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A New Investigation into a Linear Operator Connected to Gaussian Hypergeometric Functions

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Abstract.

In this work we introduce new subclasses of meromorphic functions in punctured unit disk and analyzes numerous connections and various features of these subclasses by making use of the linear operator that is connected to Gaussian hypergeometric functions.

Keywords: Meromorphic functions; Gaussian hypergeometric functions; Hadamard product; Hurwitz-Lerch-Zeta function; Linear operator.
AMS Subject Classification: 30C45, 30C50

دراسة جديدة لمؤثر خطي مرتبط بدوال كاسين فوق الهندسية

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الخلاصة:

في هذه الدراسة قدمنا اصناف جزئية جديدة من الدوال $\{1 > |\mathfrak{I}| > 0 : \mathfrak{I}\} = \mathfrak{U}^*$ وتم مناقشة العديد من المرومورفيك في قرص الوحدة المثقوب
الخواص للدوال التي تنتمي لهذا الصنف المعرف باستخدام مؤثر خطي بدلالة دوال كاسين فوق الهندسية.

1. Introduction

The investigation of specific families of regular or meromorphic, univalent functions in specified domains that may be simply or multiply linked, as well as extreme problems for their coefficients in power series expansions, function values, and derivatives, are the main components of today's research on univalent functions. These issues are often closely related to, and in many instances have been caused by, issues with conformal mapping. Numerous more unique subclasses of univalent/multivalent functions have been taken into consideration throughout the years. Their formulations often immediately point to certain analytic circumstances from which the relevant features may be inferred with relative ease [1]. On the other side, the study of geometric function theory, which forms the basis of the theory of univalent functions, includes consideration of the theory of hypergeometric functions. After

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being used by Branges [2] in the verification of Bieberbach's conjecture, a significant issue in geometric function theory, it has attracted more attention as a study subject that is active and relevant to current work. Numerous applications and generalizations have been made to this theory by well-known complex analyzers, who have also improved and enhanced it [3]. Several Gaussian hypergeometric functions were explored for their starlikeness in 1961 [4], while in 1984 [5], constructed results for both starlike and prestarlike functions were obtained using a convolution operator with an incomplete beta function.

Assume that Λ comprises the collection of all analytic, convex, and univalent functions in $h(\Im)$ in the open unit disk $\mathfrak{U} = \{\Im : |\Im| < 1\}$ satisfying $h(0) = 1$ and

$$\Re\{h(\Im)\} > 0, |\Im| < 1. \tag{1.1}$$

Consider that there are two functions $f, g \in \Lambda$ where f is *subordinate* to g or g is *superordinate* to f in $\mathfrak{U}$, so $f \prec g, \Im \in \mathfrak{U}$, if there exists a Schwarz function ω, analytic in $\mathfrak{U}$ with $\omega(0) = 0$ and $|\omega(\Im)| \leq 1$ when $\Im \in \mathfrak{U}$ such that $f(\Im) = g(\omega(\Im)), \Im \in \mathfrak{U}$. The following equivalence will be obtained if the function g is univalent in $\mathfrak{U}$:

$$f(\Im) \prec g(\Im) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathfrak{U}) \subset g(\mathfrak{U}), \qquad (\Im \in \mathfrak{U}).$$

In this study, we are going to get some requirements that are essential for a normalized analytic function. A function $f(\Im)$ which defining by the linear operator and Hurwitz-Lerch-Zeta in $\mathfrak{U}^* = \{z : 0 < |z| < 1\}$. The objective of this article is to demonstrate various features for the complex operator that meet particular subordination findings.

2. Preliminaries

Let Σ represent the category of meromorphic functions as defined below in the complex plane $\mathbb{C}$:

$$f(\Im) = \frac{1}{\Im} + \sum_{n=1}^{\infty} a_n \Im^n; \qquad (\Im \in \mathfrak{U}^*). \tag{2.1}$$

For $\mathfrak{P} \geq 0$, let

$\Sigma S^*(\mathfrak{P}) = \{f \in \Sigma : f \text{ is starlike of order } \mathfrak{P} \text{ in } \mathfrak{U}^*\}$.
And
$\Sigma K(\mathfrak{P}) = \{f \in \Sigma : f \text{ is convex of order } \mathfrak{P} \text{ in } \mathfrak{U}^*\}$.

For more information, see, for example, [6 -11].

Convolution, often known as the Hadamard product, of the functions

$$f_j(\Im) = \frac{1}{\Im} + \sum_{n=1}^{\infty} a_{n,j} \Im^n; \quad j = 1,2. \tag{2.2}$$

Which is defined by

$$(f_1 * f_2)(\Im) = \frac{1}{\Im} + \sum_{n=1}^{\infty} a_{n,1}\, a_{n,2} \Im^n. \tag{2.3}$$

The generic Pochhammer symbol for $t \in \mathbb{C}$ is denoted by the notation, which is defined as

$$(t)_k \coloneqq \frac{\Gamma(t+k)}{\Gamma(t)} = \begin{cases} t(t+1)\ldots(t+n-1), & k = n \in \mathbb{N}; t \in \mathbb{C} \\ 1, & k = 0;\ t \in \mathbb{C}. \end{cases} \tag{2.4}$$

For $\mathfrak{X} \in \mathbb{C}$ and $\mathfrak{P} \in \mathbb{C}\backslash\{0, -1, -2, ...\}$, $\widetilde{\Phi}(\mathfrak{X}, \mathfrak{P}: \mathfrak{I})$ will refer to the following function

$$\widetilde{\Phi}(\mathfrak{X}, \mathfrak{P}: \mathfrak{I}) \coloneqq \sum_{n=0}^{\infty} \frac{(\mathfrak{X})_{n+1}}{(\mathfrak{P})_{n+1}} a_n \mathfrak{I}^n. \tag{2.5}$$

Let $\Phi(\mathfrak{I}, s, a)$ stand for the Hurwitz-Lerch zeta function, which may be expressed as

$$\Phi(\mathfrak{I}, s, a) \coloneqq \sum_{n=0}^{\infty} \frac{\mathfrak{I}^n}{(n+a)^s} \tag{2.6}$$

where

$a \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $s \in \mathbb{C}$ when $|\mathfrak{I}| = 1$

and

$R(s) > 1$ when $|\mathfrak{I}| = 1.$

For the definition and certain features of this zeta function, we direct the reader to [12 -17].

Using the Hadamard product, the authors [14], [18], [19], and [20] have studied a linear operator denoted by $L_a^t(\mathfrak{X}, \mathfrak{P})$ on Σ, as shown below:

$$\begin{aligned} L_a^t(\mathfrak{X}, \mathfrak{P})(f)(\mathfrak{I}) &= \widetilde{\Phi}(\mathfrak{X}, \mathfrak{P}: \mathfrak{I}) * G_{t,a}(\mathfrak{I}) \\ &= \frac{1}{\mathfrak{I}} + \sum_{n=1}^{\infty} \frac{(\mathfrak{X})_{n+1}}{(\mathfrak{P})_{n+1}} \left(\frac{a+1}{a+n}\right)^t a_n \ \mathfrak{I}^n; \qquad (\mathfrak{I} \in \mathfrak{U}^*), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} G_{t,a}(\mathfrak{I}) &\coloneqq (a+1)^t \left[\Phi(\mathfrak{I}, s, a) - a^t + \frac{1}{\mathfrak{I}\,(a+1)^t}\right] \\ &= \frac{1}{\mathfrak{I}} + \sum_{n=1}^{\infty} \left(\frac{a+1}{a+n}\right) \mathfrak{I}^n; \qquad (\mathfrak{I} \in \mathfrak{U}^*). \end{aligned} \tag{2.8}$$

It follows from (2.7) that

$$\mathfrak{I}\left(L_a^t(\mathfrak{X}, \mathfrak{P}) f(\mathfrak{I})\right)' = \mathfrak{X} L_a^t(\mathfrak{X}+1, \mathfrak{P}) f(\mathfrak{I}) - (\mathfrak{X}+1) L_a^t(\mathfrak{X}, \mathfrak{P}) f(\mathfrak{I}). \tag{2.9}$$

Suppose Ω presents the class of analytic functions $h(\mathfrak{I})$ with $h(0) = 1$, where it's convex and univalent in $\mathfrak{U} = \mathfrak{U}^* \cup \{0\}$.

Definition 2.1. If a function $f \in \Sigma$ meets the following subordination condition:

$$(1+p)\mathfrak{I}\, L_a^t(\mathfrak{X}, \mathfrak{P}) f(\mathfrak{I}) + p\, \mathfrak{I}^2 \left(L_a^t(\mathfrak{X}, \mathfrak{P}) f(\mathfrak{I})\right)' \prec h(\mathfrak{I}) \tag{2.10}$$

it is referred to as belonging to the class $\sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p; h)$, where $h(\mathfrak{I}) \in \Omega$ and p is a complex number.

Suppose we have A present a class of functions as follows:

$$f(\mathfrak{I}) = \mathfrak{I} + \sum_{n=2}^{\infty} a_n \mathfrak{I}^n. \tag{2.11}$$

If the following conditions are met, we can assert that the function $h(\mathfrak{I}) \in A$ belongs to the class $S^*(\mathfrak{X})$,

$$\Re\left\{\frac{\Im f'(\Im)}{f(\Im)}\right\} > a, \quad (\Im \in \mathfrak{U}) \tag{2.12}$$

$$\frac{\Im}{(1-\Im)^{2(1-a)}} * f(\Im) \in S^*(\mathfrak{X}) \qquad (a > 1),$$

where * represents the Hadamard product of two analytical functions in $\mathfrak{U}$. $R(a)$ presents this course; for further information, check [21], [22]. A function $f(\Im) \in A$ belongs to the class $R(0)$ if and only if it is convex and univalent in $\mathfrak{U}$ and

$$\Re\left(\frac{1}{2}\right) = S^*\left(\frac{1}{2}\right).$$

This is the essential condition for membership in the class.

3. Main Results

We shall be able to demonstrate our findings by using the following lemmas:

Lemma 3.1. [23]*:* Suppose that $g(\Im)$ is analytic in $\mathfrak{U}$, and that $h(\Im)$ is analytical, convex, and univalent in $\mathfrak{U}$ with $h(0) = g(0)$. If,

$$g(\Im) + \frac{1}{m}\Im\, g'(\Im) \prec h(\Im) \tag{3.1}$$

$m \neq 0$ and $\Re m \geq 0$, we gain

$$g(\Im) \prec \tilde{h}(\Im) = m\,\Im^{-m}\int_0^{\Im} t^{m-1}h(t)dt \prec h(\Im),$$

where $\tilde{h}(\Im)$ is the best dominant of (3.1).

Lemma 3.2. [21]*:* Let $a < 1, f(\Im) \in S^*(a), g(\Im) \in R(a)$, and for any analytic function $F(\Im)$ in $\mathfrak{U}$, then

$$\frac{g*(fF)}{g*f}(\mathfrak{U}) \subset \overline{co}(F(\mathfrak{U})),$$

$co(F(\mathfrak{U}))$ denotes the convex hull of $F(\mathfrak{U})$.

Theorem 3.1. If $a \neq 0$ and $\Re a \geq 0$, then

$$\textstyle\sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,\tilde{h}),$$

where

$$\tilde{h}(\Im) = \mathfrak{X}\Im^{-\alpha}\int_0^{\Im} t^{\alpha-1}h(t)dt \prec h(\Im).$$

Proof. If

$$g(\Im) = (1+p)\Im(L_a^t(\mathfrak{X},\mathfrak{P})f(\Im)) + p\Im^2(L_a^t(\mathfrak{X},\mathfrak{P})f(\Im))' \tag{3.2}$$

For $f(\Im) \in \Sigma$. The results of (2.9) and (3.2) are:

$$\frac{g(\Im)}{\Im} = \mathfrak{X}p(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\Im)) + (1-\mathfrak{X}p)(L_a^t(\mathfrak{X},\mathfrak{P})f(\Im). \tag{3.3}$$

The following is the result of differentiating both sides of (3.3) and applying (2.9):

$$g'(\Im) - \frac{g(\Im)}{\Im} = \mathfrak{X}p\Im(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\Im))' + (1-\mathfrak{X}p)[\mathfrak{X}(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\Im)) -(1+\mathfrak{X})(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\Im))]. \tag{3.4}$$

By (3.3) and (3.4), we get:

$$g'(\Im) - \frac{\mathfrak{X}g(\Im)}{\Im} = \mathfrak{X}p\Im(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\Im))' + \mathfrak{X}(1+p)(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\Im))$$

so,

$$g(\mathfrak{I}) + \frac{\mathfrak{I}g'(\mathfrak{I})}{\mathfrak{X}} = (1+p)\mathfrak{I}\left(L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\mathfrak{I})\right) + p\mathfrak{I}^2 L_a^t(\mathfrak{X}+1,\mathfrak{P})f(\mathfrak{I})'. \quad (3.5)$$

If $f \in \sum_{\mathfrak{X}+1,\mathfrak{P}}^{a,t}(p;h)$, then the following may be deduced from the above equation (3.5):

$$g(\mathfrak{I}) + \frac{\mathfrak{I}g'(\mathfrak{I})}{\mathfrak{X}} \prec h(\mathfrak{I}) \qquad (\Re\mathfrak{X} \geq 0, \mathfrak{X} \neq 0).$$

As a result of Lemma 3.1, we have the following:

$$g(\mathfrak{I}) \prec \tilde{h}(\mathfrak{I}) = \mathfrak{X}\mathfrak{I}^{-\alpha}\int_0^{\mathfrak{I}} t^{\alpha-1}\, h(t)dt \prec h(\mathfrak{I})$$

and so that

$f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,\tilde{h})$.

Theorem 3.2. Let $p > 0, \delta > 0$ and $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;\delta h + 1 - \delta)$. If $\leq \delta_0$, where

$$\delta_0 = \frac{1}{2}\left(1 - \frac{1}{p}\int_0^1 \frac{\mathfrak{U}^{\frac{1}{p}-1}}{1+\mathfrak{U}}d\mathfrak{U}\right)^{-1}, \quad (3.6)$$

then $f(\mathfrak{I}) \in \sum_{\mathfrak{X}+1,\mathfrak{P}}^{a,t}(p;h) \subset \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;\tilde{h})$.

Proof. Consider the following:

$$g(\mathfrak{I}) = \mathfrak{I}\, L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I}) \quad (3.7)$$

for $f(\mathfrak{I}) \in \sum_{\mathfrak{X}+1,\mathfrak{P}}^{a,t}(p;h) \subset \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;\tilde{h})$ with $p>0$, and $\delta>0$, then we have

$$g(\mathfrak{I}) + p\mathfrak{I}g'(\mathfrak{I}) = (1+p)\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})) + p\mathfrak{I}^2(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I}))' \prec \delta(h(\mathfrak{I}) - 1 + 1).$$

The following is the outcome of Lemma 3.1:

$$g(\mathfrak{I}) \prec \frac{\delta}{p}\mathfrak{I}^{-\frac{1}{p}}\int_0^{\mathfrak{I}} t^{\frac{1}{p}-1}h(t)dt + 1 - \delta = (h * \Psi)(\mathfrak{I}), \quad (3.8)$$

where

$$\Psi(\mathfrak{I}) \prec \frac{\delta}{p}\mathfrak{I}^{-\frac{1}{p}}\int_0^{\mathfrak{I}} \frac{t^{\frac{1}{p}-1}}{1-t}dt + 1 - \delta. \quad (3.9)$$

If $0 < \delta \leq \delta_0$, where $\delta_0 > 1$ is determined by (3.6), it is evident from (3.9) that:

$$\Re\Psi(\mathfrak{I}) = \frac{\delta}{p}\int_0^1 u^{\frac{1}{p}-1}\Re\left(\frac{1}{1-u\mathfrak{I}}\right)dt + 1 - \delta > \frac{\delta}{p}\int_0^1 \frac{u^{\frac{1}{p}-1}}{1+u}du + 1 - \delta \geq \frac{1}{2}, (\mathfrak{I} \in \mathfrak{U}).$$

Using the Herglotz representation for $\Psi(\mathfrak{I})$, we can derive the following from (3.7) and (3.8):

$$\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})) \prec (h * \Psi)(\mathfrak{I}) \prec h(\mathfrak{I})$$

owing to the fact that $h(z)$ is convex univalent in $\mathfrak{U}$. Thus, $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;h)$. For $\hbar(\mathfrak{I}) = \frac{1}{1-\mathfrak{I}}$ and $f(\mathfrak{I}) \in \Sigma$ defined by

$$\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})) = \frac{\delta}{p}\mathfrak{I}^{-\frac{1}{p}}\int_0^{\mathfrak{I}} \frac{t^{\frac{1}{p}-1}}{1-t}dt + 1 - \delta.$$

It is simple to demonstrate:

$$(1+p)\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})) + p\mathfrak{I}^2(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I}))' = \delta(h(\mathfrak{I}) - 1) + 1.$$

Thus, $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;\delta h + 1 - \delta)$. Also, for $\delta > \delta_0$, we have:

$$\Re\mathfrak{I}(L_{\mathfrak{X}}^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I}) \to \frac{\delta}{p}\int_0^1 \frac{\mathfrak{U}^{\frac{1}{p}-1}}{1+\mathfrak{U}} d\mathfrak{U} + 1 - \delta < \frac{1}{2}, (\mathfrak{I} \to -1).$$

This indicates that $f(\mathfrak{I}) \notin \sum_{\alpha,\beta}^{\alpha,t}(p;h)$.

4. Convolution Properties

Theorem 4.1. Let $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;h)$, $g(\mathfrak{I}) \in \Sigma$ and

$$\Re\big(\mathfrak{I}g(\mathfrak{I})\big) > \frac{1}{2} \qquad (\mathfrak{I} \in \mathfrak{U}) \tag{4.1}$$

then,

$(f * g)(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$.

Proof. For $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;h)$ and $g(\mathfrak{I}) \in \Sigma$, we have:

$$\begin{aligned}(1+p)\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})(f*g)(\mathfrak{I})) + p\mathfrak{I}^2(L_a^t(\mathfrak{X},\mathfrak{P})(f*g)(\mathfrak{I}))' &= (1+p)\mathfrak{I}g(\mathfrak{I}) * \mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})) + p\mathfrak{I}\, g(\mathfrak{I}) \\ &\quad * \mathfrak{I}^2\big(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})\big)' \\ &= \mathfrak{I}g(\mathfrak{I}) \\ &\quad * \Psi(\mathfrak{I}),\end{aligned} \tag{4.2}$$

where

$$\Psi(\mathfrak{I}) = (1+p)\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I})) + p\mathfrak{I}^2(L_a^t(\mathfrak{X},\mathfrak{P})f(\mathfrak{I}))' \prec \hbar(\mathfrak{I}). \tag{4.3}$$

Given (4.1), the function $\mathfrak{I}g(\mathfrak{I})$ has the Herglotz representation:

$$\mathfrak{I}g(\mathfrak{I}) = \int_{|x|=1} \frac{dm(x)}{1-x\mathfrak{I}}, \qquad (\mathfrak{I} \in \mathfrak{U}) \tag{4.4}$$

$m(x)$ is a probability measure that is defined on the unit circle $|x| = 1$ and $\int_{|x|=1} dm(x) = 1$.

It is evident from (4.2), (4.3), and (4.4) that $h(\mathfrak{I})$ is a convex univalent in and that:

$$(1+p)\mathfrak{I}\big(L_a^t(\mathfrak{X},\mathfrak{P})(f*g)(\mathfrak{I})\big) + p\mathfrak{I}^2(L_a^t(\mathfrak{X},\mathfrak{P})(f*g)(\mathfrak{I}))' = \int_{|x|=1} \Psi(x\mathfrak{I})dm(x) \prec \hbar(\mathfrak{I}).$$

The theorem is therefore proved by the fact that $(f*g)(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;h)$.

Corollary 4.1. If we assume that $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$ is determined by (2.1) and

$$\omega_m(\mathfrak{I}) = \frac{1}{\mathfrak{I}} + \sum_{n=1}^{m-1} a_n \mathfrak{I}^{n-1}, (m \in N\backslash\{1\})$$

then we can see that the function

$$\sigma_m(\mathfrak{I}) = \int_0^1 t\omega_m(t\mathfrak{I})dt$$

belongs to the class $\sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;h)$.

Proof. Let

$$\sigma_m(\mathfrak{I}) = \frac{1}{\mathfrak{I}} + \sum_{n=1}^{m-1} \frac{a_n}{n+1} \mathfrak{I}^{n-1} = (f * g_m)(\mathfrak{I}), \quad (m \in N\backslash\{1\}), \qquad (4.5)$$

where

$$f(\mathfrak{I}) = \frac{1}{\mathfrak{I}} + \sum_{n=1}^{\infty} a_n \mathfrak{I}^{n-1} \in \sum_{\alpha,\beta}^{\alpha,t}(p,h)$$

and

$$g_m(\mathfrak{I}) = \frac{1}{\mathfrak{I}} + \sum_{n=1}^{m-1} \frac{\mathfrak{I}^{n-1}}{n+1} \in \Sigma.$$

and with regard to $m \in N\backslash\{1\}$, we can conclude from [24] that:

$$\Re\{\mathfrak{I} g_m(\mathfrak{I})\} = \Re\left\{1 + \sum_{n=1}^{m-1} \frac{\mathfrak{I}^n}{n+1}\right\} > \frac{1}{2} \quad (\mathfrak{I} \in \mathfrak{U}). \qquad (4.6)$$

An application of Theorem 4.1 and based on equations (4.5) and (4.6) leads to $\sigma_m(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p;h)$.

Theorem 4.2. Let $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$, $g(\mathfrak{I}) \in \Sigma$ and $\mathfrak{I}^2 g(\mathfrak{I}) \in R(a), (a < 1)$, then $(f * g)(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$.

Proof. For $(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$, $g(\mathfrak{I}) \in \Sigma$, from (4.2), we get

$$(1+p)\mathfrak{I}(L_a^t(\mathfrak{X},\mathfrak{P})(f*g)(\mathfrak{I})) + p\mathfrak{I}^2(L_a^t(\mathfrak{X},\mathfrak{P})(f*g)(\mathfrak{I}))' = \frac{\mathfrak{I}^2 g(\mathfrak{I}) * \mathfrak{I}\Psi(\mathfrak{I})}{\mathfrak{I}^2 g(\mathfrak{I}) * \mathfrak{I}}, (\mathfrak{I} \in \mathfrak{U}) \qquad (4.7)$$

where $\Psi(\mathfrak{I})$ refers to the value specified in (4.3). As $h(\mathfrak{I})$ is convex univalent in $\mathfrak{U}, \Psi(\mathfrak{I}) \prec h(\mathfrak{I})$, $\mathfrak{I}^2 g(\mathfrak{I}) \in R(a)$ and that $\mathfrak{I} \in S^*(a), (a < 1)$, consequently, we are able to get the solution by using equation (4.7) and Lemma 3.2.

The following corollary can be derived from Theorem 4.2 if the values $a = 0$ and $a = \frac{1}{2}$, are selected:

Corollary 4.2. Suppose we have $f(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$ and fulfill any of the following conditions:

i. $\mathfrak{I}^2 g(\mathfrak{I})$ is convex univalent in $\mathfrak{U}$ or,

ii. $\mathfrak{I}^2 g(\mathfrak{I}) \in S^*(\frac{1}{2})$ then,

$(f * g)(\mathfrak{I}) \in \sum_{\mathfrak{X},\mathfrak{P}}^{a,t}(p,h)$.

5. Conclusions

The goal of this study was to look at the geometric properties of a subclass of meromorphic functions in terms of a complex linear operator, including those related to Gaussian hypergeometric functions. Based on the convolution principle and the proposed operator, we were able to investigate the interesting characteristics of this subclass. Our results can potentially assist in further developing these ideas and their applications.

In summary, this research provides an important improvement in our knowledge of the geometric features of a subclass of meromorphic functions and their relationship to other special functions. We hope that our findings will motivate more study and investigation in this area.

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