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Strongly irreducible ring and its S spectrum

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Abstract

Let K be a proper ideal of a commutative ring $\boldsymbol{S}$. Then K is a strongly irreducible (SI) ideal if for any two ideals A and B of $\boldsymbol{S}, A \cap B \subseteq K$ implies $A \subseteq K$ or $B \subseteq K$. We say a ring is strongly irreducible (SI) if all its proper ideals are strongly irreducible. In this paper, some properties and characterizations of such rings are given. The relations between SI rings and some types of rings are also studied. For an SI ring $\boldsymbol{S}$, the S strongly irreducible spectrum $X = S.spec(\boldsymbol{S})$ of $\boldsymbol{S}$ is the set $S.spec(\boldsymbol{S})= \{I: I$ is an ideal of $\boldsymbol{S}\}$ and the S variety of a subset E of $\boldsymbol{S}$ is the set $V_s(E) = \{I \in S.spec(\boldsymbol{S}): E \subseteq I\}$. Then the family $F = \{V_s(E): E \subseteq \boldsymbol{S}\}$ satisfies the axioms for closed sets of a topology on $X = S.spec(\boldsymbol{S})$. Consequently, if $X_s(E) = S.spec(\boldsymbol{S}) \backslash V_s(E)$, then the family $H = \{X_s(E): E \subseteq \boldsymbol{S}\}$ forms a topology on $X = S.spec(\boldsymbol{S})$. This topology is said to be $S.spec(\boldsymbol{S})$ topology or S Zariski topology. In this work, some properties of $S.spec(\boldsymbol{S})$ topology are also studied.

Keywords: Strongly irreducible ideal; Strongly irreducible ring; S spectrum; $S.spec(\boldsymbol{S})$ topology; S Zariski topology;

حلقة غير القابلة للاختزال بقوة و طيفها

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الخلاصة

ليكن K مثالي فعلي للحلقة الابدالية $\boldsymbol{S}$. عندئذ K هو مثالي غير قابل للاختزال بقوة إذا كان لأي مثاليين A و B من $\boldsymbol{S}$، $A \cap B \subseteq K$ يؤدى الى $A \subseteq K$ او $B \subseteq K$. نسمى لحلقة ما انها حلقة غير القابلة للاختزال بقوة و طيفها إذا كانت كل المثاليات الفعلية للحلقة غير قابلة للاختزال بقوة. في هذا البحث تم إعطاء بعض خصائص و توصيفات هذه الحلقة. تمت دراسة العلاقات بين حلقة غير القابلة للاختزال بقوة وبعض أنواع الحلقات. لتكن $\boldsymbol{S}$ حلقة غير القابلة للاختزال بقوة نعرف المجموعة $S.spec(\boldsymbol{S}) = \{J: J$ is an ideal of $\boldsymbol{S}\}$. لكل مجموعة جزئية E من $\boldsymbol{S}$ يرمز المجموعة المتنوعة ل E من نمط S ب $V_s(E)$ بأنها المجموعة $V_s(E) = \{I \in S.spec(\boldsymbol{S}): E \subseteq I\}$ عندئذ العائلة $F = \{V_s(E): E \subseteq \boldsymbol{S}\}$ مغلقة تحت اتحاد عدد منتهي من المجموعات والتقاطعات العشوائية, ولذا F تحقق بديهيات المجموعات المغلقة لتوبولوجيا على $X = S.spec(\boldsymbol{S})$ وأن تجمع المجموعات الجزئية $X_s(E) = S.spec(\boldsymbol{S}) \backslash V_s(E)$ من $S.spec(\boldsymbol{S})$ ولكل المجموعات الجزئية E من $\boldsymbol{S}$ يجعل من $S.spec(\boldsymbol{S})$ فضاء توبولوجي. نسمي هذا التوبولجي ب توبولوجي $\boldsymbol{S.spec}(\boldsymbol{S})$. درسنا بعض خواص هذا التوبولوجي.

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1. Introduction

Throughout this paper, we consider that $\mathcal{S}$ is a commutative ring with identity. A proper ideal K of $\mathcal{S}$ is an SI ideal if for any two ideals C and D of $\mathcal{S}, C \cap D \subseteq K$ implies $C \subseteq K$ or $D \subseteq K$ [1], [2]. In this work, we introduce and study the concept of an SI ring. In section two, some characterizations of such rings are given. In Proposition 2.3 and Theorem2.4, some properties of SI rings are discussed. Furthermore, the relations between SI ring and each the arithmetical ring, local ring, Bezout ring, clean ring, r-clean ring, divided ring and pseudo-valuation ring are studied in Proposition 2.5, Corollary 2.6 and Proposition 2.8. For a ring $\mathcal{S}$, the S strongly irreducible spectrum $X = S.spec(\mathcal{S})$ of $\mathcal{S}$ is the set $S.spec(\mathcal{S})= \{I: I$ is a strongly irreducible ideal of $\mathcal{S}$ } and the S variety of a subset E of $\mathcal{S}$ is the set $V_s(E) = \{I \in S.spec(\mathcal{S}): E \subseteq I\}$. Then the family $F = \{V_s(E): E \subseteq \mathcal{S}\}$ is closed under finite union and arbitrary intersection. Clearly that $V_s(\emptyset) = S.spec(\mathcal{S})$ and $V_s(\mathcal{S}) = \emptyset$ so that F satisfies the axioms for closed sets of a topology on $X = S.spec(\mathcal{S})$. Let $X_s(E) = S.spec(\mathcal{S}) \setminus V_s(E)$. Then the family $= \{X_s(E): E \subseteq \mathcal{S}\}$ forms a topology on $X = S.spec(\mathcal{S})$. This topology is said to be $S.spec(\mathcal{S})$ topology or S Zariski topology [2]. In section three, $S.spec(\mathcal{S})$ of an SI ring $\mathcal{S}$ and $S.spec(\mathcal{S})$ topology are studied. It is shown that for an SI ring $\mathcal{S}$, the $S.spec(\mathcal{S})$ topology is connected, locally connected, ultraconnected, hyperconnected, strongly zero-dimensional, normal space and sober space but it is not a zero-dimensional space and it does not satisfy most of separation axioms.

2. Strongly irreducible rings

In this section, some properties and characterizations of SI rings are studied and given as well as the relations between SI rings and some other types of rings are studied.

Definition 2.1. A ring $\mathcal{S}$ is said to be an SI ring if every proper ideal of $\mathcal{S}$ is an SI ideal.

Recall that if R is a commutative ring with identity and S is multiplicatively closed subset of R containing 1 ($0 \notin S$, $1 \in S$ and if $a, b \in S$, then $ab \in S$), then the localization of R at S is the ring $R_S = \{\frac{r}{s}: r \in R, s \in S\}$ where the addition and the multiplication of the formal fractions $\frac{r}{s}$ are defined according to the natural rules, $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$, respectively [3]. In fact R is a subring of R_S. Now, let P be a prime ideal of R. Then $R - p$ is multiplicatively closed subset of R. Then the localization of R at the prime ideal P is the ring $R_P=\{\frac{a}{b}: a \in R, b \in R - P\}$ where the addition and multiplication are defined as the same as above.

Examples 2.2.

1. The localization of $\mathbb{Z}$ at the prime ideal $Q =< q >$ of $\mathbb{Z}$, q is a prime number is $\mathbb{Z}_{<q>} = \{\frac{a}{b} \in \mathbb{Q} : q \nmid b\}$. The nonzero proper ideals of $\mathbb{Z}_{<q>}$ are of the form $I_k =< q^k >$ where $k \in \mathbb{Z}^+$, these ideals are ordered by inclusion as follows: $< q >\supset< q^2 >\supset \cdots \supset< q^k >\supset< q^{k+1} >\supset \cdots$ and clearly $\mathbb{Z}_{<q>}$ is an SI ring and $< q >$ is its maximal ideal [3, p. 706].
2. Consider the ring $\mathcal{S} = \mathbb{Z}_2[t]/< t^3 >= \{0, 1, t, 1 + t, t^2, 1 + t^2, t + t^2, 1 + t + t^2\}$. The proper ideals of $\mathcal{S}$ are $I_1 = \{0\}$, $I_2 =< t^2 >= \{0, t^2\}$, and $I_3 =< t >= \{0, t, t^2, t + t^2\}$. Clearly, each of these ideals is an SI ideal. Therefore, $\mathcal{S}$ is an SI ring.

The following is the multiplication table of the ring $\mathcal{S} = \mathbb{Z}_2[t]/< t^3 >$.

$*$	0	1	t	$1+t$	t^2	$1+t^2$	$t+t^2$	$1+t+t^2$
0	0	0	0	0	0	0	0	0
1	0	1	t	$1+t$	t^2	$1+t^2$	$t+t^2$	$1+t+t^2$
t	0	t	t^2	$t+t^2$	0	t	t^2	$t+t^2$
$1+t$	0	$1+t$	$t+t^2$	$1+t^2$	t^2	$1+t+t^2$	t	1
t^2	0	t^2	0	t^2	0	t^2	0	t^2
$1+t^2$	0	$1+t^2$	t	$1+t+t^2$	t^2	1	t^2	$1+t$
$t+t^2$	0	$t+t^2$	t^2	t	0	t	t^2	t
$1+t+t^2$	0	$1+t+t^2$	$t+t^2$	1	t^2	$1+t$	t	$1+t^2$

Proposition 2.3. Let $\mathcal{S}$ be a ring. Then the following statements are equivalent:
1. $\mathcal{S}$ is an SI ring.
2. Every two ideals of $\mathcal{S}$ are comparable(The two ideals I and J of a ring are said to be comparable if $I \subseteq$ J or $J \subseteq$ I).
3. For each $a, b \in \mathcal{S}$, $< a > \subseteq <$ b $>$ or $<$ b $> \subseteq < a >$.
4. If $a, b \in \mathcal{S}$, then $a|b$ or $b|a$.
5. For any two ideals C and B of $\mathcal{S}$, $C \cap D$ is an SI ideal of $\mathcal{S}$.
6. For any two ideals C and D of $\mathcal{S}$, $C \cup D$ is an ideal of $\mathcal{S}$.

Proof.
(1) $\Leftrightarrow$ (2). This follows from [4, pp. 150, Lemma 3.5].
(2) $\Leftrightarrow$ (3). If every two ideals of $\mathcal{S}$ are comparable, then for each $a, b \in \mathcal{S}$, $< a > \subseteq <$ b $>$ or $<$ b $> \subseteq < a >$. For the converse, let I and J be any two ideals of $\mathcal{S}$ and suppose $I \nsubseteq J$. Then there is $a \in J$ and $a \notin I$. Hence, $a \notin <$ b $>$ for each $b \in J$, consequently $< a > \nsubseteq <$ b $>$. Then $<$ b $> \subseteq < a >$ for each $b \in J$. So that $J \subseteq I$.
(3) $\Leftrightarrow$ (4). It is obvious.
(2) $\Leftrightarrow$ (5) Let C and D be any two ideals of $\mathcal{S}$ and $C \cap D$ be an SI ideal. Since $C \cap D \subseteq C \cap D$, then $C \subseteq C \cap D$ or $D \subseteq C \cap D$. So that $C \subseteq D$ or $D \subseteq$ C. For the converse, suppose that C and D are any two ideals of $\mathcal{S}$ and let I and J be two ideals with $I \cap J \subseteq C \cap D$. From the assumption $I \subseteq J$ or $J \subseteq I$, consequently $I \cap J = I \subseteq C \cap D$ or $I \cap J = J \subseteq C \cap D$.
(2) $\Leftrightarrow$ (6) is well known.

Theorem 2.4. Let $\mathcal{S}$ be an SI ring. Then
1. Any localization of $\mathcal{S}$ is an SI ring.
2. Every finitely generated ideal of $\mathcal{S}$ is principal.
3. If $\mathcal{S}$ is Noetherian, then $\mathcal{S}$ is a principal ideal ring.
4. If I is a proper ideal of $\mathcal{S}$, then $\mathcal{S}/I$ is an SI ring.
5. The homomorphic image of an SI ring is an SI ring.
6. Every nonmaximal proper ideal is contained in a proper principal ideal.
7. If $\mathcal{S}$ is an Artinian ring, then $\mathcal{S}$ has a unique prime ideal which is the maximal ideal.
8. The only idempotents of $\mathcal{S}$ are 0 and 1.
9. If I is an ideal of $\mathcal{S}$, then I is not an n-sequence prime ideal of $\mathcal{S}$.

Proof.
1. Let S be a nonzero multiplicatively closed subset of $\boldsymbol{S}$ and M, N be two ideals in $\boldsymbol{S}_S$. Then there are two ideals I, J in $\boldsymbol{S}$ such that $M = I_S$ and $N = J_S$. Since $\boldsymbol{S}$ is an SI ring, then $I \subset J$ or $J \subset I$. If $I \subset J$ and $x \in M$. Then $x = \frac{a}{s}$ where $a \in I \subset J, s \in S$. This means $x \in N$. Therefore, $M \subset N$. Similarly if $J \subset I$, then $N \subset M$. Therefore, by Proposition 2.3, $\boldsymbol{S}$ is an SI ring.
2. It is enough to show that every ideal generated by two elements is principal. So let $I = < f, g >$ be an ideal of R. Then $< f > \subseteq < g >$ or $< g > \subseteq < f >$, so $I = < g >$ or $I = < f >$.
3. Let $\boldsymbol{S}$ be Noetherian. Then every ideal of $\boldsymbol{S}$ is finitely generated. By part 2, $\boldsymbol{S}$ is a principal ideal ring.
4. Let I be a proper ideal of $\boldsymbol{S}$ and $< a + I >$, $< b + I >$ be two ideals of $\boldsymbol{S}/I$. Since $\boldsymbol{S}$ is an SI ring, then $< a > \subseteq < b >$ or $< b > \subseteq < a >$. So that $< a + I > \subseteq < b + I >$ or $< b + I > \subseteq < a + I >$. By Proposition 2.3, $\boldsymbol{S}$ is an SI ring.
5. Let $f: \boldsymbol{S}_1 \to \boldsymbol{S}_2$ be a ring homomorphism and $\boldsymbol{S}_1$ be an SI ring. Since $ker(f)$ is an ideal of $\boldsymbol{S}_1$, then $\boldsymbol{S}_1/ker(f) \cong f(\boldsymbol{S}_1)$ and by part 4, $\boldsymbol{S}_1/ker(f)$ is an SI ring, consequently, $f(\boldsymbol{S}_1)$ is an SI ring.
6. Let I be a non maximal proper ideal of $\boldsymbol{S}$. Then there is a maximal ideal M such that $I \subset M$. Then there is $x \in M$ and $x \notin I$. Hence, $x \notin < a >$ for each $a \in I$, consequently $< x > \nsubseteq < a >$. By Proposition 2.3, $< a > \subseteq < x >$ for each $a \in I$. Therefore, $I \subseteq < x >$.
7. Let P and Q be two prime ideals of an SI Artinian ring $\boldsymbol{S}$. By [3, pp. 752, Theorem 3(3)], both P and Q are maximal ideals, but it is well known that two maximal ideals cannot be comparable. So that there is a unique prime ideal.
8. Let $\boldsymbol{S}$ be an SI ring and $e \in \boldsymbol{S}$ be a non-trivial idempotent($e \neq 0, 1$). Then $1 - e$ is a non-trivial idempotent. Then by Proposition 2.3, either $< e > \subseteq < 1 - e >$ or $< 1 - e > \subseteq < e >$. This implies either $1 \in < 1 - e >$ or $1 \in < e >$. Consequently $< 1 - e > = \boldsymbol{S}$ or $< e > = \boldsymbol{S}$, contradiction.
9. Let I be a proper ideal of $\boldsymbol{S}$. By [5, pp. 3677, Theorem 3.10], I is not an n-sequence prime ideal of $\boldsymbol{S}$.

Recall from [6], a ring $\boldsymbol{S}$ is said to be arithmetical if for all ideals I, J and K of $\boldsymbol{S}$, we have $(I + J) \cap K = (I \cap K) + (J \cap K)$.

Proposition 2.5. Let $\boldsymbol{S}$ be a ring and M be a maximal ideal of $\boldsymbol{S}$. Then $\boldsymbol{S}_M$, the localization of $\boldsymbol{S}$ at M is an SI ring if and only if $\boldsymbol{S}$ is an arithmetical ring.

Proof. This obtained by [7, pp. 321, Exercises 19] and Proposition 2.3.

Corollary 2.6. Let $\boldsymbol{S}$ be a ring. Then
1. If $\boldsymbol{S}$ is an SI ring, then it is arithmetical.
2. If $\boldsymbol{S}$ is an arithmetical ring and its ideals are irreducible, then $\boldsymbol{S}$ is an SI ring.

Proof.
1. Let I, J and K be three ideals of an SI ring $\boldsymbol{S}$. There are six cases, without loss of generality, we suppose $I \subset J \subset K$. Then $I \cap (J + K) = I \cap K = I$ and $(I \cap J) + (I \cap K) = I + I = I$. The proof of the other cases are similar. Therefore, $I \cap (J + K) = (I \cap J) + (I \cap K)$.
2. The proof follows from [6, pp. 269, Lemma 2.2(3)].

Remark 2.7. An arithmetical ring does not need to be an SI ring. For example, the ring of integers is arithmetical, but it is not an SI ring.

Now we recall some definitions that are used throughout this work:

1. A ring $\boldsymbol{S}$ is called Bezout if every finitely generated ideal I of $\boldsymbol{S}$ is principal [8].
2. A ring $\boldsymbol{S}$ is called a local ring if it has a unique maximal ideal [3].

3. A ring $\mathcal{S}$ is called clean if each element of $\mathcal{S}$ can be expressed as the sum of a unit element and an idempotent element and $\mathcal{S}$ is called r-clean if each element of $\mathcal{S}$ can be expressed as the sum of a regular element and an idempotent element [9].
4. A ring $\mathcal{S}$ is an exchange ring if for any $a \in \mathcal{S}, a - e \in (a^2 - a)\mathcal{S}$ for some $e^2 = e \in \mathcal{S}$ [10].
5. A ring $\mathcal{S}$ is said to be semipotent if each ideal of $\mathcal{S}$ that is not contained in its Jacobson radical contains a nonzero idempotent. A semipotent ring $\mathcal{S}$ is said to be potent if idempotents lift modulo its Jacobson radical [9].
6. A prime ideal P of a ring $\mathcal{S}$ is called divided if P is comparable to every principal ideal of $\mathcal{S}$. If every prime ideal of $\mathcal{S}$ is divided, then $\mathcal{S}$ is called a divided ring [11].
7. A prime ideal P of $\mathcal{S}$ is called strongly prime if aP and $b\mathcal{S}$ are comparable for every $a, b \in \mathcal{S}$. If every prime ideal of a ring $\mathcal{S}$ is strongly prime, then $\mathcal{S}$ is called a pseudo-valuation ring [11].

In the following proposition, we give a relation between an SI ring and some other types of rings.

Proposition 2.8. If $\mathcal{S}$ is an SI ring, then
1. $\mathcal{S}$ is a Bezout ring.
2. $\mathcal{S}$ is a local ring.
3. $\mathcal{S}$ is a clean ring (Resp. exchange ring, semipotent ring, potent ring, r-clean ring).
4. $\mathcal{S}$ is a divided ring.
5. $\mathcal{S}$ is a pseudo-valuation ring.

Proof.
1. It follows from Theorem 2.4(2).
2. Suppose $\mathcal{S}$ has at least two maximal ideals I and J. Then by Proposition 2.3, $I \subset J$ or $J \subset I$, we get a contradiction since two maximal ideals cannot be comparable. Therefore, $\mathcal{S}$ has a unique maximal ideal.
3. Let $\mathcal{S}$ be an SI ring. Then from Part 2, $\mathcal{S}$ is a local ring. Let M be the maximal ideal of $\mathcal{S}$ and $a \in \mathcal{S}$. If $a \in M$, then clearly $a - 1 \notin M$ which means $u = a - 1$ is a unit. Then $a = u + 1$ is a clean element. If $a \notin M$, then a is a unit. Hence, $\mathcal{S}$ is a clean ring, consequently, $\mathcal{S}$ is r-clean. By [9, pp. 3, Proposition 1.4], [9, pp. 5, Proposition 1.8] and [9, pp. 5, Corollary 1.9], $\mathcal{S}$ is an exchange ring, semipotent ring and a potent ring.

Parts 4 and 5 are obvious.

The following examples show that the converse of none of the statements given in Proposition 2.8 is true in general.

Examples 2.9.
1. A Bezout ring does not need to be an SI ring. In particular, a principal ideal ring does not need to be an SI ring. For example, every finitely generated ideal of $\mathbb{Z}$ is a principal ideal, but $\mathbb{Z}$ is not an SI ring.
2. Consider the ring $\mathcal{S} = \mathbb{Z}_2[x, y]/< x^2, y^2 >= \{0, 1, x, y, xy, 1 + x, 1 + y, 1 + xy, x + y, x + xy, y + xy, 1 + x + y, 1 + x + xy, 1 + y + xy, x + y + xy, 1 + x + y + xy\}$. The nonzero proper ideals of $\mathcal{S}$ are $I_1 =< x >= \{0, x, xy, x + xy\}$, $I_2 =< y >= \{0, y, xy, y + xy\}$, $I_3 =< x + y >= \{0, x + y, xy, x + y + xy\}$, $I_4 =< xy >= \{0, xy\}$ and $I_5 =< x, y >= \{0, x, y, xy, x + xy, y + xy, x + y, x + y + xy\}$. Since $\mathcal{S}$ has only six proper ideals which are $< 0 >$, I_1, I_2, I_3, I_4 and I_5, and $< 0 >$, I_1, I_2, I_3 and I_4 are contained in I_5, then I_5 is a

maximal ideal of $\boldsymbol{S}$ and there are no other maximal ideals of $\boldsymbol{S}$. Hence, $\boldsymbol{S}$ is a local ring but $\boldsymbol{S}$ is not an SI ring since the two ideals I_1 and I_2 are not comparable.
3. Consider the group ring $\boldsymbol{S} = \mathbb{Z}_2(G)$ where G is a cyclic group of order 3 generated by g. Then $\boldsymbol{S} = \mathbb{Z}_2(G) = \{0, 1, g, g^2, 1 + g, 1 + g^2, g + g^2, 1 + g + g^2\}$. The elements $1, g$ and g^2 are units and each of $0, 1, g + g^2, 1 + g + g^2$ is an idempotent element. This means that $\boldsymbol{S}$ is a clean ring, consequently, $\boldsymbol{S}$ is r-clean. The proper ideals of $\boldsymbol{S}$ are $I_1 =< 0 >$, $I_2 =< g + g^2 >= \{0, g + g^2, 1 + g^2, 1 + g\}$ and $I_3 =< 1 + g + g^2 >= \{0, 1 + g + g^2\}$. Clearly, I_1 is not an SI ideal. Hence, $\boldsymbol{S}$ is not an SI ring.

The following is the multiplication table of the ring $\boldsymbol{S} = \mathbb{Z}_2(G)$

$*$	0	1	g	$1+g$	g^2	$1+g^2$	$g+g^2$	$1+g+g^2$
0	0	0	0	0	0	0	0	0
1	0	1	g	$1+g$	g^2	$1+g^2$	$g+g^2$	$1+g+g^2$
g	0	g	g^2	$g+g^2$	1	$1+g$	$1+g^2$	$1+g+g^2$
$1+g$	0	$1+g$	$g+g^2$	$1+g^2$	$1+g^2$	$g+g^2$	$1+g$	0
g^2	0	g^2	1	$1+g^2$	g	$g+g^2$	$1+g$	$1+g+g^2$
$1+g^2$	0	$1+g^2$	$1+g$	$g+g^2$	$g+g^2$	$1+g$	$1+g^2$	0
$g+g^2$	0	$g+g^2$	$1+g^2$	$1+g$	$1+g$	$1+g^2$	$g+g^2$	0
$1+g+g^2$	0	$1+g+g^2$	$1+g+g^2$	0	$1+g+g^2$	0	0	$1+g+g^2$

4. Let $\boldsymbol{S} = \mathbb{Z}_4[x]/< 2x, x^2 >= \{0, 1, 2, 3, x, 1 + x, 2 + x, 3 + x\}$. The proper ideals of $\boldsymbol{S}$ are $I_1 =< 0 >$, $I_2 =< 2 >= \{0, 2\}$, $I_3 =< x >= \{0, x\}$, $I_4 =< 2 + x >= \{0, 2 + x\}$ and $I_5 =< 2, x >= \{0, 2, x, 2 + x\}$. The only prime ideal of $\boldsymbol{S}$ is I_5 which is the maximal ideal of $\boldsymbol{S}$ and contains all other ideals. So $\boldsymbol{S}$ is a divided ring and $\boldsymbol{S}$ is not an SI ring, since $< 0 >$ is not an SI ideal. Furthermore, $\boldsymbol{S}$ is a pseudo-valuation ring.

Recall that a ring $\boldsymbol{S}$ is said to be regular if for every element $r \in \boldsymbol{S}$ there is some element $x \in \boldsymbol{S}$ such that $rxr = r$ [9].

Proposition 2.10. If $\boldsymbol{S}$ is an SI ring which is not a field, then $\boldsymbol{S}$ is not a regular ring.
Proof. Let $\boldsymbol{S}$ be an SI ring which is not a field, then $\boldsymbol{S}$ has a nonzero proper ideal I. By Theorem 2.4(8), $\boldsymbol{S}$ has no non-trivial idempotent. Now, if $\boldsymbol{S}$ is a regular ring, then by [12, pp. 2671, Corollary 2.9], I is generated by a non-trivial idempotent, then we get a contradiction. Therefore, $\boldsymbol{S}$ is not a regular ring.

Recall that a ring $\boldsymbol{S}$ is said to be a Zerlegung prime ideal ring (ZPI-ring) if every proper ideal of $\boldsymbol{S}$ can be written as a product of prime ideals of $\boldsymbol{S}$ [4], and a commutative ring $\boldsymbol{S}$ is said to be a prime ring if $ab\boldsymbol{S} =< 0 >$ implies that $a = 0$ or $b = 0$ such that $a, b \in \boldsymbol{S}$ [13].

Remark 2.11.
1. If $\boldsymbol{S}$ is a local ZpI ring, then it is an SI ring. This follows from [4, pp. 150, Theorem 3.7] and Proposition 2.3.
2. A subring of an SI ring does not need to be an SI ring. For example, the subring $\mathbb{Z}$ of $\mathbb{Q}$ is not an SI ring since $\mathbb{Z}$ has at least two non-comparable ideals but $\mathbb{Q}$ is an SI ring since the zero ideal is the only proper ideal of $\mathbb{Q}$.
3. An SI ring does not need to be a prime ring. For example, the ring $\mathbb{Z}_8$ is an SI ring but it is not a prime ring since $2, 4 \in \mathbb{Z}_8\backslash\{0\}$ and $(2)(4)\mathbb{Z}_8 =< 0 >$.

Proposition 2.12. Let $\mathcal{S}=\mathbb{Z}_m$, the ring of integers modulo m. Then $\mathcal{S}$ is an SI ring if and only if $m=p^n$ for some prime p and positive integer n.

Proof. Let $m=p^n$ for some prime p and positive integer n that is $\mathcal{S}=\mathbb{Z}_{p^n}$. Clearly, the ideals of $\mathbb{Z}_{p^n}$ are
$<p>, <p^2>, \dots, <p^{n-1}>, <0>$ and all these ideals are comparable. Therefore, $\mathcal{S}$ an SI ring. Now, for the converse, suppose that $\mathbb{Z}_m$ is an SI ring. Let $m={p_1}^{h_1}{p_2}^{h_2}\dots{p_r}^{h_r}$ be the prime factorization of m. Since the additive group H of the ring $\mathcal{S}$ is finite, then H$\simeq \mathbb{Z}_{{p_1}^{h_1}}\times\mathbb{Z}_{{p_2}^{h_2}}\dots\mathbb{Z}_{{p_r}^{h_r}}$. We claim that r is equal to one. If r is greater than one, put $A=<{p_1}^{h_1}>\times \underbrace{\{0\}\times\{0\}\times\dots\times\{0\}}_{(r-1)-\,times}$ and $B=\{0\}\times<{p_2}^{h_2}>\times \underbrace{\{0\}\times\{0\}\times\dots\times\{0\}}_{(r-2)-\,times}$, then $A\cap B=\{0\}\times\{0\}\dots\times\{0\}$ but neither $A=0$ nor $B=0$. Then the zero ideal is not an SI ideal. Consequently $r=1$. Then $\mathcal{S}\simeq\mathbb{Z}_{p^n}$ for some $n\in\mathbb{Z}^+$ and a prime number p.

Remark 2.13. If $\mathcal{S}$ is a ring with p^n elements where p is a prime number and $n>1$, and the additive group of $\mathcal{S}$ is isomorphic to the additive group $(\mathbb{Z}_{p^n}, +_{p^n})$, then $\mathcal{S}$ is an SI ring. This statement is true since the additive group of every ideal of $\mathcal{S}$ is isomorphic to a subgroup of the group $(\mathbb{Z}_{p^n}, +_{p^n})$ and the family of these subgroups is ordered by inclusion. Now, the following question impose itself " Is there any SI ring of characteristic p^n different from the ring $(\mathbb{Z}_{p^n}, +_{p^n}, \cdot_{p^n})$ whose additive group is isomorphic to the group $(\mathbb{Z}_{p^n}, +_{p^n})$?". In [14], it is shown that for each prime p and positive integer $n\leq 5$ there is only one ring of characteristic p^n which is $(\mathbb{Z}_{p^n}, +_{p^n}, \cdot_{p^n})$ itself. So that a partial answer for the previous question is given as follows: For each prime p and positive integer $n\leq 5$ the only ring with characteristic p^n is the ring $(\mathbb{Z}_{p^n}, +_{p^n}, \cdot_{p^n})$.

The following result is clear so the proof is omitted.
Proposition 2.14. Let $\mathcal{S}$ be a local ring of order p^n whose maximal ideal is principal. If $\mathcal{S}$ is an SI ring, then for each $1\leq k<n$ the ring $\mathcal{S}$ has at most one ideal of order p^k.

Now, we have to mention that there is a finite ring that is not isomorphic to the ring $\mathbb{Z}_{p^n}$ for each prime p and $n>1$ as it is shown in the following example.

Examples 2.15.
1. Consider the quotient ring $\mathcal{S}=\mathbb{Z}_2[x]/<x^2>=\{0,1,\ x,\ 1+x\}$ with $x^2=0$. The ring $\mathcal{S}$ has three proper ideals $I_1=<0>=\{0\}$, $I_2=<x>=\{0,x\}$, which are ordered by inclusion but it is not isomorphic with the ring $\mathbb{Z}_{p^n}$ for each prime p and a positive integer n.
2. Consider the quotient ring $\mathcal{S}=\mathbb{Z}_2[x]/<x^3>=\{0,1,\ x,x^2,1+x,1+x^2,x+x^2,1+x+x^2\}$ with $x^3=0$. The proper ideals of $\mathcal{S}$ are $I_1=\{\mathbf{0}\}$, $I_2=<x^2>=\{\mathbf{0},x^2\}$, $I_2=<x>=\{\mathbf{0},\ x,\ x^2,\ x+x^2\}$. Note that $\mathcal{S}$ has 2^3 elements, one ideal of order 2 and one ideal of order 2^2. It is not difficult to show that the additive group $(\mathcal{S}, +)$ is isomorphic to the additive group of the ring $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ with usual addition and multiplication. Clearly that $\mathcal{S}$ is an SI ring, and the characteristic of $\mathcal{S}$ is 2.

3. S spectrum of *SI* rings

Let $\mathcal{S}$ be a commutative ring and $X=Spec(\mathcal{S})=\{J: J \text{ is a prime ideal of } \mathcal{S}\}$. Then for any subset E of $\mathcal{S}$, $V(E)$ is defined to be the set $V(E)=\{P: P \text{ is a prime ideal of } \mathcal{S} \text{ and } E\subseteq P\}$ and $X(E)=Spec(\mathcal{S})\backslash V(E)$ [3, p. 731]. In [2], this concept was generalized to the strongly

irreducible spectrum $X = S.spec(\boldsymbol{S})$ of $\boldsymbol{S}$ which is the set $S.spec(\boldsymbol{S}) = \{J: J$ is a strongly irreducible ideal of $\boldsymbol{S}\}$. For any subset E of $\boldsymbol{S}$, the S variety of E, denoted by $V_s(E)$, is the set $V_s(E) = \{J \in S.spec(\boldsymbol{S}): E \subseteq J\}$ and $X_s(E) = \{J \in S.spec(\boldsymbol{S}): E \nsubseteq J\}$. Then by [2, pp. 125, Proposition 3.2], the family F=\{ V_s $(< E >): E \subseteq R\}$ satisfies the axioms for closed sets of a topology on $X = S.spec(\boldsymbol{S})$ [1]. Now, if $\boldsymbol{S}$ is an SI ring, then each of its proper ideals is an SI ideal. Then $S.spec(\boldsymbol{S}) = \{J: J$ is a proper ideal of $\boldsymbol{S}\}$. Consequently, the S variety of E, denoted by $V_s(E)$, is the set $V_s(E) = \{J: J$ is a proper ideal of $\boldsymbol{S}$ and $E \subseteq J$ \} and $X_s(E) = S.spec(\boldsymbol{S})\backslash V_s(E) = \{J: J$ is a proper ideal of $\boldsymbol{S}$ and $I \subset < E >$ \}.
From the references [15], [3] and [16] , the topological concepts are taken which are used in this work such as open set, closed set, clopen set, basis, connected space, locally connected space, hyperconnected space, ultraconnected space, regular space, normal space, T_0 space, T_1 space, variety, spectrum of a ring, sober space, and generic point.

Remark 3.1. Let $\boldsymbol{S}$ be an SI ring. Then
1. $X = S.spec(\boldsymbol{S}) = \{I: I$ is a proper ideal of $\boldsymbol{S}\}$
2. For each subset E of $\boldsymbol{S}$, $V_s(E) = \{J: J$ is a proper ideal of $\boldsymbol{S}$ and $E \subseteq J\}$ equivalently $X_s(E) = \{J: J$ is a proper ideal of $\boldsymbol{S}$ and $J \subset < E >\}$ where $< E >$ is the ideal generated by E.
3. For each ideal I of $\boldsymbol{S}$, $V_s(I) = \{J: J$ is a proper ideal of $\boldsymbol{S}$ and $I \subseteq J\}$ equivalently $X_s(I) = \{J: J$ is a proper ideal of $\boldsymbol{S}$ and $J \subset I\}$.
4. Let B be a basis for the $S.spec(\boldsymbol{S})$ topology. For each subset E of $\boldsymbol{S}$, if $\emptyset \neq X_s(E) \subseteq S.spec(\boldsymbol{S})$ is an open set, then $X_s(E) \in B$.

Proposition 3.2. Let $\boldsymbol{S}$ be an SI ring. Then
1. Every nonempty open subset of $S.spec(\boldsymbol{S})$ contains the zero ideal.
2. Every nonempty closed subset of $S.spec(\boldsymbol{S})$ contains the maximal ideal of $\boldsymbol{S}$.
3. If $< 0 > \in V_s(E)$ for some $E \subseteq \boldsymbol{S}$, then $V_s(E) = S.spec(\boldsymbol{S})$.
4. $< 0 >$ is a generic point in $S.spec(\boldsymbol{S})$.

Proof.
1. Let $X_s(E) \neq \emptyset$ be an open subset of $S.spec(\boldsymbol{S})$. Then there is an ideal J of $\boldsymbol{S}$, $J \in X_s(E)$. By Remark 3.1, $X_s(E) = \{I: I$ is a proper ideal of $\boldsymbol{S}$ and $I \subset < E >\}$, consequently $J \subset < E >$. Since $< 0 > \subseteq J \subset < E >$, then $< 0 > \in X_s(E)$.
2. Let $V_s(E)$ be a nonempty closed subset of $S.spec(\boldsymbol{S})$. Then there is a proper ideal J of $\boldsymbol{S}$ such that $J \in V_s(E)$. By Remark 3.1, $V_s(E) = \{S: S$ is a proper ideal of $\boldsymbol{S}$ and $E \subseteq S\}$, consequently $E \subseteq J$. If J is not the maximal ideal of $\boldsymbol{S}$, then $J \subset M$ where M is the maximal ideal. So that $E \subseteq M$, consequently $M \in V_s(E)$.
3. Let $< 0 > \in V_s(E)$. Then $=< 0 >$. Therefore, $V_s(E) = S.spec(\boldsymbol{S})$.
4. Clearly $\overline{\{< 0 >\}} = \bigcap_{<0> \in V_s(I)} V_s(I)$. Then by part (3), $\overline{\{< 0 >\}} = S.spec(\boldsymbol{S})$. This means that $< 0 >$ is dense in $S.spec(\boldsymbol{S})$. Thus $< 0 >$ is a generic point in $S.spec(\boldsymbol{S})$.

Remark 3.3. Let $\boldsymbol{S}$ be an SI ring. Then the family of all closed subsets of $S.spec(\boldsymbol{S})$ and the family of all open subsets of $S.spec(\boldsymbol{S})$ are well ordered by inclusion.
Proposition 3.4. Let $\boldsymbol{S}$ be an Artinian SI ring. Then
1. The family of all ideals of $\boldsymbol{S}$ constructs an ascending chain of ideals of the form:
$I_0 \subset I_1 \subset \cdots \subset I_{n-1}$.
2. The family of closed subsets of $S.spec(\boldsymbol{S})$, construct an ascending chain of closed subsets of the form: $V_s(I_{n-1}) \subset \cdots \subset V_s(I_1) \subset V_s(I_0)$.
3. The family of open sets of $S.spec(\boldsymbol{S})$, construct an ascending chain of open subsets of the form:
$X_s(I_0) \subset X_s(I_1) \subset \cdots \subset X_s(I_{n-1})$.

Proof.
1. Let $T_0 = \{S_\lambda\}_{\lambda \in \Lambda}$ be the family of all ideals of $\boldsymbol{S}$. Then T_0 has a minimal ideal under inclusion which is unique since the ideals are ordered by inclusion. Suppose that I_0 is the minimal ideal of T_0. For each $k \in \mathbb{Z}^+$, let I_k is the minimal ideal of T_k where $T_k = T_{k-1} \backslash \{I_{k-1}\}$. Then clearly $I_{k-1} \subset I_k$ and there is no ideal J in $T_0 = \{S_\lambda\}_{\lambda \in \Lambda}$ such that $I_{k-1} \subset J \subset I_k$. Consequently, we obtain a chain of ideals $I_0 \subset I_1 \subset I_2 \subset \cdots$ of T_0. Since $\boldsymbol{S}$ is Artinian, there exists $t \in \mathbb{Z}^+$ such that $I_t = I_{t+1} = \cdots$. Therefore, the ideals in T_0 form the following ascending chain of ideals: $I_0 \subset I_1 \subset \cdots \subset I_{n-1}$.
2 and 3 follow from Remark 3.3.
The following Proposition is easy to prove so the proof is omitted.

Proposition 3.5. Let $\boldsymbol{S}$ be an SI ring. Then
1. The space $S.spec(\boldsymbol{S})$ is connected.
2. The space $S.spec(\boldsymbol{S})$ is locally connected.
3. The space $S.spec(\boldsymbol{S})$ is hyperconnected.
4. The space $S.spec(\boldsymbol{S})$ is ultraconnected.

Recall from [12], a space X is zero-dimensional if it has a basis consisting of clopen sets, and it is strongly zero-dimensional if for any closed set A and an open set U containing A, there exists a clopen set V such that $A \subseteq V \subseteq U$.

Proposition 3.6. Let $\boldsymbol{S}$ be an SI ring. Then
1. If $\boldsymbol{S}$ is not a field, then the space $S.spec(\boldsymbol{S})$ is not zero-dimensional.
2. The space $S.spec(\boldsymbol{S})$ is strongly zero-dimensional.

Proof.
1. Let B be a basis of the $S.spec(\boldsymbol{S})$ topology and $X_s(E) \in B$ for some $E \subseteq \boldsymbol{S}$. By Proposition 3.2(1), either $X_s(E) = \phi$ or $< 0 > \in X_s(E)$. This means that the space $S.spec(\boldsymbol{S})$ has only two clopen subsets which are ϕ and $S.spec(\boldsymbol{S})$. Therefore, if $\boldsymbol{S}$ is not a field, then the space $S.spec(\boldsymbol{S})$ is not zero-dimensional.
2. Let $A = V_s(E)$ be closed set contained in an open set $U = X_s(F)$ where $E, F \subseteq \boldsymbol{S}$. If $V_s(E) = \emptyset$, take $V = \emptyset$, then $A \subseteq V \subseteq U$. If $A \neq \emptyset$, then by Proposition 3.2, the maximal ideal M of $\boldsymbol{S}$ belongs to $V_s(E)$. So that $M \in X_s(F)$, consequently $M \subset < F >$. Therefore, $X_s(F) = U = S.spec(\boldsymbol{S})$. Put $V = S.spec(\boldsymbol{S})$, then $\phi \neq A \subseteq V \subseteq U$.

Recall from [9], a ring $\boldsymbol{S}$ is said to be π-regular if for every element $r \in \boldsymbol{S}$ there is some element $x \in \boldsymbol{S}$ such that $r^n x r^n = r^n$ and recall from [3, p. 750], the Krull dimension of a commutative ring $\boldsymbol{S}$ is the maximum possible length of a chain $P_1 \subset P_2 \subset \ldots \subset P_n$ of distinct prime ideals in $\boldsymbol{S}$.

Proposition 3.7. Let $\boldsymbol{S}$ be an SI ring with Krull dimension zero, then
1. $\boldsymbol{S}$ is π-regular.
2. The space $Spec(\boldsymbol{S})$ is zero-dimensional.

Proof.
1. By Proposition 2.8, $\boldsymbol{S}$ is a clean ring. Since $\boldsymbol{S}$ has Krull dimension zero, then by [12, pp. 2670, Corollary 2.8], $\boldsymbol{S}$ is π-regular.
2. This is a direct consequence of [12, pp. 2669, Theorem 2.3].

Proposition 3.8. Let $\boldsymbol{S}$ be an SI ring which is not a field. Then
1. The space S.$spec(\boldsymbol{S})$ is T_0.

2. The space S.$spec(\boldsymbol{S})$ is not T_1.
3. The space S.$spec(\boldsymbol{S})$ is not regular.
4. The space S.$spec(\boldsymbol{S})$ is normal.
5. The space S.$spec(\boldsymbol{S})$ is sober.

Proof.

1. It follows from the definition of a T_0 space.
2. Since $\boldsymbol{S}$ is not a field, then it has at least two proper ideals I and J. Suppose that there are two open sets $X_s(E)$ and $X_s(F)$ such that $I \in X_s(E)$ and $J \in X_s(F)$. By Remark 3.3, $X_s(E) \subseteq X_s(F)$ or $X_s(F) \subseteq X_s(E)$. Therefore, the space S.$spec(\boldsymbol{S})$is not T_1.
3. Since $\boldsymbol{S}$ is not a field, then $M \neq < 0 >$ where M is the maximal ideal of . Clearly $V_s(M) = \{M\}$, then $< 0 > \notin V_s(M)$, but by Remark 3.3, there are no disjoint open subsets $X_s(E)$ and $X_s(F)$ such that $< 0 > \in X_s(E)$ and $M \in X_s(F)$.
4. By Remark 3.3, every two closed subsets of $S.spec(\boldsymbol{S})$ are comparable, then the space S.$spec(\boldsymbol{S})$ is normal.
5. Every closed subset of $S.spec(\boldsymbol{S})$ is irreducible since the only clopen subsets of $S.spec(\boldsymbol{S})$ are $\emptyset$ and $S.spec(\boldsymbol{S})$. Now, let A be a closed subset of $S.spec(\boldsymbol{S})$. So that there is an ideal I of $\boldsymbol{S}$ such that $A = V_s(I) = \{S: S$ is a proper ideal of $\boldsymbol{S}$ and $I \subseteq S\} \neq \emptyset$. Then clearly $\overline{\{I\}} = \bigcap_{I \in V_s(J)} V_s(J) = V_s(I)$. If there is another ideal $J \neq I$ such that $\overline{\{J\}} = V_s(I)$, then $V_s(I) = V_s(J)$. So that $I \in V_s(J)$ and $J \in V_s(I)$ consequently, $I \subseteq J$ and $J \subseteq I$, contradiction. Therefore, the space S.$spec(\boldsymbol{S})$ is sober.

Conclusions

In this paper, the concept of a strongly irreducible ring is introduced. Some properties and characterizations of strongly irreducible ring are given. Relations between such rings and some types of other rings are discussed. It is shown that the ring $= \mathbb{Z}_m$ is an SI ring if and only if $m = p^n$ for some prime p and positive integer n. For an SI ring $\boldsymbol{S}$, the concepts of $V_s(E)$, $X_s(E)$ for a subset E of $\boldsymbol{S}$ and $S.spec(\boldsymbol{S})$ topology are introduced. Some properties of $V_s(E)$ and $X_s(E)$ are discussed. The maximal ideal of $\boldsymbol{S}$ belongs to every nonempty closed subset $V_s(E)$ of $S.spec(\boldsymbol{S})$ and the zero ideal of $\boldsymbol{S}$ belongs to every nonempty open subset $X_s(E)$ of $S.spec(\boldsymbol{S})$. Some properties of $S.spec(\boldsymbol{S})$ topology are investigated. It is shown that the family of all open subsets (resp. closed subsets) of $S.\ s(\boldsymbol{S})$ is well ordered by inclusion. Moreover, $S.spec(\boldsymbol{S})$ topology is T_0, normal and sober but it is neither T_1 nor regular space.

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