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P-Rational Submodules

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Abstract

A submodule N is called rational in M if $Hom_R(\frac{M}{N}, E(M))=0$, where E(M) is the injective hull of M. Rational submodules have been studied and discussed by many authors such as H.H. Storrer, H. Khabazian, E. Ghashghaei, A. Hajikarimi and A.R. Naghipour, M.S. Abbas and M.A. Ahmed. The main objective of this paper is to give a new class of submodules named P-rational submodules. This class is contained properly in the class of rational submodules. Several properties of this concept are introduced. The relationships between this class of submodules and some other related concepts are discussed such as essential and quasi-invertible submodules. Other characterizations of the P-rational submodule analogous to those which is known in the concept of the rational submodule are given.

Keywords: Rational submodules, Pure submodules, P-rational submodules, Essential submodules

المقاسات النسبية من النمط –P

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الخلاصة

يُقال للمقاس الجزئي N بأنه نسبي اذا كان $Hom_R(\frac{M}{N}, E(M)) = 0$. تم دراسة المقاسات الجزئية النسبية ومناقشتها من قبل العديد من الباحثين مثلاً:

H.H. Storrer, E. Ghashghaei, H. Khabazian, A.R. Naghipour & A. Hajikarimi، and M.S. Abbas & M.A.Ahmed

ان الهدف الرئيس من هذا البحث هو إعطاء نوع جديد من المقاسات يُدعى بالمقاسات الجزئية النسبيية من النمط –P. ان هذا الصنف من المقاسات الجزئية محتوى بشكل فعلي في المقاسات الجزئية النسبية. العديد من الخصائص المهمة قُدمت حول المقاسات الجزئية النسبية من النمط –P. كما نوقشت علاقة هذا الصنف من المقاسات الجزئية مع عدد من المقاسات الأخرى. كما تم إعطاء عدد من التشخيصات للمقاسات الجزئية النسبية من النمط –P مناظرة للتشخيصات المعروفة الخاصة بالمقاسات الجزئية النسبية.

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1. Introduction:

Rational submodules played a large and important role in the module theory. Many authors studied the concept of rational submodules such as H.H. Storrer, H. Khabazian, E. Ghashghaei, A. Hajikarimi & A.R. Naghipour, M.S. Abbas & M.A. Ahmed. Other features of rational submodules have also been explored. Furthermore, those researchers found new types containing or contained in the class of rational submodules. In addition to that, the rationality property has appeared as an additional condition with other concepts to introduce other new concepts. However, there are few findings on the concept of rational submodules compared to its important role in module theory.

An R-module M is called injective if for every monomorphism f: M ⟶B and every homomorphism g: M ⟶C there exists a homomorphism h: B ⟶C with g=h∘f, [1, P.116]. A non-zero submodule N of M is said to be essential (briefly $N\leq_e M$) if N∩L≠0 for every non-zero submodule L of M, [2, P.15]. An injective hull of any R-module M is denoted by E(M), and it is defined as a monomorphism f: M⟶E(M) such that E(M) is an injective module and f is an essential monomorphism, [1, P.124]. A submodule N of an R-module M is called rational (simply, $N\leq_r M$), if $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, [3, P.274].

Our main goal is to introduce a class of submodules which contained properly in the class of rational submodules, which we named the P-rational submodule. A submodule N of M is called pure if N∩IM=IN for every ideal I of R, [4, P.18]. We define the P-rational submodule as a pure submodule N which satisfies $\mathrm{Hom}_R(\frac{M}{N}, E(M)) =0$.

This paper is divided into three sections. In Section Two, we study P-rational submodules. Various characteristics of the P-rational submodule are given and discussed that are analogous to the results which are known in the concept of the rational submodule. Among these results:

- Let $A\leq_{pr}B\leq C$ with B having PIP. If $A_1 \leq_{pr} B_1\leq C$ then $A\cap A_1 \leq_{pr} B\cap B_1$, see Propositions 2.8.
- For any chain of modules, A≤ B≤ C with B is a pure submodule of C, $A\leq_{pr}C$ if and only if $A\leq_{pr}B$ and $B\leq_{pr}C$, see Theorem 2.11.
- Let L be a non-zero pure submodule of an R-module M. If for any 0≠m∈M, $\mathrm{ann}_R(\frac{M}{L}) \nsubseteq \mathrm{ann}_R(m)$, then $L\leq_{pr}M$, see Proposition 2.21.

In addition, we present several characterizations for this type of submodule such as the following:

- Let N be a pure submodule of an R-module M. Then $N\leq_{pr}M$ if and only if for all submodules L of M with N≤L≤M, $\mathrm{Hom}_R(\frac{L}{N}, M)=0$, see Theorem 2.7.
- Let N be a pure submodule of M. Then the following statements are equivalent.

  i. $\mathrm{Hom}_R(\frac{L}{N}, M)=0$ for each submodule L of M with N⊆L⊆M.

  ii. For all y∈M and x∈M\{0} there exists r∈R such that rx≠0 and ry∈N.

  See Theorem 2.13.
- For any R-module M, with N is a pure submodule of M, the following statements are equivalent:

i. For all y∈M and x∈M\{0}, there exists r∈R such that rx≠0 and ry∈N.

ii. $N \leq_{pr}M$.

iii. For any submodule P of M with N⊆P⊆M, $\mathrm{Hom}_R(\frac{P}{N}, M)=0$.

See Theorem 2.16.

In Section Three we study the relationships between this class of modules and other related concepts, among these results are the following:

- Let N be a submodule of an R-module M. Consider the following statements.

a. $N \leq_{pr} M$.
b. $N \leq_{r} M$.
c. $N \leq_{e} M$.

Then (a) ⇒ (b) ⇒ (c) and if M is a nonsingular module and N pure then (c) ⇒ (a).
See Proposition 3.4.

- Let M be a multiplication module with a prime annihilator. Consider the following statements.

i. $N \leq_{pr} M$.
ii. $N \leq_{pqu} M$.
iii. $N \leq_{qu} M$.
iv. $N \leq_{e} M$.
v. $N \leq_{pe} M$.

Then (i) ⇒ (ii) ⇒ (iii) ⇔ (iv) ⇒(v), and if M is a fully P-essential module then (v) ⇒ (iv).
See Proposition 3.7.

It is worth remembering that all rings R in this paper are commutative with identity, and all modules are unitary left R-modules.

2. P-Rational Submodules

This section is devoted to examining a new concept that we call the P-rational submodules. It is a special class of rational submodules. The comparison of the results according to the results in our new concept has been found which are analogues of the properties which satisfied in rational submodules.

Definition 2.1: A submodule N of an R-module M is said to be P-rational if N is pure and $\mathrm{Hom}_R(\frac{M}{N}, E(M))=0$, where E(M) is the injective hull of M. It denoted by $N \leq_{pr} M$. An ideal I of a ring R is said to be P-rational if I is a P-rational R-submodule.

Remark 2.2: It is clear that every P-rational submodule is rational, but the converse is not correct, as the following examples show:

1. Consider the ring of rational numbers $\mathbb{Q}$, and let $\mathbb{Q}$(x,y) be the polynomial ring of two independent variables x and y. Since $\mathbb{Q}$ is a field then $\mathbb{Q}$(x,y) is an integral domain. Let A=<(x,y)> be an ideal of $\mathbb{Q}$(x,y) generated by (x,y), so that $A=\{xf_1 + yf_2 | f_1, f_2 \in \mathbb{Q}(x,y)\}$, $A \neq 0$. We claim that $A \leq_r \mathbb{Q}(x,y)$. In fact, $\mathrm{Hom}_R\left(\frac{\mathbb{Q}(x,y)}{A}, E\big(\mathbb{Q}(x,y)\big)\right)=\mathrm{Hom}_R\left(\frac{\mathbb{Q}(x,y)}{A}, \mathbb{Q}(x,y)\right)$. Note that, $\mathbb{Q}(x,y)$ is an integral domain. This implies that $\mathrm{Hom}_R\left(\frac{\mathbb{Q}(x,y)}{A}, \mathbb{Q}(x,y)\right)=0$, [5, Example 1.3 (1), P.6], that is $A \leq_r \mathbb{Q}(x,y)$. In contrast, A is not P-rational in $\mathbb{Q}(x,y)$, to show that: let $B=\{f \in \mathbb{Q}(x,y) | f(x,y)=a$, where $x,y \in \mathbb{Q}$ and $a \in 2\mathbb{Z}\}$, then $AB=\{axf_1+ayf_2 | f_1,f_2 \in \mathbb{Q}(x,y)\}$. It is clear that $AB \neq (0)$, but $A \cap B\mathbb{Q}=0$. Therefore, A is not pure ideal of $\mathbb{Q}(x,y)$, hence A is not P-rational ideal in $\mathbb{Q}(x,y)$.

2. Consider the $\mathbb{Z}$-module $\mathbb{Q}$, where $\mathbb{Q}$ is the set of all rational numbers. We claim that, $\mathbb{Z} \leq_r \mathbb{Q}$. Let L be a submodule of $\mathbb{Q}$ with $\mathbb{Z} \leq L \leq \mathbb{Q}$, we have to show that $\mathrm{Hom_R}\left(\frac{\mathbb{Q}}{\mathbb{Z}}, E(\mathbb{Q})\right)=0$. Note that $E(\mathbb{Q})=\mathbb{Q}$ since $\mathbb{Q}$ is injective. Let f: $\frac{\mathbb{Q}}{\mathbb{Z}} \longrightarrow \mathbb{Q}$ be a homomorphism. Note that $f(\frac{n}{m}+\mathbb{Z})=nf(\frac{1}{m}+\mathbb{Z})$, for all $\frac{n}{m} \in \mathbb{Q}$, m≠0. We are done if we can show that $f(\frac{1}{m}+\mathbb{Z})=0$. Note that $mf(\frac{1}{m}+\mathbb{Z})=f(\frac{m}{m}+\mathbb{Z})=f(\mathbb{Z})=0$. But m and $f(\frac{1}{m}+\mathbb{Z}) \in \mathbb{Q}$, m≠0, thus $f(\frac{1}{m}+\mathbb{Z})=0$, hence $f(\frac{n}{m}+\mathbb{Z})=0$. That is f=0. In contrast, $\mathbb{Z} \nleq_{Pr} \mathbb{Q}$, since $\mathbb{Z}$ is not pure submodule of $\mathbb{Q}$.

Examples and Remarks 2.3:

1. Any non-zero R-module M is a P-rational submodule of itself since M is pure in itself and $\mathrm{Hom_R}\left(\frac{M}{M}, E(M)\right)=0$.

2. If 0≠M, then (0) is not P-rational in M. In fact, (0) is pure in M but $\mathrm{Hom_R}\left(\frac{M}{(0)}, E(M)\right)\neq 0$.

3. N=2$\mathbb{Z}$ is not P-rational in M=$\mathbb{Z}$. In fact, any P-rational submodule must be pure and rational, but N=2$\mathbb{Z}$ is not pure in M=$\mathbb{Z}$. In fact, if I=<6> is an ideal of R=$\mathbb{Z}$ then IN=<6>(2$\mathbb{Z}$)=12$\mathbb{Z}$ and N∩IM=2$\mathbb{Z}$ ∩<6>$\mathbb{Z}$ =6$\mathbb{Z}$, so that IN≠ N∩IM.

4. N=$<\bar{2}>$ is not P-rational in the $\mathbb{Z}$-module M=$\mathbb{Z}_4$, since $<\bar{2}>$ is not pure submodule of $\mathbb{Z}_4$ and $\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_4}{<\bar{2}>}, E(\mathbb{Z}_4)\right)= \mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_4}{<\bar{2}>}, \mathbb{Z}_{2^\infty}\right)$, [6, P.21] and clearly $\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_4}{<\bar{2}>}, \mathbb{Z}_{2^\infty}\right)\neq 0$. Now, to show that N=$<\bar{2}>$ is not pure in $\mathbb{Z}_4$, consider the ideal I=$<2>$ of $\mathbb{Z}$, note that $\bar{2} \in 2\mathbb{Z}_4 \cap <\bar{2}> = <\bar{2}>$ but $\bar{2} \notin 2<\bar{2}>=(\bar{0})$.

5. Every P-rational submodule is essential. This follows from Remark 2.2.

6. A direct summand of any R-module may not be P-rational, for example: let M=$\mathbb{Z}_6$, A=$<\bar{2}>$, since $<\bar{2}>$ is a direct summand of $\mathbb{Z}_6$, then $<\bar{2}>$ is a pure submodule of $\mathbb{Z}_6$. But $<\bar{2}>$ is not P-rational of $\mathbb{Z}_6$ since $\mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}_6}{<\bar{2}>}, E(\mathbb{Z}_6)\right)=\mathrm{Hom}_{\mathbb{Z}}\left(\frac{<\bar{2}>\oplus<\bar{3}>}{<\bar{2}>}, E(\mathbb{Z}_6)\right) = \mathrm{Hom}_{\mathbb{Z}}(<\bar{3}>, E(<\bar{2}>\oplus<\bar{3}>)) = \mathrm{Hom}_R(<\bar{3}>, 2^\infty \oplus 3^\infty)\neq 0$.

7. A direct summand N of any R-module M is P-rational in M if and only if N=M.

Proof: For the necessity, assume that N is a direct summand of M, since $N \leq_{pr} M$, then $N \leq_e M$. But it is known that any module M is equal to any essential and direct summand of M, thus N=M. The sufficiently is straightforward.

An R-module M is called semisimple if every submodule of M is a direct summand of M, [1, P.107].

8. If M is a semisimple module, then M is the only P-rational submodule of M.

Proof: Since every proper submodule N of M is a direct summand, then $\frac{M}{N} \cong L$, where L is a direct summand of M. But L≤M≤E(M), therefore $\mathrm{Hom_R}\left(\frac{M}{N}, E(M)\right) \cong \mathrm{Hom_R}(L, E(M))\neq 0$, so that M is the only P-rational submodule of itself.

Recall that a non-zero module M is called pure simple if the only pure submodules of M are: (0) and M itself, [7].

9. If M is a pure simple R-module, then the only P-rational submodules in M is M itself.

Proof: Let N be a P-rational submodule of M. Since M is a pure simple module, then the only pure submodules are (0) and M itself, if N=(0), then by (2), N is not P-rational. If N=M, then by (1), N is a P-rational submodule.

An R- module M is called F-regular if every submodule of M is pure, [8].

10. Let M be an F-regular module. Then N$\leq_r$M if and only if N$\leq_{pr}$M.
Proof: Assume that N$\leq_r$M. Since M is F-regular, then every submodule of M is pure, thus N$\leq_{pr}$M. The converse is clear.

A ring R is said to be regular if ∀r∈R there exists x∈R such that r=rxr, [2, P.10].

11. Let M be an R-module over a regular ring R. Then N is a rational submodule in M if and only if N is P-rational in M.
Proof: Since R is a regular ring, then M is F-regular R-module, [4, Remark 1.2 (2), P.29]. By (10), the result is obtained. The converse is clear.
12. Homomorphic image of a P-rational submodule may not be P-rational. For example: consider the $\mathbb{Z}$-modules $\mathbb{Z}_4$ and $\mathbb{Z}_6$. Let $f: \mathbb{Z}_6 \longrightarrow \mathbb{Z}_4$ be a homomorphism defined by f($\bar{0}$)=f($\bar{2}$)=f($\bar{4}$)=$\bar{0}$, f($\bar{1}$)=f($\bar{3}$)=f($\bar{5}$)=$\bar{2}$, let N=$\mathbb{Z}_6$, it is clear that N is P-rational in $\mathbb{Z}_6$, but f(N)=<{$\bar{0}$,$\bar{2}$}>=$<\bar{2}>$ is not P-rational submodule of $\mathbb{Z}_4$.
13. Let A and B be submodules of an R-module M with A≅B. If A is P-rational in M, then B may not be P-rational in M, for example: in the $\mathbb{Z}$-module $\mathbb{Z}$, note that $\mathbb{Z}\cong 2\mathbb{Z}$, and clearly $\mathbb{Z}$ is P-rational in $\mathbb{Z}$, but $2\mathbb{Z}$ is not P-rational in $\mathbb{Z}$, since $2\mathbb{Z}$ is not pure submodule of $\mathbb{Z}$.
14. It is known that in any integral domain R, every non-zero ideal is rational. However, this statement is not satisfying for a P-rational submodule, to show that, take R≡$\mathbb{Z}$, and the ideal $2\mathbb{Z}$ of $\mathbb{Z}$. Since $\mathbb{Z}$ is pure simple so by (9), the only non-zero P-rational ideal in $\mathbb{Z}$ is only $\mathbb{Z}$, thus $2\mathbb{Z}$ is not P-rational ideal in $\mathbb{Z}$.
15. Let M=$M_1\oplus M_2$ be an R-module and let N be a P-rational submodule of M_i for some i=1,2.
16. Then N is not necessarily P-rational in M. For example, consider the $\mathbb{Z}$-module M=$\mathbb{Z}\oplus\mathbb{Z}_4$ and the submodule N=$\mathbb{Z}_4$ of the $\mathbb{Z}$-module $\mathbb{Z}_4$. We observe that $\mathbb{Z}_4$ is P-rational in $\mathbb{Z}_4$, but not P-rational in $\mathbb{Z}\oplus\mathbb{Z}_4$, since $\mathrm{Hom}_{\mathbb{Z}}(\frac{\mathbb{Z}\oplus\mathbb{Z}_4}{(0)\oplus\mathbb{Z}_4},(\mathrm{E}(\mathbb{Z}\oplus\mathbb{Z}_4))$, which is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z},\mathrm{E}(\mathbb{Z}\oplus\mathbb{Z}_4)\big)$. In addition, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathrm{E}(\mathbb{Z}\oplus\mathbb{Z}_4))\cong\mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z},\mathrm{E}(\mathbb{Z})\oplus\mathrm{E}(\mathbb{Z}_4)\big)$, [3, P.77]. But $\mathrm{E}(\mathbb{Z}_4)=\mathbb{Z}_{2^\infty}$, [3, P.21], therefore $\mathrm{Hom}_{\mathbb{Z}}\big(\mathbb{Z},\mathrm{E}(\mathbb{Z})\oplus\mathrm{E}(\mathbb{Z}_4)\big)=\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}\oplus\mathbb{Z}_{2^\infty})$. It is clear that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Q}\oplus\mathbb{Z}_{2^\infty})\neq 0$. Thus $\mathbb{Z}_4$ is not P-rational submodule of $\mathbb{Z}\oplus\mathbb{Z}_4$.

Recall that an R-module M is called multiplication, if every submodule of M is of the form IM, for some ideal I of R, [9] and M is called faithful if for any non-zero r∈R there is an element m∈M such that rm≠0, [2, P.4].

Proposition 2.4: Let M be a finitely generated multiplication and faithful module over an integral domain R. If N is a P-rational submodule of M, then (N:$_R$M) is a P-rational ideal of R, where (N:$_R$M)={r∈R\ rM ⊆ N}, [10].
Proof: Since R is an integral domain, and it is known that every non-zero ideal over any integral domain is rational, so that (N:$_R$M)$\leq_r$R. On the other hand, N is a pure submodule of M, and M is a finitely generated multiplication and faithful module, therefore (N:$_R$M) is a pure ideal of R, [10] hence (N:$_R$M)$\leq_{pr}$R.

Proposition 2.5: The annihilator of any P-rational submodule of an R-module M is equal to annihilator M.
Proof: Assume that N is a P-rational submodule of M, that is $\mathrm{Hom}_{\mathrm{R}}(\frac{\mathrm{M}}{\mathrm{N}},\mathrm{E}(\mathrm{M}))$ =0. It is clear that $\mathrm{ann}_{\mathrm{R}}(\mathrm{M})\leq\mathrm{ann}_{\mathrm{R}}(\mathrm{N})$, to prove $\mathrm{ann}_{\mathrm{R}}(\mathrm{N})\leq\mathrm{ann}_{\mathrm{R}}(\mathrm{M})$: let r∈$\mathrm{ann}_{\mathrm{R}}$(N), and define $f:\frac{\mathrm{M}}{\mathrm{N}}\rightarrow$ E(M) by f(m+N)=rm, for all m+N∈ $\frac{\mathrm{M}}{\mathrm{N}}$. Clearly, f is well-defined and homomorphism. Because

N is P-rational in M, then f=0, therefore rm=0, hence r $\in \mathrm{ann_R}(M)$. So that $\mathrm{ann_R}(N) = \mathrm{ann_R}(M)$.

The condition of Proposition 2.5 is necessary but not sufficient to imply that $N \leq_{pr} M$, as we take the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}$, and the submodule $\mathbb{Z}\oplus(0)$. Note that $\mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z}) = \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}\oplus(0))=0$. While $\mathbb{Z}\oplus(0)$ is not P-rational in $\mathbb{Z}\oplus\mathbb{Z}$. In fact, $\mathbb{Z}\oplus(0)$ is pure in $\mathbb{Z}\oplus\mathbb{Z}$, but $\mathrm{Hom}_{\mathbb{Z}}(\frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}\oplus(0)}, E(\mathbb{Z}\oplus\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(0\oplus\mathbb{Z}, \mathbb{Q}\oplus\mathbb{Q}) \neq 0$.

Since $\mathrm{ann_R}(R)=0$, so as a consequence of Proposition 2.5, we have the following.
Corollary 2.6: Let I be an ideal of a ring R. If I is a P-rational ideal in R then $\mathrm{ann_R}(I)=0$.

The converse of Corollary 2.6 is not true, for example: let $R=\mathbb{Z}$, $I=2\mathbb{Z}$, then $\mathrm{ann}_{\mathbb{Z}}(2\mathbb{Z})=0$, while $2\mathbb{Z}$ is not P-rational submodule in $\mathbb{Z}$ as we saw in Example 2.3 (3).

In the following theorem, we introduce another characterization of the definition of P-rational submodules. Compare with [11, Proposition 4.8, P.33].

Theorem 2.7: Let N be a pure submodule of an R-module M. Then $N \leq_{pr} M$ if and only if for all submodules L of M with $N \leq L \leq M$, $\mathrm{Hom_R}(\frac{L}{N}, M)=0$.

Proof: For the necessity, let $L \leq M$ such that $N \leq L \leq M$. Suppose there exists a non-zero homomorphism f: $\frac{L}{N} \longrightarrow M$. Consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \frac{L}{N} \arrow[r, "i"] \arrow[d, "f"] & \frac{M}{N} \arrow[ddl, "g"] \\
& M \arrow[d, "j"] & \\
& E(M) &
\end{tikzcd}$$

where E(M) is the injective hull of M and i, j are the inclusion homomorphism. Since E(M) is injective, then there exists a homomorphism g: $\frac{M}{N} \longrightarrow E(M)$ such that $g \circ i = j \circ f$. Now, since $f \neq 0$ so there exists a non-zero element $w \in \frac{L}{N}$ such that $f(w) \neq 0$, hence $(j \circ f)(w) \neq 0$. Since the diagram is commutative, then $(g \circ i)(w) \neq 0$, hence $g \neq 0$. That is $\mathrm{Hom_R}(\frac{M}{N}, E(M)) \neq 0$, but this contradicts our assumption. Thus $\mathrm{Hom_R}(\frac{L}{N}, M)=0$. For sufficiency, suppose there exists a non-zero homomorphism g: $\frac{M}{N} \longrightarrow E(M)$. Now, $g^{-1}(M)$ is a submodule of $\frac{M}{N}$. Put $g^{-1}(M) \equiv \frac{L}{N}$ where $N \leq L \leq M$. Define h: $\frac{L}{N} \longrightarrow M$ by $h(x+N)=g(x+N)$ for all $x+N \in \frac{L}{N}$. Note that the function h is well-defined and homomorphism. In fact, $x+N \in g^{-1}(M)$, $x \in L$, so that $g(x+N) \in M$. Moreover, $g \neq 0$, so there exists $m+N \in \frac{M}{N}$ such that $g(m+N) \neq 0$. Now, $g(m+N) \in E(M)$, and since $M \leq_e E(M)$, then there exists $r \in R$ such that $0 \neq rg(m+N) \in M$, [2, P.55]. This implies that $0 \neq g(rm+N) \in M$, hence $(r+m)+N \in g^{-1}(M) \equiv \frac{L}{N}$. So that $h(rm+N)=g(rm+N) \neq 0$. We conclude that $h \neq 0$, that is $\mathrm{Hom_R}(\frac{L}{N}, M) \neq 0$, which is a contradiction. Thus g=0. Additionally, N is pure, therefore N is a P-rational submodule in M.

An R-module M has the Pure Intersection Property (simply, PIP) if the intersection of any two pure submodules is a pure submodule, [12, P.33].

Proposition 2.8: Let $A\leq_{pr}B\leq C$ with B having PIP. If $A_1 \leq_{pr} B_1\leq C$ then $A\cap A_1 \leq_{pr} B\cap B_1$.
Proof: Let $A\cap A_1\subseteq N\subseteq B\cap B_1$, and let f: $\frac{N}{A\cap A_1} \to B\cap B_1$ be a homomorphism, consider the chain of submodules, $A_1\subseteq(N\cap A)+A_1\subseteq B_1$, and define g: $\frac{(N\cap A)+A_1}{A_1} \to B_1$ by $g(x+A_1)=f(x+A\cap A_1)$ $\forall x\in N\cap A$. To show that g is well-defined, let $x+A_1=x_1+A_1$, x, $x_1\in N\cap A$, then $x-x_1\in A_1$. Since $x\in N\cap A$, then $x-x_1\in A\cap A_1$, hence $x+A\cap A_1=x_1+A\cap A_1$, that is $f(x+A\cap A_1)=f(x_1+A\cap A_1)$. Also, g is a homomorphism. Since $A_1 \leq_{pr} B_1$, then $g(x+A)=0$ $\forall x\in N\cap A$, hence
$f(x+A\cap A_1)=0$ $\forall x\in N\cap A$ ………… (1)
which will be used later. Let $A\subseteq N+A\subseteq B$. Define h: $\frac{N+A}{A}\to B$ by $h(y+A)=f(y+A\cap A_1)$ $\forall y+A\in\frac{N+A}{A}$. To show that h is well-defined, let $y+A=y_1+A$, then $y-y_1\in N\cap A$. By (1), $f[(y-y_1+A\cap A_1]=0$, so that $f[(y+A\cap A_1-y_1+A\cap A_1]$. This implies that $f(y+A\cap A_1)=f(y_1+A\cap A_1)$, therefore $h(y+A)=h(y_1+A)$. In addition, h is a homomorphism. Since $A\leq_{pr}B$ then h=0, hence f=0. On the other hand, B has PIP, then $A\cap A_1$ is a pure submodule of B, and then $A\cap A_1$is pure in $B\cap B_1$, [13, Remark 1.4, P.37]. So we have $\mathrm{Hom}_R\left(\frac{N}{A\cap A_1}, B\cap B_1\right)=0$ and $A\cap A_1$ is a pure submodule of $B\cap B_1$, thus $A\cap A_1 \leq_{pr} B\cap B_1$.

As a consequence of Proposition 2.8, we conclude the following.

Corollary 2.9: Let M be a module satisfying the PIP. If both A and B are P-rational submodules in M, then $A\cap B\leq_{pr}M$.
Since every multiplication module has PIP then we have the following.

Corollary 2.10: Let $A\leq_{pr}B\leq C$ and $A_1 \leq_{pr} B_1\leq C$ then $A\cap A_1 \leq_{pr} B\cap B_1$, provided that B is a multiplication module.

Theorem 2.11: For any chain of modules $A\leq B\leq C$ with B is a pure submodule of C, $A\leq_{pr}C$ if and only if $A\leq_{pr}B$ and $B\leq_{pr}C$.
Proof: For the necessity, assume that $A\leq_{pr}C$, and let f: $\frac{K}{A}\to B$, where $A\leq K\leq B$. Consider the following sequence of homomorphisms:

$$\frac{K}{A} \xrightarrow{f} B \xrightarrow{i} C$$

where i is the inclusion homomorphism. Since $A\leq_{pr}C$, then $i\circ f=0$, so that $(i\circ f)\left(\frac{K}{A}\right)=f\left(\frac{K}{A}\right)=0$, hence f=0. On the other hand, A is a pure submodule of C, this implies that A is pure in B, [13, Remark 1.4, P.37]. Thus $A\leq_{pr}B$. Now, we have to show that $B\leq_{pr}C$, let h: $\frac{L}{B}\to C$ be a homomorphism, and define j: $\frac{L}{A}\to\frac{L}{B}$ by $j(x+A)=x+B$ for each $x+A\in\frac{L}{A}$. It can be easily show that j is well-defined and homomorphism. Consider the following:

$$\frac{L}{A} \xrightarrow{j} \frac{L}{B} \xrightarrow{h} C$$

$h\circ j\in\mathrm{Hom}_R\left(\frac{L}{A}, C\right)$. Since $A\leq_{pr}C$, then $h\circ j=0$, this means $(h\circ j)\left(\frac{L}{A}\right)=h\left(\frac{L}{B}\right)=0$, thus h=0. Besides that B is a pure submodule of C, therefore $B\leq_{pr}C$. For the converse direction, Suppose that $A\leq_{pr}B$ and $B\leq_{pr}C$, and f: $\frac{M}{A}\to C$ be any homomorphism set $f^{-1}(B)\equiv\frac{K}{A}$ where $K\leq M$. Now, A∩B⊆M∩B∩K⊆B means A⊆M∩B∩K⊆B. We will construct a map that has a codomain B. The homomorphism f restricts to a map g: $\frac{M\cap B\cap K}{A}\rightarrow$ B by g(x+A)=f(x+A) ∀x∈ M∩B∩K. Note that g is well-defined, in fact, x∈K, implies x+A∈$\frac{K}{A}\equiv f^{-1}$(B), hence f(x+A)∈B, that is the image of any element of $\frac{M\cap B\cap K}{A}$ belongs to B. Also, g is a homomorphism. Now, since A$\leq_{pr}$B, then g=0, therefore f(x+A)=0 ∀x∈ M∩B∩K, hence:

$$f\left(\frac{M\cap B\cap K}{A}\right)=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we claim that f$\left(\frac{M\cap B}{A}\right)$∩B=0. Let w∈f$\left(\frac{M\cap B}{A}\right)$∩B, then w∈B and w∈f $\left(\frac{M\cap B}{A}\right)$. This implies that w=f(t+A), where t∈M∩B, therefore t+A∈f^{-1}(B), since w∈B. So that t+A∈$\frac{K}{A}$, that is t∈K, hence t∈M∩B∩K. By (1), f(t+A)∈f$\left(\frac{M\cap B\cap K}{A}\right)$=0, therefore f(t+A)=0, which means w=0. Thus, f $\left(\frac{M\cap B}{A}\right)$∩B=0. Since B$\leq_{pr}$C, then by Remark (2.3)(5), B$\leq_{e}$C, hence

$$f\left(\frac{M\cap B}{A}\right)=0, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Now, define h: $\frac{M+B}{B}\rightarrow$C by h(m+b+B)=h(m+B)=f(m+A)∀m∈M. To prove h is well-defined, let m_1+B=m_2, then m_1-m_2∈B∩M. By (2), we obtain f(m_1-m_2)+A=0, therefore f (m_1+A) = f(m_2+A), thus h is well-defined. Since B$\leq_{pr}$C, then h=0, that is f(m+A)=0 ∀m∈M, hence f=0, that is $\mathrm{Hom}_R\left(\frac{M}{A},C\right)$=0. On the other hand, A is pure in B and B is pure in C, then A is pure in C, [13, Remark 1.4, P.37]. Thus A$\leq_{pr}$C.

Corollary 2.12: For any two of submodules A and B of M with B is a pure submodule of M, if A∩B$\leq_{pr}$M then A∩B$\leq_{pr}$B.
Proof: Since A∩B⊆B⊆M, and B pure in M then the result follows directly from Theorem 2.11.

Note: In Corollary **2.12**, if we replace the condition (B is pure in M) with the condition (A is pure in M), then we conclude that A∩B$\leq_{pr}$A.

Compare the following with [2, Proposition 2.25, P.55].

Theorem 2.13: Let N be a pure submodule of M. Then the following statements are equivalent.
1. $\mathrm{Hom}_R(\frac{L}{N}$, M)=0 for each submodule L of M with N⊆L⊆M.
2. For all y∈M and x∈M\{0} there exists r∈R such that rx≠0 and ry∈N.
Proof:
(1) ⇒ (2): Given (1), and let y∈M, x∈M\{0}. Set J={r∈R such that ry∈N}. Note that J≠∅, since 0∈J. We are done if we can show that M_1={b∈M: Jb=0} is equal to zero. Suppose that M_1≠0, so that there exists a non-zero element m∈M such that Jm=0. Define Ψ: $\frac{N+Ry}{N}\longrightarrow$ M by Ψ(ry+N)=rm for all r∈R. Ψ is well-defined, to verify that, let ry+N=sy+N, then ry-sy∈N. This implies that (r-s)y∈N, and according to the constriction of J we deduce that r-s∈J. But Jm=0, thus (r-s)m=0, so that rm=sm. Also, Ψ is a homomorphism, so by assumption Ψ=0, but this is a contradiction, since Ψ(y+N)=m≠0. Thus M_1=0, so Jx≠0, hence for a suitable element r∈J we obtain rx≠0 and ry∈N.
(2) ⇒ (1) Suppose the converse, that is there exists a submodule K of M with N⊆K⊆M and $\mathrm{Hom}_R(\frac{K}{N}$, M)≠0. This means the existence of a non-zero homomorphism Φ:$\frac{K}{N}\longrightarrow$M, that is there

exists y∈K such that 0≠ Φ(y+N)∈M. Put Φ(y+N)≡x≠0. Let r∈R, rΦ(y+N) = Φ(ry+N)=rx. If ry∈N then Φ(ry+N) = Φ(N)=0, therefore rx=0. But according to (2), rx≠0, so we have a contradiction, thus Φ=0, and hence $\mathrm{Hom_R}(\frac{K}{N}$, M)=0.

Note that Proposition 2.9 can be obtained as a conclusion of Theorem 2.13 as follows.

Corollary 2.14: Let M be a module having PIP. If A and B are P-rational submodules of M, then A∩B$\leq_{pr}$M.

Proof: Let x,y ∈M, where x≠0. Since A$\leq_{pr}$M, so by Theorem 2.13, there exists r∈R such that rx≠0 and ry∈A. Now, rx, ry∈M and rx≠0, since B$\leq_{pr}$M, again by Theorem 2.13, there exists t∈R such that trx≠0 and try∈B. But try∈A, therefore try∈A∩B. Thus ∀x,y∈M we find tr∈R such that trx≠0 and try∈A∩B. Moreover, both A and B are pure submodules of M, and M has PIP, therefore A∩B is pure in M, thus A∩B$\leq_{pr}$M.

Corollary 2.15: In any multiplication module, the intersection of any two P-rational submodules is also a P-rational submodule.

Proof: Since any multiplication module has PIP, [12, Proposition 2.3, P.33] then the result follows directly from Proposition 2.14.

The following Theorem is an analogue of [3, Proposition 8.6, P.274]

Theorem 2.16: For any R-module M, with N is a pure submodule of M, the following statements are equivalent:

1. For all y∈M and x∈M\{0}, there exists r∈R such that rx≠0 and ry∈N.

2. N $\leq_{pr}$M.

3. For any submodule P of M with N⊆P⊆M, $\mathrm{Hom_R}(\frac{P}{N}$, M)=0.

Proof:

(1) ⇒ (2): If N$\nleq_{pr}$M, that is $\mathrm{Hom_R}(\frac{M}{N}$, E(M))≠0, then there exists a non-zero homomorphism f: M→ E(M) with f(N)=0. Since f(M)≤ E(M), and M$\leq_e$E(M)), Then M∩f(M)≠0, so that there exist x,y∈M\{0}such that f(y)=x. By (1), there exists r∈R such that rx≠0 and ry∈N. Since f(N)=0, then f(ry)=0, and we obtain the following:

0=f(ry)=rf(y)=rx,

hence rx=0, which is a contradiction. Thus $\mathrm{Hom_R}\left(\frac{M}{N}, \mathrm{E(M)}\right)$=0.

(2) ⇒ (3): It immediately follows from Theorem 2.7.

(3) ⇒ (1): It follows directly from Theorem 2.13.

As an application of Theorem 2.16, we have the following.

Example 2.17: Consider the ℤ-module M=$\mathbb{Z}\oplus\mathbb{Z}_2$ and the submodule N=$2\mathbb{Z}\oplus\mathbb{Z}_2$ of M=$\mathbb{Z}\oplus\mathbb{Z}_2$. $2\mathbb{Z}\oplus\mathbb{Z}_2 \nleq_{pr} \mathbb{Z}\oplus\mathbb{Z}_2$. In fact, if we take the non-zero element y=$(0,\bar{1})\in 2\mathbb{Z}\oplus\mathbb{Z}_2$ and x=$(0,\bar{1})\in\mathbb{Z}\oplus\mathbb{Z}_2$, so for each r∈R, if $r(0,\bar{1})\in 2\mathbb{Z}\oplus\mathbb{Z}_2$, then r must be even. Therefore $r(0,\bar{1})$=0, so by Theorem 2.16, $2\mathbb{Z}\oplus\mathbb{Z}_2 \nleq_r \mathbb{Z}\oplus\mathbb{Z}_2$, and hence $2\mathbb{Z}\oplus\mathbb{Z}_2 \nleq_{pr} \mathbb{Z}\oplus\mathbb{Z}_2$.

Proposition 2.18: Let f: M→M` be an R-monomorphism. If L$\leq_{pr}$M´ then $f^{-1}$(L)$\leq_{pr}$ M, provided that the inverse image of any pure submodule of M` is pure in M.

Proof: Assume that L$\leq_{pr}$M, and let x,y∈M with x≠0, so that f(x)f(y)∈M`. Since f is a monomorphism, then f(x)≠0. By Theorem 2.16, there exists r∈R such that rf(y)∈L and rf(x)≠0, so we conclude f(ry)∈L and f(rx)≠0. This implies that ry∈f^{-1}(L) and rx∉ f^{-1}(0)=kerf=0, that is rx≠0, hence f^{-1}(L)$\leq_r$M, [2, Proposition 2.25, P.55]. In addition, f^{-1}(L) is a pure submodule of M by assumption. Thus f^{-1}(L)$\leq_{pr}$M.

In the following proposition, we use a different condition to get the same result in Proposition 2.18.

Proposition 2.19: For any isomorphism f: M→ M′, where M and M′ be R-modules, if L$\leq_{pr}$ M′ then f^{-1}(L)$\leq_{pr}$M.
Proof: Assume that L$\leq_{pr}$M. By the same argument of Proposition 2.18, we obtain f^{-1}(L)$\leq_r$M. On the other hand, we have L is pure in M′ and since f is epimorphism then f^{-1}(L) is a pure submodule of M, [14, Lemma 2.8]. Thus f^{-1}(L)$\leq_{pr}$M.

An R-module M is said to be cohopfian if every injective endomorphism of M is an isomorphism, [3, P.17].

Corollary 2.20: Let f: M→M′ be a monomorphism, where M, M′ be R-modules, and M is cohopfian. If L$\leq_{pr}$M′ then f^{-1}(L)$\leq_{pr}$M.
Proof: Since f is a monomorphism and M is cohopfian then f is an isomorphism, and by Proposition 2.19, we deduce f^{-1}(L)$\leq_{pr}$M.

Compare the following with [15, Lemma 2.10].

Proposition 2.21: Let L be a non-zero pure submodule of an R-module M. If for any 0≠m∈M, $\mathrm{ann}_R(\frac{M}{L}) \not\subseteq \mathrm{ann}_R(m)$, then L$\leq_{pr}$M.
Proof: We depend on Theorem 2.16, so let m,s∈M with m≠0. $\mathrm{ann}_R(\frac{M}{L})$={r∈Rl rM⊆L}, this means there exists r∈R such that rM⊆L. It follows that rs∈L. Since $\mathrm{ann}_R(\frac{M}{L}) \not\subseteq \mathrm{ann}_R(m)$ and r$\in\mathrm{ann}_R(\frac{M}{L})$, then r$\notin \mathrm{ann}_R$(m), that is rm≠0. Thus, we deduce that rs∈L and rm≠0. On the other hand, L is a pure submodule of M, so by Theorem 2.16, $\mathrm{Hom}_R(\frac{M}{L}$, E(M))=0, that is L$\leq_{pr}$M.

Corollary 2.22: Let M be an R-module. If I is a pure ideal of R such that $(0:_M I)=0$ (i.e ann_R(I)=0). Then IM$\leq_{pr}$M.
Proof: Assume that IM is not P-rational submodule in M. By the contrapositive of Proposition 2.21, $\mathrm{ann}_R(\frac{M}{IM})\subseteq\mathrm{ann}_R$(x) for some 0≠x∈M. This implies that Ix=0. But $(0:_M I)=0$, so we get a contradiction, therefore IM$\leq_{pr}$M.

3. P-Rational Submodules and Related Concepts

In this section, the relationships of P-rational submodules with some classes of related submodules are investigated such as quasi-invertible, purely quasi-invertible, essential, P-essential and SQI submodules.

A submodule N of an R-module M is called purely quasi-invertible (briefly we use the symbol N$\leq_{pqu}$ M) if N is pure and $\mathrm{Hom}_R(\frac{M}{N}$, M)=0, [14].

Proposition 3.1: Every P-rational submodule is purely quasi-invertible.

Proof: Let N be a P-rational submodule of an R-module M, by Remark 2.2, N$\leq_r$M. This implies that N $\leq_{qu}$M, [5, Proposition 3.3, P.14]. But N is pure in M, then N$\leq_{pqu}$M, [14].

Recall that A submodule N of an R-module M is P-essential if for every pure submodule L of M with N∩L=(0), implying that L=(0), [16].

Following [16], every essential submodule is P-essential, so we have the following.

Proposition 3.2: Each P-rational submodule is P-essential.
Proof: Let N be a P-rational submodule of M, so N is a rational submodule. This implies that N is essential**.** But every essential submodule is P-essential, thus the result follows.

An R-module M is nonsingular if Z(M)=0, where Z(M)={x∈M \ $\mathrm{ann}_{\mathrm{R}}(x)\leq_{\mathrm{e}}$R}, [2, P.31].

Proposition 3.3: Let M be a nonsingular module, and N be a pure submodule of M. Then N$\leq_{pr}$M if and only if N$\leq_e$M.
Proof: The necessity follows by Remark 2.3 (5). For the converse, let K≤M with N⊆K⊆M, we depend on the Theorem 2.7, so we have to show that $\mathrm{Hom}_{\mathrm{R}}(\frac{\mathrm{K}}{\mathrm{N}}$, M)=0. Since N$\leq_e$M then N$\leq_e$K, [2, Proposition 1.1, P.16]. Therefore, $\frac{\mathrm{K}}{\mathrm{N}}$ is singular. But M is nonsingular, then $\mathrm{Hom}_{\mathrm{R}}$ ($\frac{\mathrm{K}}{\mathrm{N}}$, M)=0, [2, Proposition 1.20, P.31]. Besides that, N is a pure submodule of M, so by Theorem 2.7, N$\leq_{pr}$M.

Proposition 3.4: Let N be a submodule of an R-module M. Consider the following statements.
1. N$\leq_{pr}$M.
2. N$\leq_r$M.
3. N$\leq_e$M.
Then (1) ⇒ (2) ⇒ (3) and if M is nonsingular and N is a pure submodule of M then (3) ⇒ (1)
Proof:
(1) ⇒(2): It is obvious.
(2) ⇒(3): It is clear.
(3) ⇒(1): Since M is a nonsingular module and N is pure in M, so by Proposition 3.3, the result follows.

We need to introduce the following definition.

Definition 3.5: An R-module M is called fully P-essential if every P-essential submodule of M is essential in M.

Remark 3.6: If M is a fully P-essential module then N$\leq_e$M if and only if N$\leq_{pe}$M.
Proof: It is straightforward.

Proposition 3.7: Let M be a multiplication module with a prime annihilator. Consider the following statements.
i.N$\leq_{pr}$M.
ii.N$\leq_{pqu}$M.
iii.N$\leq_{qu}$M.
iv.N$\leq_e$M.
v.N$\leq_{pe}$M.

Then (i) ⇒ (ii) ⇒ (iii) ⇔ (iv) ⇒ (v), and if M is a fully P-essential module then (v) ⇒ (iv).

Proof:

(i) ⇒ (ii): It is just Proposition 3.1.
(ii) ⇒ (iii): It is obvious.
(iii) ⇒ (iv): Since M is multiplication with a prime annihilator, then the two submodules quasi-invertible and essential coincide, [5, Theorem 3.11, P.18].
(iv) ⇔ (v): It is clear

Remember that a submodule N of M is called SQI-submodule if for each $f \in \mathrm{Hom}_R(\frac{M}{N}, M)$, then $f(\frac{M}{N}) \ll M$, [17, P.44].

Proposition 3.8: Every P-rational is an SQI submodule.
Proof: Since every P-rational is purely quasi-invertible, and every purely quasi-invertible is an SQI submodule, [14]. So, the result is obtained.

The converse of Proposition 3.8 is not true in general, for example, the submodule $< \bar{2} >$ of the $\mathbb{Z}$-module $\mathbb{Z}_4$ is SQI, [17] but it is not P-rational as we saw in Example 2.3 (4).

Finally, it is important to remember that, as we noted earlier, there are relatively few references that have been concerned with studying rational submodules such as [15] and [18-22], which is what motivated us to carry out this study.

4. Conclusions:

In this article, the class of rational submodules has been restricted to a new class of submodules. It is called P-rational submodules. The main results of this work can be summarized as follows:

1. The main characteristics of the P-rational submodules are studied, and the emphasis is on the analogue of the known results in the concept of rational submodules.
2. Other characterizations of P-rational submodules are investigated and they were compared with those in the concept of rational submodules.
3. Sufficient conditions under which P-rational and rational submodules are identical are given.

The connections between P-rational and other related concepts were established, such as essential, P-essential, quasi-invertible, purely quasi-invertible and SQI submodules. However, all of these relationships can be represented in the following diagram:

P-Rational submodule ⇒ Rational submodule ⇒Essential submodule ⇒ P-essential submodule

Rational submodule ⇒ Purely quasi-invertible submodule ⇒ Quasi-invertible submodule ⇒ SQI

In our future work, we will obtain more results about the class of P-Rational submodules and study its important influence on module theory.

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References

[1] F. Kasch, *"Modules and Rings"*, *Academic Press, London*, 1982.
[2] K.R. Goodearl, *"Ring Theory, Nonsingular Rings and Modules"*, Marcel Dekker, Inc. New York, Basel, 1976.
[3] T.Y. Lam, *"Lectures on Modules and Rings", California Springer, Berkeley*, 1998.
[4] S.M. Yaseen, "F-Regular Modules", M.Sc Thesis, College of Science, University of Baghdad, 1993.
[5] A.S. Mijbass, "Quasi-Dedekind Modules", Ph.D. Thesis, College of Science University of Baghdad, 1997.
[6] A.H. Hussein, "Ker-Injective and L-Injective Modules", M.Sc.Thesis, College of Science, University of Baghdad, 1993.
[7] D.J. Fieldhouse, "*Pure Simple and Indecomposable Rings*", *Can. Math. Bull.*, vol.13, no.1, pp.71-78, 1970.
[8] D.J. Fieldhouse, "Pure Theories", *Math Ann*, no.184, pp.1-18, 1969.
[9] A. Barnard, "Multiplication Modules", *J. Algebra,* vol.71, no.1, pp.174-178, 1981.
[10] M.A. Majid and J.S. David, "Pure Submodules of Multiplication Modules", *Beitrage Zur Algebra Geom.*, vol.45, no.1, pp.61-74, 2004.
[11] N.V. Dung, D.V. Huynh, P.F. Smith, and R. Wisbauer, *"Extending Modules"*, Pitman Research Notes in Mathematics Series 313, New York, 1994.
[12] B.H. Al-Bahraany, "Modules With the Pure Intersection Property", Ph.D. Thesis, College of Science, University of Baghdad, 2000.
[13] Z.T. Salman, "On Purely Extending Modules", M.Sc. Thesis, College of Science, University of Baghdad, 2005.
[14] Th.Y Ghawi, "Purely Quasi-Dedekind Modules and Purely Prime Modules", *AL-Qadisiya J. Sci.*, vol.16, no.4, pp.30-45, 2011.
[15] A. Hajikarimi and A.R. Naghipour, "A Note on Monoform Modules", *Bull. Korean Math. Soc.* vol.56, no.2, pp.505-514, 2019.
[16] N.M. Al-Thani, "Pure Baer Injective Modules", *Int. J. Math. Math. Sci.*, vol.20, no.3, pp.529-538, 1997.
[17] A.G. Naoum and I.M.A. Hadi, "SQI Submodules and SQD Modules", *Iraqi J. Sc*, vol.1, no.2, pp.43-54, 2002.
[18] H.H. Storrer, "Rational Extensions of Modules", *Pacific Journal of Mathematics* vol.38, no.3, pp.785-794, 1971.
[19] H. Khabazian and S. Safaeeyan, " P-Dense Submodules", *Journal of Mathematical Extension* vol.3, no.1, pp.95-105, 2008.
[20] E. Ghashghaei and M. Namdari, "On Strongly Dense Submodules", *Bull. Iranian Math. Soc*. vol.42, no.3, pp.731-747, 2016.
[21] M.S. Abbas and M.A. Ahmed, "Rationally Extending Modules and Strongly Quasi-Monoform Modules", *Al-Mustansiriya J. Sci*, vol.22, no.3, pp.31-38, 2011.
[22] Z.M. Abd Al-Majeed and M.S. Nayef, " Tr-Rationally Closed Submodules and RCLS-Modules", *2nd International Conference on Physics and Applied Sciences, J. Phys.: Conf. Ser*, 2021.