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The Homomorphisms of Distributive Semimodules

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Abstract
Many researchers discussed distributive modules and their properties. In this work, the distributive property will be studied over semimodules. Some of the results obtained in homomorphisms of distributive modules are generalized. Some conditions were needed like concept subtractive, i-regular, k-regular, and k-cyclic, to get good results. The relationship between the distributive semimodule over the local semiring and the hollow semimodule was obtained, and the relationship between the distributive semimodule and the homomorphisms distributed over the intersection process or the inverse image distributed over the addition.

Keywords: Distributive semimodule, homomorphisms, subtractive, local semiring, k-cyclic.

التماثل لشبه المقاسات التوزيعية

علي حسن عبدالخالق الحامتي , اسعد محمد علي الحسيني
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الخلاصة
الكثير من البحوث ناقشت المقاسات التوزيعية وخواصه. في هذا العمل ندرس هذه الخاصية بالنسبة لشبه المقاسات. بعض النتائج التي تم الحصول عليها في تماثلات المقاسات التوزيعية تم تعميمها بإضافة بعض الشروط مثل مفهوم (مطروح، i–المنتظم، k–المنتظم، k–الدوري) لنحصل على بعض النتائج الجيدة . العلاقة بين شبه المقاس التوزيعي بالنسبة لشبه حلقه محليه وشبه المقاس المجوف تم الحصول عليها، وعلاقة شبه المقاس التوزيعي وخاصية توزيع التماثلات على عملية التقاطع او توزيع الصورة العكسية على الجمع.

1. Introduction

An R-semimodule U is distributive if $T \cap (Z + F) = (T \cap Z) + (T \cap F)$ for any $T, Z,$ and $F \in L(U)$. This definition was given by Saffar Ardabili in [1] , without more investigation into this property. In this work, this concept will be studied in detail.

In the modules, researchers have been interested since the seventies in the distributive feature like W. Stephenson and Victor Camillo in [2] and [3], respectively. Where it was defined, finding some equivalents for it, and find Some properties of distributive modules, and researching its applications. In [4] P. Vamos found the relationship between distributive

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and cyclic modules. Then the conditions under which it is distributive modules are a direct sum of cyclic submodules by V. Erdogdu in [5]. The researcher A. A. Tuganbaev studied a concept named "distributive extensions "over rings not necessarily commutative [6]. Naser Zamani found the relationship between distributive modules and primal submodules in [7], while Engin Büyükaşık in [8] gave and studied a generalization of distributive property by defining weakly distributive module. The relationship between the distributive module and the Armendariz module was gotten by Yiqiang [9]. In this work, some of these concepts and results will be converted to semimodules. The preliminaries are in section two for the convenience of the reader. In Section three, we study the homomorphisms of distributive semimodules and the relationship between the distributive, local semiring, and artinian semimodule.

2. Preliminaries.

Definition 2. 1. [10]

An R -semimodule U is called uniserial if for any two subsemimodules H and L of U, either $H \subseteq L$ or $L \subseteq H$.

Definition 2.2. [11]

An $R-$ subsemimodule E of R - semimodule U is superfluous if any $K \in \mathrm{L}(U)$, $K + E = U$ imply $K = U$. The R - semimodule U is said to be hollow if every proper $E \in \mathrm{L}(U)$ is superfluous.

Definition 2.3. [11]

An $R -$ subsemimodule E of R - semimodule U is subtractive if for all $m, n \in U$, $m,\ m + n \in E$ implies $n \in E$.

Definition 2.4. [12]

Let R be semiring if it has only one maximal ideal then called local.

Definition 2.5. [13]

Let A and U be R – semimodules the map μ: $A \rightarrow U$ is a homorphism . Then

1- μ is $i -$ regular, if $\mu(A) = Im\,(A)$.

2- μ is $k -$ regular, if $\mu(e) = \mu(\acute{e})$ implies $e + h = \acute{e} + \acute{h}$ for some $h, \acute{h} \in \ker(\mu)$.

3- μ is regular if is *i*-regular, and *k*-regular.

Definition 2.6. [14]

A semimodule U over semiring R is artinian if any non-empty set $M \subseteq L(U)$ has minimal with respect to inclusion. It's mean there are subsemimodule $0 \neq T \in M$, such that if $P \subseteq T$ where $P \in M$ implies either $P = T$ or $P = 0$

Lemma 2.7. [14]

If U is artinian R – semimodule and $A \in L\,(U)$ then $U/_A$ is an artinian.

Lemma 2.8. [13]

Let U, A be R – semimodules and $\mu \in Hom\,(U, A)$ then

i) $\mu(U)$ is subtractive if and only if μ is *i*- regular.

ii) μ is a monomorphism, then μ is *k*- regular.

Lemma 2.9. [14]

Let U, A be R – semimodules and $\mu \in Hom\,(U, A)$. If μ is k-regular and U is subtractive, then $\mu^{-1}\,\mu\,(M) = \ker(\mu) + M$, for all $M \in L(U)$.

3- Main Results

Lemma 3.1

Let U and A be R – semimodules and $\mu \in Hom\,(U, A)$ and $M \in L(A)$, then $\mu\left(\mu^{-1}(M)\right) = M \cap \mu(U)$.

Proof.

Let $y \in M \cap \mu(U)$, then $y = \mu(a) \in M$ for some $a \in U$ so $a \in \mu^{-1} \in M$, hence $\mu(a) = y \in \mu\left(\mu^{-1}(M)\right)$ then $M \cap \mu(\acute{U}) \subset \mu\left(\mu^{-1}(M)\right)$ and it's clear $\mu\left(\mu^{-1}(M)\right) \subset M \cap \mu(U)$, now
we get $\mu\left(\mu^{-1}(M)\right) = M \cap \mu(U)$.

Lemma 3.2

Let U and A be R – semimodules. If ω, and $\mu \in Hom\,(U, A)$ where w is $i-$ regular, then $(\omega + \mu)^{-1}\left(\omega\,(U)\right) = \mu^{-1}\left(\omega(U)\right)$.

Proof

$(\omega + \mu)^{-1}\left(\omega\,(U)\right) = \{\, e \in U: (\omega + \mu)(e) \in \omega(U)\}$

$= \{\, e \in U: \omega(e) + \mu(e) \in \omega(U)\}$

$= \{\, e \in U: \mu(e) \in \omega(U)\}$ because $\omega(U)$ is subtractive by Lemma 2.8

$= \mu^{-1}\left(\omega(U)\right)$.

Proposition 3.3

Let μ be $k-$ regular a homomorphism from a subtractive R – semimodule U to R semimodule A.

i) If A is distributive, then $\mu^{-1}(W + N) = \mu^{-1}(W) + \mu^{-1}(N)$, for any $W, N \in L(A)$.

ii) If U is distributive, then $\mu(T \cap S) = \mu(T) \cap \mu(S)$, for any $T, S \in L(U)$.

Proof

i) Since $\mu^{-1}(W) + \mu^{-1}(N) = \mu^{-1}(W) + \mu^{-1}(N) + Ker\mu$, and by Lemma 2.9, we get

$\mu^{-1}(W) + \mu^{-1}(N) = \mu^{-1}\mu\left(\mu^{-1}(W) + \mu^{-1}(N)\right) = \mu^{-1}\left(\mu\,\mu^{-1}\,(W) + \mu\,\mu^{-1}\,(N)\right)$

by Lemma 3.1, we get $\mu^{-1}(W) + \mu^{-1}(N) = \mu^{-1}\,(W \cap \mu(U) + N \cap \mu(U)\,)$

$= \mu^{-1}[(W + N) \cap \mu(U)] = \mu^{-1}(W + N) \cap \mu^{-1}\left(\mu(U)\right) = \mu^{-1}(W + N) \cap U = \mu^{-1}\,(W + N)$.

ii) Since $\mu(T) \cap \mu(S) = [\mu(T) \cap \mu(S)] \cap \mu(U) = \mu\mu^{-1}(\mu\,(T) \cap \mu(S))$

$= \mu\,[\,\mu^{-1}\,(\mu\,(T) \cap \mu(S)\,)] = \mu\,[\,\mu^{-1}\mu\,(T) \cap \mu^{-1}\mu(S)]$, by Lemma (2.9) we get

$\mu(T) \cap \mu(S) = \mu\,[(\ker\mu + T\,) \cap (\ker\mu + S\,)]$

$= \mu\,(\ker\mu + (\,T \cap S)) = \mu\,(\ker\mu\,) + \mu(\,T \cap S) = 0 + \mu\,(T \cap S) = \mu\,(T \cap S)$.

If μ is a $k-$ regular homomorphism from R – semimodule U to R –semimodule A, $\ker\mu + T$ and $\ker\mu + S$ are subtractive in U, then $T, S \in L(U)$ satisfy the property in (ii).

Proposition 3.4

Let U be R - semimodule, A is a distributive R – semimodule. If $\mu, \omega \in Hom\,(U, A)$ are $i - regular$. Then,

i) $U = \omega^{-1}\,\mu(U) + \mu^{-1}\,\omega(U)$.

ii) $C = C \cap \omega^{-1}\left(\mu(C)\right) + C \cap \mu^{-1}(\,\omega(C))$, for any $C \in L(U)$.

proof
i) since $(\mu + \omega)(U) \subseteq \mu(U) + \omega(U)$, then
$(\mu + \omega)^{-1}[(\mu + \omega)(U)] \subseteq (\mu + \omega)^{-1}[\mu(U) + \omega(U)]$, so $U = (\mu + \omega)^{-1}\left(\mu(U) + \omega(U)\right) = (\mu + \omega)^{-1}\mu(U) + (\mu + \omega)^{-1}\omega(U)$.
By Lemma (3.2)
we get $U = \omega^{-1}\mu(U) + \mu^{-1}\omega(U)$.

ii) Let μi and ωi be homomorphisms such that $i: C \to U$ inclusion map.
Now, apply (i) we get

$$C = (\omega i)^{-1}\left(\mu i(C)\right) + (\mu i)^{-1}\left(\omega i(C)\right)$$

since $(\mu i)^{-1}(E) = E \cap \mu(E)$ for all $E \in L(U)$ and $\mu \in Hom\,(U, A)$,
then $C = C \cap \omega^{-1}\left(\mu(C)\right) + C \cap \mu^{-1}\left(\omega(C)\right)$.

Corollary 3.5
If U is a distributive R - semimodule, $\omega \in End\,(U)$ is $i - regular$, then $B = \left(B \cap \omega(B)\right) + \left(B \cap \omega^{-1}(B)\right)$.

Proof. By using Proposition 3.4 with $\mu = i$ the identity on U.

Proposition 3.6
Let U be subtractive and distributive R -semimodule and B be R -semimodule and $\omega, g \in Hom\,(U, B)$, such that $\omega + g$ is k- regular. Then,
i) $0 = g(\ker\omega) \cap \omega(ker\, g)$.
ii) $C = (C + \omega\left(g^{-1}(C)\right) \cap (C + g\left(\omega^{-1}(C)\right)$, for any and $C \in L(B)$.
proof.
i) Since $(\omega + g)(ker\omega \cap \ker g) = 0$, then by Proposition 3.3 we get that
$(\omega + g)(ker\omega) \cap (\omega + g)(ker g) = 0$, so
$[\omega(\ker\omega) + g(\ker\omega)] \cap [\omega(ker g) + g(\ker g)] = 0$
then $g(\ker\omega) \cap \omega(\ker g) = 0$.

ii) If Π is the natural epimorphism of B onto $B/C \ni \Pi\omega, \Pi g \in Hom\,(U, B/C)$.
Now apply (i) , we get $C = \Pi g(\ker\Pi\omega) \cap \Pi\omega(\ker\Pi g)$
$= \Pi g(\omega^{-1}(C)) \cap \Pi\omega(g^{-1}(C))$, since $\ker\Pi\omega = \omega^{-1}(C)$, and $\ker\Pi\, g = g^{-1}(C)$
$= (C + g\left(\omega^{-1}(C)\right) \cap (C + \omega(g^{-1}(C))$.

Corollary 3.7
Let U be subtractive and distributive R-semimodule and $\omega \in End\,(U)$ with $\omega + i$ is k-regular where i is the identity on U. If $B \in L(U)$, then $B = \left(B + \omega^{-1}(B)\right) \cap \left(B + \omega(B)\right)$.
Proof.
By using Proposition (3.6) we get that $B = \left(B + \omega(i^{-1}(B)\right) \cap (B + i(\omega^{-1}(B)))$.
Since $i^{-1}(B) = B$, and $i(\omega^{-1}(B)) = \omega^{-1}(B)$, then $B = \left(B + \omega^{-1}(B)\right) \cap \left(B + \omega(B)\right)$.

Proposition 3.8

Let U be subtractive and distributive R -semimodule, $B \in L\ (U)$ and $\omega \in End\ (U)$ is $k-$ regular.

i) If ω is $i-regular$ then $B = \left(B \cap \omega^{-1}(B)\right) + \omega\ (B\ \cap\ \omega^{-1}(B))$.

ii) If $\omega + i$ is k – regular where that i is identity then $B = \left(B + \omega(B)\right) \cap \omega^{-1}\ (\omega(B) + B)$.

Proof.

i) By Corollary 3.5, we get $B = \left(B \cap \omega^{-1}(B)\right) + (B \cap \omega(B))$
$= \left(B \cap \omega^{-1}(B)\right) + (B \cap [\omega(U) \cap \omega(B)]\ ) = \left(B \cap \omega^{-1}(B)\right) + (B \cap \omega(U) \cap \omega(B)\ )$
$= \left(B \cap \omega^{-1}(B)\right) + \left(\omega\left(\omega^{-1}(B)\right) \cap \omega(B)\ \right)$, and by Proposition 3.3, we get B $= B \cap \omega^{-1}(B) + \omega(\omega^{-1}(B) \cap B\ )$.

ii) By Corollary 3.7, we get $B = \left(B + \omega(B)\right) \cap \left(B + \omega^{-1}(B)\right)$, then
B $= \left(B + \omega(B)\right) \cap \left(B + \omega^{-1}(0 + B)\right)$. Now, by using Proposition 3.3, we have
B $= \left(B + \omega(B)\right) \cap \left(B + \omega^{-1}(0) + \omega^{-1}(B)\right)$
$= \left(B + \omega(B)\right) \cap [(B + \ker \omega) + \omega^{-1}(B)]$
$= \left(B + \omega(B)\right) \cap \left(\omega^{-1}\omega(B) + \omega^{-1}(B)\right)$
$= \left(B + \omega(B)\right) \cap \omega^{-1}(\omega(B) + B)$.

Corollary 3.9

Let U be subtractive and distributive R -semimodule, $B \in L(U)$, suppose that $\omega \in End\ (U)$ and ω is regular, then

i) $\omega^{-1}(B) \cap \omega(B) \subseteq B \subseteq \omega^{-1}(B) + \omega(B)$.

ii) $B \cap \omega^2(B) \subseteq \omega(B) \subseteq B + \omega^2(B)$.

Proof. (i)

Let $B \in L(U)$ and $\omega \in End(U)$, since $B \cap \omega^{-1}(B) \subseteq B$, hence $\omega\left(B \cap \omega^{-1}(B)\right) \subseteq \omega(B)$, then, by Proposition(3.8) we get $B = \left(B \cap \omega^{-1}(B)\right) + \omega\ \left(B\ \cap\ \omega^{-1}(B)\right)$, so $B \subseteq \left(B \cap \omega^{-1}(B)\right) + \omega(B)$ hence $B \subseteq \omega^{-1}(B) + \omega(B)$.

Also $\omega^{-1}(B) \cap \omega(B) \subseteq B$ when ω is k- regular, since $\omega^{-1}(B)\ \subseteq \omega^{-1}\left(B + \omega(B)\right)$ and $\omega(B) \subseteq B + \omega(B)$ then $\omega^{-1}(B) \cap \omega(B) \subseteq \omega^{-1}\left(B + \omega(B)\right) \cap\ B + \omega(B)$. Now by Proposition (3.8) we get $\omega^{-1}(B) \cap \omega(B) \subseteq B$.

(ii)

By (i) we get $\omega^{-1}(B) \cap \omega(B) \subseteq B \subseteq\ \omega^{-1}(B) + \omega(B)$
$\omega(\omega^{-1}(B) \cap \omega(B)) \subseteq \omega(B) \subseteq\ \omega(\omega^{-1}(B) + \omega(B))$
$\omega(\omega^{-1}(B))\ \cap \omega^2(B) \subseteq \omega(B) \subseteq\ \omega(\omega^{-1}(B)) + \omega^2(B)$
$B \cap \omega^2(B) \subseteq \omega(B) \subseteq B \cap \omega(B) +\ \omega^2(B)$
$B \cap \omega^2(B) \subseteq \omega(B) \subseteq B \cap \omega(B) +\ \omega^2(B) \subseteq B + \omega^2(B)$
$B \cap \omega^2(B) \subseteq \omega(B) \subseteq B + \omega^2(B)$.

If U is a subtractive and distributive semimodule and $\mu\ \in End\ (U)$ is regular then by Proposition (3.8) any subsemimodule $B \in L(U)$, can be written in form $B = T\ \cap \mu^{-1}(T)$ and if $\mu, \omega + i$ are k- regular then by Proposition (3.8) can be written in form $B = W + \mu(W)$ for some $T, W \in L(U)$. These representations are unique as we shall see in the following corollary.

Corollary 3.10

Suppose that W, T are subsemimodules of a distributive and subtractive semimodule U and $\mu \in End\ (U)$.

i) If μ and $\mu + i$ are k- regular and $W + \mu(W) = T + \mu(T)$ then $W = T$.

ii) If μ is regular and $W \cap \mu^{-1}(W) = T \cap \mu^{-1}(T)$ then $W = T$.

Proof.

i) By Proposition 3.8, we get

$$W = (W + \mu(W)) \cap \mu^{-1}(W + \mu(W)) = (T + \mu(T)) \cap \mu^{-1}(T + \mu(T)) = T.$$

ii) Follows similarly.

Corollary 3.11

Let U be a distributive semimodule and $N \in L\ (U)$ and $ȡ \in End\ (U)$.

i) If $ȡ, ȡ + i$ is k- regular and $ȡ^m(N) \subseteq \sum_{i=0}^{m-1} ȡ^i(N)$ for some $m \geq 1,$ then $ȡ(N) \subseteq N$.

ii) If $ȡ$ is regular and $\bigcap_{i=0}^{m-1} ȡ^{-i}(N) \subseteq ȡ^{-m}(N)$ for some $m \geq 1,$ then $ȡ(N) \subseteq N$.

Proof.

i) if m=1 is trivial, we assume that the case is true when m, by induction, we prove that is true when m+1, suppose that

$ȡ^{m+1}(N) \subseteq \sum_{i=0}^{m} ȡ^i(N)$ and define $N_m = \sum_{i=0}^{m} ȡ^i(N)$.

Then

$$ȡ(N_{m-1}) = ȡ(N) + ȡ^2(N) + \cdots + ȡ^m(N).$$

$$N_{m-1} + ȡ(N_{m-1}) = N + ȡ(N) + ȡ^2(N) + \cdots + ȡ^m(N) = N_m.$$

Now

$$ȡ^{m+1}(N) \subseteq \sum_{i=0}^{m} ȡ^i(N) = N_m .$$

$$ȡ(N_m) = N_m + ȡ^{m+1}(N) \subseteq N_m + N_m$$

$$then\ ȡ(N_m) \subseteq N_m$$

So, $N_m = N_m + ȡ(N_m)$ .Then $N_m = N_{m-1}$. By Corollary3.10(i)

$$\text{then } ȡ^n(N) \subseteq N_m \subseteq N_{m-1} = \sum_{i=0}^{n-1} ȡ^i(N)$$

$$ȡ^n(N) \subseteq \sum_{i=0}^{n-1} ȡ^{-i}(N) \text{ by hyopesis then } ȡ\ (N) \subseteq N.$$

ii) Also we can use induction on m.

Define $N_m = \bigcap_{i=0}^{m} ȡ^{-i}(N)$ and suppose that $\bigcap_{i=0}^{m} ȡ^{-i}(N) \subseteq ȡ^{-m-1}(N)$

now

$$ȡ^{-1}(N_{m-1}) = ȡ^{-1}(N) \cap ȡ^{-2}(N) \cap \ldots \cap ȡ^{-m}(N)$$

$$N_{m-1} \cap ȡ^{-1}(N_{m-1}) = N \cap ȡ^{-1}(N) \cap ȡ^{-2}(N) \cap \ldots \cap ȡ^{-m}(N) = N_m.$$

$$\text{then } N_m = N_{m-1} \cap ȡ^{-1}(N_{m-1}) \ldots .*$$

Since $N_m = \bigcap_{i=0}^{m} ȡ^{-i}(N) \subseteq ȡ^{-m-1}(N),$

$$N_m = N_m \cap N_m \subseteq ȡ^{-m-1}(N) \cap N_m$$

$$N_m \subseteq N \cap [\, ȡ^{-1}(N) \cap ȡ^{-2}(N) \cap \ldots \cap ȡ^{-m}(N) \cap ȡ^{-m-1}(N)]$$

$$N_m \subseteq N \cap ȡ^{-1}(N_m)$$

$$\text{implies that } N_m \subseteq N \cap ȡ^{-1}(N_m) \ldots **$$

using Corollary (3.10). on * and ** we get $N_m = N_{m-1}$

so $\bigcap_{i=0}^{m-1} ȡ^{-1}(N) = N_{m-1} = N_m \subset ȡ^{-n}(N)$. By induction, then $ȡ\ (N) \subseteq N$.

Proposition 3.12

If U and B are semimodules and $ȡ, g \in Hom\,(U,B)$, such that ȡ and g are $i-regular$. If U is hollow and B a distributive then either $ȡ(U) \subseteq g(U)$ or $g(U) \subseteq ȡ(U)$.

Proof.

By Proposition (3.4) we get $U = g^{-1}ȡ(U) + ȡ^{-1}g(U)$ also, by hypothesis we have U is hollow, hence $g^{-1}ȡ(U) = U$ or $ȡ^{-1}g(U) = U$.

If $g^{-1}ȡ(U) = U$ implies that $g(\,g^{-1}ȡ(U)) = g(\,U)$, since $g\,g^{-1}ȡ(U) \subseteq ȡ(\,U)$ therefore $g(U) \subseteq ȡ(U)$. Similarly when $ȡ^{-1}g(U) = U$ we get $ȡ(U) \subset g(U)$.

In order to get some results, a condition weaker than subtractive semimodule is needed, that is, only some kind of subsemimodules to be subtractive. In the following such condition will be introduced. A semimodule is k- cyclic if any cyclic subsemimodule of it is subtractive. For example the ℕ-semimodule ℕ is k-cyclic which is not subtractive.

Corollary 3.13

If R is a local semiring and B is a k-cyclic distributive R - semimodule, then B is uniserial semimodule.

Proof.

Since R is local then every proper subsemimodule of the R - semimodule is superfluous, putting $U = R$ in Proposition (3.12) then $ȡ(R) \subset g(R)$ or $g(R) \subset ȡ(R)$ for any $ȡ, g \in Hom\,(R,B)$.

Now let I, W any two subsemimodules of B and $I \nsubseteq W$, this mean $\exists\, x \in I$ and $x \notin W$ and define ȡ and g such that $ȡ(\mathrm{r}) = rx, \forall r \in R$ and $\forall\, y \in W$, $g(\mathrm{r}) = ry, \forall r \in R$.

By Lemma (2.9) we get ȡ and g are $i-$ regular and by Proposition (3.12) we get $Ry \subset Rx, \forall\, y \in W$ implies that $W \subseteq Rx$ hence $W \subseteq I$.

Corollary 3.14

Let R be local semiring and R is a k-cyclic semimodule over R, then R is a distributive semimodule if and only if the R-semimodule R is uniserial.

Proof.

For the first direction is verified by Corollary 3.13. The other direction its clear any uniserial is distributive semimodule.

Theorem 3.15

Let U and B be semimodules and $ȡ, g \in Hom\,(U,B\,)$ when U is a subtractive and artinian , B is distributive, and $ȡ + g$ is k-regular. If $\ker ȡ \subseteq ker\ g$, then $g\,(U) \subseteq ȡ\,(U)$.

Proof.

Let $\ker ȡ \subseteq \ker g$ and $g\,(U) \nsubseteq ȡ\,(U), \Omega = \{\,N \in L\,(U) : g(N) \subseteq ȡ\,(N)\}$
define $M = \sum N\,, s.t\ N \in \Omega$,

$$g\,(M) = g\,(\sum N) = \sum g(N) \subseteq \sum ȡ(N) = ȡ\,(M)$$

then M the largest subsemimodule of U in Ω, and M proper in U because of $g\,(U) \subseteq ȡ\,(U)$, Now $0 = g\,(\ker g\,) \subseteq ȡ\,(ker g\,)$,then $\ker g \in \Omega\,, \Omega \neq \varnothing$.

By hypothesis U is artinian then $U/_M$ is artinian, so $U/_M$ has a minimal subsemimodule say $C/_M$ implies that $C/_M \neq 0$ and $C/_M$ is simple . Since $M \subseteq C \subseteq U$, then $g(M) \subseteq ȡ\,(M) \subseteq ȡ\,(C)$, and $M \subseteq g^{-1}\big(g\,(M)\big) \subseteq g^{-1}\big(ȡ\,(M)\big) \subseteq g^{-1}\big(ȡ\,(C)\big)$, implies that $M \subseteq g^{-1}\big(ȡ\,(C)\big)$ and $M \subseteq C$, then $M \subseteq C \cap g^{-1}\big(ȡ\,(C)\big) \subseteq C$ and $[C \cap g^{-1}\big(ȡ\,(C)\big)]/M \subseteq$

$C/_M$. Since $C/_M$ is simple, then $[C \cap g^{-1}(ȡ(C))]/M = \bar{0}$, hence $C \cap g^{-1}(ȡ(C)) = M$. By Proposition (3.6) we get $C = (C \cap g^{-1}ȡ(C)) + (C \cap ȡ^{-1}g(C)$. Now,

$ȡ(C) = ȡ(C \cap g^{-1}ȡ(C)) + ȡ(C \cap ȡ^{-1}g(C))$
$\subseteq ȡ(M) + (ȡ(C) \cap ȡ\,ȡ^{-1}g(C)) = ȡ(M) + [ȡ(C) \cap (g(C) \cap ȡ(A))]$
$= ȡ(M) + [ȡ(C) \cap ȡ(A) \cap g(C)] = ȡ(M) + [ȡ(C) \cap g(C)]$
$= ȡ(M) + [ȡ(C) \cap g(C) \cap g(A)] = ȡ(M) + g\,g^{-1}[ȡ(C) \cap g(C)]$
$= ȡ(M) + g\,[g^{-1}ȡ(C) \cap g^{-1}g(C)] = ȡ(M) + g\,[g^{-1}ȡ(C) \cap (C + \ker g)]$
$= ȡ(M) + g\,[g^{-1}ȡ(C) \cap C] = ȡ(M) + g(M) = ȡ(M)$
then $ȡ(C) \subseteq ȡ(M)$ so $ȡ(C) = ȡ(M)$
implies that $ȡ^{-1}ȡ(C) = ȡ^{-1}ȡ(M)$ hence $C + \ker ȡ = M + \ker ȡ$
and C = M [Since $\ker ȡ \subseteq M \subseteq C$] this is a contradiction our assumption is false, hence $g(U) \subseteq ȡ(U)$. In addition if U and B be semimodules and $ȡ, g \in Hom(U, B)$ when U is artinian, B is distributive, and $ȡ + g$ is $k-$regular if $\ker ȡ = \ker g$ then $g(U) = ȡ(U)$.

Corollary 3.16

Let U be artinian distributive and subtractive semimodule and $H \in L(U), f: H \to U$ monomorphism then $f(H) = H$.

Proof.

Since U artinian then H is artinian, and if i: $H \to U$ is the inclusion map, then $f + i$ is a monomorphism .

By Lemma (2.9) we get $f + i$ is $k-$regular and by monomorphism $ker\ i = 0$, also $ker\ f = 0$.

By Theorem 3.15, we get $f(H) = i(H) = H$.

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