underline{x}_{k_s}^{\alpha}(W_s),\bar{x}_{k_s}^{\alpha}(W_s)\right)-g\left(s,\underline{x}_{(k-1)_s}^{\alpha}(W_s),\bar{x}_{(k-1)_s}^{\alpha}(W_s)\right)\right]ds\right|^2 \quad (13)$$

$\leq$
$2E\int_0^t\left|f\left(s,\underline{x}^{\alpha}_{k_s}(W_s),\bar{x}^{\alpha}_{k_s}(W_s)\right)-f\left(s,\underline{x}^{\alpha}_{(k-1)_s}(W_s),\bar{x}^{\alpha}_{(k-1)_s}(W_s)\right)\right|^2ds+$
$2E\int_0^t\left|g\left(s,\underline{x}^{\alpha}_{k_s}(W_s),\bar{x}^{\alpha}_{k_s}(W_s)\right)-g\left(s,\underline{x}^{\alpha}_{(k-1)_s}(W_s),\bar{x}^{\alpha}_{(k-1)_s}(W_s)\right)\right|^2ds$
$\leq$
$2K_*^2t\int_0^t E\left(\left|\underline{x}^{\alpha}_{k_s}(W_s)-\underline{x}^{\alpha}_{(k-1)_s}(W_s)\right|^2+\left|\underline{x}^{\alpha}_{k_s}(W_s)-\bar{x}^{\alpha}_{(k-1)_s}(W_s)\right|^2\right)ds+$
$2K_*^2\int_0^t E\left(\left|\underline{x}^{\alpha}_{k_s}(W_s)-\underline{x}^{\alpha}_{(k-1)_s}(W_s)\right|^2+\left|\underline{x}^{\alpha}_{k_s}(W_s)-\bar{x}^{\alpha}_{(k-1)_s}(W_s)\right|^2\right)ds$
$\leq(2K_*^2t+2K_*^2)E\left[\left|\underline{x}^{\alpha}_{k_s}(W_s)-\underline{x}^{\alpha}_{(k-1)_s}(W_s)\right|^2+\left|\underline{x}^{\alpha}_{k_s}(W_s)-\bar{x}^{\alpha}_{(k-1)_s}(W_s)\right|^2\right]t^k$
$\leq\frac{(Mt)^{k+1}}{(k+1)!}$, for $0\leq k\leq m-1$.

This implies that $\underline{x}^{\alpha}_{m+1_t}(W_t)\in\mu^2_{\omega}[0,T]$ and hence the proof of inductive assumption for $m+1$ is complete.

Now, from inequality (13), we have also:
$E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right|^2\leq$
$2E\left|\int_0^t\left[f\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)-f\left(s,\underline{x}^{\alpha}_{m-1_s}(W_s),\bar{x}^{\alpha}_{m-1_s}(W_s)\right)\right]ds\right|^2+$
$2E\left|\int_0^t\left[g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)-g\left(s,\underline{x}^{\alpha}_{m-1_s}(W_s),\bar{x}^{\alpha}_{m-1_s}(W_s)\right)\right]ds\right|^2.$

Hence:
$Sup_{0\leqslant t\leqslant T}E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right|^2\leqslant 2TK_*\left(\left|\underline{x}^{\alpha}_{m_t}(W_t)-\underline{x}^{\alpha}_{m-1_t}(W_t)\right|^2+\left|\bar{x}^{\alpha}_{m_t}(W_t)-\right.\right.$
$\left.\bar{x}^{\alpha}_{m-1_t}(W_t)\right|^2+$
$\left.2Sup_{0\leqslant t\leqslant T}\left|\int_0^t\left[g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)-g\left(s,\underline{x}^{\alpha}_{m-1_s}(W_s),\bar{x}^{\alpha}_{m-1_s}(W_s)\right)\right]dW_s\right|^2\right),$

using Theorem 2:
$Sup_{0\leqslant t\leqslant T}E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right|^2\leqslant 2K_*^2\int_0^T\left(E\left|\underline{x}^{\alpha}_{m_s}(W_s)-\underline{x}^{\alpha}_{m-1_s}(W_s)\right|^2+E\left|\bar{x}^{\alpha}_{m_s}(W_s)-\right.\right.$
$\left.\left.\bar{x}^{\alpha}_{m-1_s}(W_s)\right|^2\right)ds+8K_*^2\int_0^t\left(E\left|\underline{x}^{\alpha}_{m_s}(W_s)-\underline{x}^{\alpha}_{m-1_s}(W_s)\right|^2+E\left|\bar{x}^{\alpha}_{m_s}(W_s)-\bar{x}^{\alpha}_{m-1_s}(W_s)\right|^2\right)ds$
$\leqslant(2TK_*^2+8K_*^2)T.$

Therefore:
$E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right|^2\leq\frac{C(MT)^m}{m!}.$

Now, to prove the convergence of the sequence $\left\{\underline{x}^{\alpha}_{m_t}\right\}_{m=1}^{\infty}$ uniformly in $t\in[0,T]$, which means that to prove the following sequence of partial sums must also be converge uniformly:
$\underline{x}^{\alpha}_{k_t}(W_t)=\underline{x}^{\alpha}_0+\sum_{m=0}^{k-1}\left(\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right),\forall\, k=1,2,...$
$=\underline{x}^{\alpha}_0+\left(\underline{x}^{\alpha}_{1_t}(W_t)-\underline{x}^{\alpha}_{0_t}(W_t)\right)+\cdots+\left(\underline{x}^{\alpha}_{k_t}(W_t)-\underline{x}^{\alpha}_{k-1_t}(W_t)\right).$

From (4), we have:
$E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)\right|^2=E\left|\underline{x}^{\alpha}_0+\int_0^t f\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)ds+\int_0^t g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)dW_s\right|^2$
$\leq E\left|\underline{x}^{\alpha}_0\right|^2+E\left|\int_0^t f\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)ds\right|^2+E\left|\int_0^t g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)dW_s\right|^2$
$\leq E\left|\underline{x}^{\alpha}_0\right|^2+K^2t\int_0^t\left((1+E)\left|\underline{x}^{\alpha}_{m_s}(W_s)\right|^2+E\left|\bar{x}^{\alpha}_{m_s}(W_s)\right|^2\right)ds+K^2\int_0^t\left(1+E\left|\underline{x}^{\alpha}_{m_s}(W_s)\right|^2+\right.$
$\left.E\left|\bar{x}^{\alpha}_{m_s}(W_s)\right|^2\right)ds$
$\leq E\left|\underline{x}^{\alpha}_0\right|^2+(K^2t+K^2)\int_0^t\left(E\left|\underline{x}^{\alpha}_{m_s}(W_s)\right|^2+E\left|\bar{x}^{\alpha}_{m_s}(W_s)\right|^2\right)ds$
$\leq C\left(1+E\left|\underline{x}^{\alpha}_0\right|^2\right)+C\int_0^t\left(E\left|\underline{x}^{\alpha}_{m_s}(W_s)\right|^2+E\left|\bar{x}^{\alpha}_{m_s}(W_s)\right|^2\right)ds.$

Now, carrying the last inequality recursively, yields to:

$E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)\right|^2 \leq C\left(1+E\left|\underline{x}^{\alpha}_0\right|^2\right)+Ct\left[C\left(1+E\left|\underline{x}^{\alpha}_0\right|^2+Ct[C(1+E\left|\underline{x}^{\alpha}_0\right|^2\right)+\right.$
$C\int_0^t\left(E\left|\underline{x}^{\alpha}_{m-1_s}(W_s)\right|^2+E\left|\bar{x}^{\alpha}_{m-1_s}(W_s)\right|^2\right)ds]].$
So, carrying the last inequality m-times will produce:
$E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)\right|^2 \leq \left(1+C+C^2t+\cdots+C^m\frac{t^m}{m!}+\cdots\right)\left(1+E\left|\underline{x}^{\alpha}_0\right|^2\right)$
$=e^{Ct}\left(1+E\left|\underline{x}^{\alpha}_0\right|^2\right).$
Therefore,
$E\left|\underline{x}^{\alpha}_{m+1_t}(W_t)\right|^2 \leqslant c\left(1+E\left|\underline{x}^{\alpha}_0\right|^2\right)e^{Ct}.$
To prove $\underline{x}^{\alpha}_{k_t}(W_t)$ converge to $\underline{x}^{\alpha}_t(W_t)$ as $k\to\infty$, for each Weiner process W_t, i.e., to prove $\lim_{k\to\infty}\underline{x}^{\alpha}_{k_t}(W_t) = \underline{x}^{\alpha}_t(W_t)$, given any $\varepsilon>0$, there exists $N\in\mathbb{N}$, such that:
$\left|\underline{x}^{\alpha}_{k_t}(W_t)-\underline{x}^{\alpha}_t(W_t)\right|<\varepsilon,\ \forall\, k\geqslant N,$
and so:
$\left|\underline{x}^{\alpha}_0+\sum_{m=0}^{k-1}\left(\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right)-\underline{x}^{\alpha}_t(W_t)\right| = \left|\underline{x}^{\alpha}_0+\left(\underline{x}^{\alpha}_{1_t}(W_t)-\underline{x}^{\alpha}_{0_t}(W_t)\right)+\right.$
$\left.\left(\underline{x}^{\alpha}_{2_t}(W_t)-\underline{x}^{\alpha}_{1_t}(W_t)\right)+\cdots+\left(\underline{x}^{\alpha}_{m_t}(W_t)-\underline{x}^{\alpha}_{m-1_t}(W_t)\right)\right|<\varepsilon,$
which implies:
$\left|\underline{x}^{\alpha}_{k_t}(W_t)-\underline{x}^{\alpha}_t(W_t)\right|<\varepsilon.$
Since $\underline{x}^{\alpha}_{k_t}(W_t)=\underline{x}^{\alpha}_0+\sum_{m=0}^{k-1}\left(\underline{x}^{\alpha}_{m+1_t}(W_t)-\underline{x}^{\alpha}_{m_t}(W_t)\right)$, for all $k=1,2,\ldots$ converges on a compact interval $[0,T]$, then the sequence converges uniformly.
Now, to prove $\underline{x}^{\alpha}_t(W_t)$ is continuous, that means we have to prove $\lim_{h\to 0}\underline{x}^{\alpha}_{t+h}(W_t)=\underline{x}^{\alpha}_t(W_t)$. Since:
$\left|\underline{x}^{\alpha}_{t+h}(W_t)-\underline{x}^{\alpha}_t(W_t)\right| = \left|\underline{x}^{\alpha}_{t+h}(W_t)-\underline{x}^{\alpha}_{k_{t+h}}(W_t)+\underline{x}^{\alpha}_{k_{t+h}}(W_t)-\underline{x}^{\alpha}_{k_t}(W_t)+\underline{x}^{\alpha}_{k_t}(W_t)-\right.$
$\left.\underline{x}^{\alpha}_t(W_t)\right|$
$\leq \left|\underline{x}^{\alpha}_{t+h}(W_t)-\underline{x}^{\alpha}_{k_{t+h}}(W_t)\right|+\left|\underline{x}^{\alpha}_{k_{t+h}}(W_t)-\underline{x}^{\alpha}_{k_t}(W_t)\right|+\left|\underline{x}^{\alpha}_{k_t}(W_t)-\underline{x}^{\alpha}_t(W_t)\right|$
$=\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$
Therefore, $\lim_{h\to 0}\underline{x}^{\alpha}_{t+h}(W_t)=\underline{x}^{\alpha}_t(W_t)$ and therefore $\underline{x}^{\alpha}_t(W_t)$ is continuous.
To prove $\underline{x}^{\alpha}_t(W_t)$ is a solution, that means to show that $\underline{x}^{\alpha}_t(W_t)$ satisfies the stochastic integral equation:
$\underline{x}^{\alpha}_t(W_t)=\underline{x}^{\alpha}_0+\int_0^t f\left(s,\underline{x}^{\alpha}_s(W_t),\bar{x}^{\alpha}_s(W_t)\right)ds+\int_0^t g\left(s,\underline{x}^{\alpha}_s(W_t),\bar{x}^{\alpha}_s(W_t)\right)dW_s,$
and since:
$\underline{x}^{\alpha}_{m+1_t}(W_t)=\underline{x}^{\alpha}_0+\int_0^t f\left(s,\underline{x}^{\alpha}_{m_s}(W_t),\bar{x}^{\alpha}_{m_s}(W_t)\right)ds+\int_0^t g\left(s,\underline{x}^{\alpha}_{m_s}(W_t),\bar{x}^{\alpha}_{m_s}(W_t)\right)dW_s,$
and, as $m\longrightarrow\infty$, then:
$\underline{x}^{\alpha}_t(W_t)=$
$\underline{x}^{\alpha}_0+lim_{m\to\infty}\int_0^t f\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)ds+lim_{m\to\infty}\int_0^t g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)dW_s.$
Therefore, it is enough to prove the following:
$lim_{m\to\infty}\int_0^t f\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)ds=\int_0^t f\left(s,\underline{x}^{\alpha}_s(W_s),\bar{x}^{\alpha}_s(W_s)\right)ds,$
$lim_{m\to\infty}\int_0^t g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)dW_s=\int_0^t g\left(s,\underline{x}^{\alpha}_s(W_s),\bar{x}^{\alpha}_s(W_s)\right)dW_s.$
Now:
$E\left|\int_0^t\left[f\left(s,\underline{x}^{\alpha}_s(W_s),\bar{x}^{\alpha}_s(W_s)\right)-f\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)\right]ds\right|^2+$
$E\left|\int_0^t\left[g\left(s,\underline{x}^{\alpha}_s(W_s),\bar{x}^{\alpha}_s(W_s)\right)-g\left(s,\underline{x}^{\alpha}_{m_s}(W_s),\bar{x}^{\alpha}_{m_s}(W_s)\right)\right]ds\right|^2 \leq 2K_*^2tE\int_0^t\left(\left|\underline{x}^{\alpha}_s(W_s)-\right.\right.$
$\left.\left.\underline{x}^{\alpha}_{m_s}(W_s)\right|^2+\left|\bar{x}^{\alpha}_s(W_s)-\bar{x}^{\alpha}_{m_s}(W_s)\right|^2\right)ds+2K_*^2tE\int_0^t\left(\left|\underline{x}^{\alpha}_s(W_s)-\underline{x}^{\alpha}_{m_s}(W_s)\right|^2+\left|\bar{x}^{\alpha}_s(W_s)-\right.\right.$
$\left.\left.\bar{x}^{\alpha}_{m_s}(W_s)\right|^2\right)ds,$

and taking $m \longrightarrow \infty$ and using Fatou's lemma [22], we conclude that:
$E\left|\underline{x}_t^\alpha(W_t)\right|^2 \leqslant C\left(1+E\left|\underline{x}_0^\alpha\right|^2\right)$.
Thus $\underline{x}_t^\alpha(W_t)$ is a solution of the integral equations (6) and hence it is asolution of differential equations (4). □

Theorem 4 (Uniqueness Theorem). Under the same hypotheses of Theorem 3, there exists a unique solution of Eqs. (4) and (6).
Proof. Suppose that $\underline{x}_{1_t}^\alpha(W_t)$ and $\underline{x}_{2_t}^\alpha(W_t)$ are any two solutions which belonging to $\mu_\omega^2[0,T]$ of Eqs.(4) and (6). Hence:
$\underline{x}_{1_t}^\alpha(W_t)=\underline{x}_0^\alpha+\int_0^t f\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)ds+\int_0^t g\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)dW_s$,
$\underline{x}_{2_t}^\alpha(W_t)=\underline{x}_0^\alpha+\int_0^t f\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)ds+\int_0^t g\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)dW_s$.
Therefore,
$\left|\underline{x}_{1_t}^\alpha(W_t)-\underline{x}_{2_t}^\alpha(W_t)\right|=\left|\int_0^t\left[f\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)-f\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)\right]ds+\right.$
$\left.\int_0^t\left[g\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)-g\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)\right]dW_s\right|$.
Taking the expectation and using inequalities (10), we get:
$E\left|\underline{x}_{1_t}^\alpha(W_t)-\underline{x}_{2_t}^\alpha(W_t)\right|=E\left|\int_0^t\left[f\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)-f\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)\right]ds\right|+$
$E\left|\int_0^t\left[g\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)-g\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)\right]dW_s\right|$.
Hence,
$E\left|\underline{x}_{1_t}^\alpha(W_t)-\underline{x}_{2_t}^\alpha(W_t)\right|^2\leq 2E\left|\int_0^t\left[f\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)-f\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)\right]ds\right|^2$
$+2E\left|\int_0^t\left[g\left(s,\underline{x}_{1_s}^\alpha(W_s),\bar{x}_{1_s}^\alpha(W_s)\right)-g\left(s,\underline{x}_{2_s}^\alpha(W_s),\bar{x}_{2_s}^\alpha(W_s)\right)\right]dW_s\right|^2$
$\leq 2K_*^2t\int_0^t E\left[\left|\underline{x}_{1_s}^\alpha(W_s)-\underline{x}_{2_s}^\alpha(W_s)\right|^2+\left|\bar{x}_{1_s}^\alpha(W_s)-\bar{x}_{2_s}^\alpha(W_s)\right|^2\right]ds+2K_*^2t\int_0^t E\left[\left|\underline{x}_{1_s}^\alpha(W_s)-\right.\right.$
$\left.\left.\underline{x}_{2_s}^\alpha(W_s)\right|^2+\left|\bar{x}_{1_s}^\alpha(W_s)-\bar{x}_{2_s}^\alpha(W_s)\right|^2\right]ds$
$\leq(2K_*^2t+2K_*^2t)\int_0^t E\left[\left|\underline{x}_{1_s}^\alpha(W_s)-\underline{x}_{2_s}^\alpha(W_s)\right|^2+\left|\bar{x}_{1_s}^\alpha(W_s)-\bar{x}_{2_s}^\alpha(W_s)\right|^2\right]ds$.
and using Gronwall's inequality, thus the function:
$\underline{\varphi}_t^\alpha(W_t)=E\left[\left|\underline{x}_{1_t}^\alpha(W_t)-\underline{x}_{2_t}^\alpha(W_t)\right|^2+\left|\bar{x}_{1_t}^\alpha(W_t)-\bar{x}_{2_t}^\alpha(W_t)\right|^2\right]$,
will satisfies:
$\underline{\varphi}_t^\alpha(W_t)\leq(2K_*^2t+2K_*^2t)\left(\int_0^t\underline{\varphi}_s^\alpha(W_s)ds\right)$,
i.e.,
$\underline{\varphi}_t^\alpha(W_t)\leq C\left(\int_0^t\underline{\varphi}_s^\alpha(W_s)ds\right)$,
and hence
$\underline{\varphi}_0^\alpha(W_t)\leq C\left(\int_0^0\underline{\varphi}_s^\alpha(W_s)ds\right)=0$,
which means that $\underline{\varphi}_0^\alpha(W_t)=0$, for every Brownian motion W_t.
Therefore, $\underline{\varphi}_t^\alpha(W_t)=0$, for all $t\in[0,T]$ and every Brownian motion W_t and so:
$E\left[\left|\underline{x}_{1_t}^\alpha(W_t)-\underline{x}_{2_t}^\alpha(W_t)\right|^2+\left|\bar{x}_{1_t}^\alpha(W_t)-\bar{x}_{2_t}^\alpha(W_t)\right|^2\right]=0$,
i.e., $\underline{x}_{1_t}^\alpha(W_t)=\underline{x}_{2_t}^\alpha(W_t)$ and $\bar{x}_{1_t}^\alpha(W_t)=\bar{x}_{2_t}^\alpha(W_t)$, for every Brownian motion W_t and for all $t\in[0,T]$, i.e., the solution of Eqs.(4) and (6) is unique. ■

4. Conclusions

Among the most important tasks in solving fuzzy stochastic ordinary differential equations, is the need to investigate before that the existence and uniqueness theorems for the solutions obtained. Also, since the main difficulty in studying differential equations with fuzzy logic is how to deal with such equations, which contains uncertainity in their nature. Thus this uncertinity or vaguness may be overcome by using the concept of α-level sets, which proved to be effective and reliable in studying the existence and uniqueness theorems of fuzzy stochastic ordinary differential equations, because it will transform the differential equation into crisp space and using the mean square convergence of the Picard successive approximations to the related stochastic integral equation, where the critical discussions regarding these theorems for the solutions of fuzzy stochastic ordinary differential equations are given in Theorems 3 and 4, respectively.

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