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New Classes of Filters in Delta−Algebras

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Abstract

The main objective of this topic is to define new types of filters in δ −algebra, such as δ −filter and complete δ −filter. Some theorems that establish the link between these filters were also presented and validated. Any δ −filter in δ −algebra is shown it is a complete δ −filter. Furthermore, regular δ −algebra is demonstrated as a new class of δ −algebra. Additionally, this study introduces two new concepts such as complete δ −ideal and δ −normal. Also, any δ −normal in regular δ −algebra is demonstrated that it is a δ −subalgebra. In addition, the connections between the complete δ −ideal and the δ −subalgebra is investigated. However, some instances are provided to clarify and bolster our results and findings.

Keywords: δ −algebra, complete δ – filters, δ −ideals, regular δ −algebra, δ −subalgebra.

فئات جديدة من المرشحات في جبر – دلتا

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الخلاصة:

الهدف الرئيسي من هذه الدراسة هو تقديم انواع جديد من المرشحات في جبر – دلتا مثل مرشحات – دلتا و مرشحات – دلتا الكاملة , كذلك بعض النظريات تم تقديمها التي تحدد العلاقة بين هذه المرشحات و التي تم توضيحها و التحقق منها , كذلك تم توضيح ان أي مرشحة – دلتا في جبر – دلتا هي مرشحة – دلتا الكاملة , بالإضافة الى ذلك تم تقديم و دراسة مفهوم جبر – دلتا الطبيعي كفئة جديدة من جبر – دلتا, كذلك في هذه الدراسة تم تقديم مفهومين جديدين هما مثالية – دلتا الكاملة و الطبيعي–دلتا , كذلك تم توضيح ان الطبيعي –دلتا في جبر – دلتا المنتظم يكون الجبر الجزئي– دلتا بالإضافة إلى ذلك ، تم تحديد العلاقة بين الجبر–دلتا الكامل و الجبر الجزئي–دلتا كذلك ، تم تقديم بعض الامثلة لتوضيح وتعزيز نتائجنا.

1. Introduction:

In the framework of bounded implicative BCK-algebra, E. Y. Deeba produced quotient algebra by applying a filter in 1980 [1]. The BCK-algebra class is a legitimate subclass of the BCI-algebra class, as is well known. In BCK-algebra, Meng [2] created the idea of a BCK-filter in 1996. The idea of d-algebra, a helpful extension of BCK-algebra, was put forth in 1999 by J. Naggers and H. S. Kim [3].

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The ideal theory is a key principle in d-algebra. The idea of d-ideals is therefore introduced by Naggers, Jun, and Kim [4]. In 2017 [5], Mahmood and Abud Alradha introduced the idea of generation permutation topological $\rho-$algebra and characterizations of $\rho-$algebra using symmetric group permutation. The idea of $\delta-$algebra is then introduced [6]. These classes of of $\rho/\delta-$algebra and others, which have been studied in great detail, are currently of interest to many academics who also research their applications in soft, fuzzy, intuitionistic, permutations, and other contexts [7-12].

The structures of this work, in section 2, the background of some basic notions are recalled. Next, in section 3, the main objective of this topic is to define two novel filter types: the $\delta-$filter and complete $\delta-$filter. We also discussed and proved a few theorems that show the relationship between these concepts.

2. The Basic Definitions

The following definitions were applied to arrive at the information and characteristics acquired in this section.

Definition 2.1: [6]

Let $\mho \neq \emptyset$ be a set with a constant 0, and a binary operation $*$.The system $(\mho, *, 0)$ is a $\delta-$algebra if such that:

i. $t * t = 0, \forall t \in \mho$.
ii. $0 * t = 0$,
iii. $t * r = 0$ and $r * t = 0$ imply that $t = r, \forall t, r \in \mho$.
iv. $t * r = r * t \neq 0, \forall\ t \neq r \in \mho - \{0\}$
v. $(t * (t * r)) * (r * t) = 0, \forall\ t \neq r \in \mho - \{0\}$.

Definition 2.2 [6]

Let $\emptyset \neq S \subseteq \mho$, where $(\mho, *, 0)$ is δ-algebra. The set S is said to be $\delta-$subalgebra of $\mho$ if $t * r \in S$ for any $t, r \in S$.

Definition 2.3 [6]

For any $\emptyset \neq S \subseteq \mho$, where $(\mho, *, 0)$ is $\delta-$algebra. The set S is $\delta-$ideal of $\mho$ if satisfies:

(i) $t, r \in S \Longrightarrow t * r \in S$.
(ii) $t * r \in S$ & $r \in S \Longrightarrow t \in S$.

3- $\delta-$ filter and complete $\delta-$ filter

This section examines concept $\delta-$filter, complete $\delta-$filter, and their interrelationships.

Definition (3.1) :

Let $S \neq \emptyset$ be a subset of a $\delta-$algebra is called a $\delta-$filter if.

i) $0 \in S$
ii) $(\sigma^* * \varkappa)^* \in S, \varkappa \in S \Longrightarrow \sigma \in S$, where $\sigma^* = \sigma * 0, \forall \sigma \in S$.

Example (3.2) : Assume that $\mho = \{0, \mathcal{L}, \hbar, g\}$ and define $*$ on $\mho$ by the Table (1).

Table 1: $(\mho, \circledast, 0)$ is $\delta -$algebra

$\circledast$	0	$\mathcal{L}$	$\hbar$	$\mathscr{g}$
0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$
$\hbar$	$\hbar$	$\mathcal{L}$	0	$\mathcal{L}$
$\mathscr{g}$	$\mathscr{g}$	$\mathcal{L}$	$\mathcal{L}$	0

Hence $(\mho, \circledast, 0)$ is $\delta -$algebra and $S_1 = \{0, \hbar, \mathscr{g}\}$ is $\delta -$filter. But, in another side $S_2 = \{0, \mathcal{L}\}$ is not $\delta -$ filter, since $(\hbar^* \circledast \mathcal{L})^* = \mathcal{L} \in S_2$ but $\hbar \notin S_2$.

Proposition (3.3): If $\{\mathcal{M}_j, \mathrm{j} \in \propto\}$ is a collection of $\delta -$filter in $\delta -$algebra.Then their intersection is a $\delta -$ filter.

Proof:
Suppose that $\{\mathcal{M}_j, \mathrm{j} \in \propto\}$ is a family of $\delta -$filter in $\delta -$algebra $\mho$. Then $0 \in \mathcal{M}_\mathrm{j}, \forall\ \mathrm{j} \in \propto$, and so $0 \in \cap_{\mathrm{j}\in\propto} \forall \mathcal{M}_\mathrm{j}$. If $(\sigma^* \circledast \varkappa)^* \in \cap_{\mathrm{j}\in\propto} \mathcal{M}_j, \varkappa \in \cap_{\mathrm{j}\in\propto} \mathcal{M}_\mathrm{j}$, then $(\sigma^* \circledast \varkappa)^* \in \mathcal{M}_j$, $\mathrm{h} \in \mathcal{M}_\mathrm{j}, \forall\ \mathrm{j} \in \propto$. Since $\mathcal{M}_\mathrm{j}$ is a $\delta -$ filter,$\forall\ \mathrm{j} \in \propto$. Hence $\sigma \in \mathcal{M}_\mathrm{j}, \forall \mathrm{j} \in \propto$. Then $\sigma \in \cap_{\mathrm{j}\in\propto} \mathcal{M}_\mathrm{j}$.■

Remarks (3.4):
It is not necessary the union of $\delta -$filter and $\delta -$subalgebra give us $\delta -$filter, see example (3.5).

Example (3.5): Assume that $\mho = \{0, \mathcal{L}, \hbar, \mathscr{g}, \eth, \alpha\}$ and define $\circledast$ on $\mho$ by the Table (2).

Table 2: $S_1 \cup S_2$ is not $\delta -$ filter

$\circledast$	0	$\mathcal{L}$	$\hbar$	$\mathscr{g}$	$\eth$	α
0	0	0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\hbar$	$\mathscr{g}$	$\eth$	α
$\hbar$	$\hbar$	$\hbar$	0	$\mathscr{g}$	$\eth$	α
$\mathscr{g}$	$\mathscr{g}$	$\mathscr{g}$	$\mathscr{g}$	0	$\eth$	α
$\eth$	$\eth$	$\eth$	$\eth$	$\eth$	0	α
α	α	α	α	α	α	0

Hence $(\mho, \circledast, 0)$ is $\delta -$ algebra $\mho$. Also, $S_1 = \{0, \mathcal{L}\}$ is $\delta -$ filter and $S_2 = \{0, \alpha\}$ is $\delta -$subalgebra. However $S_1 \cup S_2 = \{0, \mathcal{L}, \alpha\}$ is not $\delta -$ filter, since $(\mathscr{g}^* \circledast \alpha)^* = \alpha \in S_1 \cup S_2$ but $\mathscr{g} \notin S_1 \cup S_2$.

Proposition (3.6):
Let $(\mho, \circledast, 0)$ be a $\delta -$algebra and S be $\delta -$ideal of $\mho$. Then S is $\delta -$filter, if $r \circledast 0 = r, \forall r \in \mho$.
Proof:
Assume S is $\delta -$ideal of $\mho$. Then $S \neq \emptyset$ and there is an element $r \in S$, for some $r \in \mho$. Since S is $\delta -$ideal thus $r \circledast r \in S$ [by Definition (2.3)-(i)], but $0 = r \circledast r$ [by Definition (2.1)-(i)]. Then $0 \in S$. Now, let $(\sigma^* \circledast \varkappa)^* \in S$ and $\varkappa \in S$. We consider that $(\sigma^* \circledast \varkappa)^* = \big((\sigma \circledast 0) \circledast \varkappa\big) \circledast 0 = (\sigma \circledast \varkappa) \circledast 0 = (\sigma \circledast \varkappa) \in S$, but $\varkappa \in S$. Therefore $\sigma \in S$ [by Definition (2.3)-(ii)]. Hence S is $\delta -$ filter ■

Definition (3.7):
Let $S \neq \emptyset$ be a subset of a δ −algebra is called a complete δ −filter c (c − δ − filter) if
i) $0 \in S$
ii) $(\sigma^* \circledast \varkappa^*)^* \in S, \forall \varkappa \in S \Longrightarrow \sigma \in S$

Example (3.8):
Assume that $\mho = \{0, \mathcal{L}, \hbar, g, ð\}$ and define $\circledast$ on $\mho$ by Table (3).

Table 3: S_1 is (c − δ − filter) and S_2 is not.

$\circledast$	0	$\mathcal{L}$	$\hbar$	g	$ð$
0	0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\hbar$	$\hbar$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$
g	$ð$	$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$
$ð$	0	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	0

Hence $(\mho, \circledast, 0)$ is δ − algebra. Also, $S_1 = \{0, g, ð\}$ is (c − δ − filter). In another side, we have $S_2 = \{0, g\}$ is not (c − δ − filter) since $(ð^* \circledast 0^*)^* = 0 \in S_2$ and $(ð^* \circledast g^*)^* = 0 \in S_2$ but $ð \notin S_2$.

Proposition (3.9):
Any δ − filter in δ − algebra S is a (c − δ − filter).
Proof:
Let S be δ − filter in δ − algebra $\mho$ and $(\sigma^* \circledast \varkappa^*)^* \in S, \forall \varkappa \in S$. Since S is δ − filter, then $\sigma \in S$. Then S is (c − δ − filter).■

Remark (3.10):
The generic opposite of proposition (3.9) cannot be valid, as shown by the following example.

Example (3.11): Suppose that $\mho = \{0, \mathcal{L}, \hbar, g, ð, \alpha\}$ and define $\circledast$ by Table (4).

Table 4: S is (c − δ − filter) but not δ − filter.

$\circledast$	0	$\mathcal{L}$	$\hbar$	g	$ð$	α
0	0	0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\hbar$	g	$ð$	α
$\hbar$	$\hbar$	$\hbar$	0	g	$ð$	α
g	0	g	g	0	$ð$	α
$ð$	$\hbar$	$ð$	$ð$	$ð$	0	α
α	$ð$	α	α	α	α	0

It is clear that $(\mho, \circledast, 0)$ is δ − algebra and $S = \{0, g\}$ is (c − δ − filter) but is not δ − filter since $(ð^* \circledast g)^* = 0 \in S$ and $ð \notin S$.

Definition (3.12):
For any $\emptyset \neq S \subseteq \mho$, where $(\mho, \circledast, 0)$ is δ −algebra. The set S is complete δ − ideal of $\mho$ if satisfies:
(i) $t, r \in S \Longrightarrow t \circledast r \in S$.

(ii) $t \circledast r \in S, \forall\, r \in S \ni r \neq 0 \Longrightarrow t \in S$.

Example (3.14):
Assume that $\mho = \{0, \mathcal{L}, \hbar, g\}$ and define $\circledast$ on $\mho$ by Table (5).

Table 5: S is complete $\delta -$ ideal.

$\circledast$	0	$\mathcal{L}$	$\hbar$	g
0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$
$\hbar$	$\hbar$	$\mathcal{L}$	0	$\mathcal{L}$
g	g	$\mathcal{L}$	$\mathcal{L}$	0

Hence $(\mho, \circledast, 0)$ is $\delta -$ algebra and $S = \{0, g\}$ is complete $\delta -$ ideal.

Lemma (3.15):
Let $(\mho, \circledast, 0)$ be $\delta -$ algebra. Then every $\delta -$ ideal is complete $\delta -$ ideal.
Proof:
Let $(\mho, \circledast, 0)$ be $\delta -$ algebra and S be $\delta -$ ideal. Then condition (i) in Definition (3.12) is hold [since S is $\delta -$ ideal]. Also, if $t \circledast r \in S, r \in S \ni r \neq 0$, thus $t \in S$ [From Definition (2.3)-(ii)]. So, condition (ii) in Definition (3.12) is hold. Hence S is a complete $\delta -$ ideal.

Remark (3.16):
It is not necessary any complete $\delta -$ ideal is $\delta -$ ideal.

Example (3.17):
Assume that $\mho = \{0, \mathcal{L}, \hbar, g\}$ and define $\circledast$ on $\mho$ by the Table (6).

Table 6: S is complete $\delta -$ ideal

$\circledast$	0	$\mathcal{L}$	$\hbar$	g
0	0	0	0	0
$\mathcal{L}$	g	0	$\mathcal{L}$	$\mathcal{L}$
$\hbar$	$\hbar$	$\mathcal{L}$	0	$\mathcal{L}$
g	g	$\mathcal{L}$	$\mathcal{L}$	0

Hence $S = \{0, g\}$ is complete $\delta -$ ideal, but it is not $\delta -$ ideal, since $\mathcal{L} \circledast 0 = g \in S, 0 \in S$, but $\mathcal{L} \notin S$.

Remark (3.18):
Let $(\mho, \circledast, 0)$ be $\delta -$ algebra and S be $\delta -$ subalgebra, then $0 \in S$. Since $r \circledast r = 0 \in S$, for any $r \in S$ [since S is $\delta -$ subalgebra].

Remark (3.19):
It is clearly any $\delta -$ ideal is $\delta -$subalgebra.

Lemma (3.20):
Every complete $\delta -$ ideal is $\delta -$subalgebra.

Proof:
Let S be a complete $\delta -$ ideal and let $r, t \in S$, then from Definition (3.12)-(i), we have $r \divideontimes t \in S$. Then S is $\delta -$subalgebra.

Remark (3.21):
It is not necessary any $\delta -$ subalgebra is complete $\delta -$ ideal.

Example (3.22):
Assume that $\mho = \{0, \mathcal{L}, \hbar, g\}$ and define $\divideontimes$ on $\mho$ by the Table (7).

Table 7: S is $\delta -$subalgebra but not complete $\delta -$ ideal.

$\divideontimes$	0	$\mathcal{L}$	$\hbar$	g
0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$
$\hbar$	$\hbar$	$\mathcal{L}$	0	$\mathcal{L}$
g	g	$\mathcal{L}$	$\mathcal{L}$	0

Here $S = \{0, \mathcal{L}\}$ is $\delta -$subalgebra but not a complete $\delta -$ ideal, since $\hbar \divideontimes \mathcal{L} = \mathcal{L} \in \mathfrak{H}, \hbar \notin S$.

Definition (3.23):
For any $\emptyset \neq S \subseteq \mho$, where $(\mho, \divideontimes, 0)$ is $\delta -$algebra. The set S is $\delta -$ normal of $\mho$ if any $t \divideontimes r$, $\mathfrak{p} \divideontimes \mathfrak{q} \in S$ implies $(t \divideontimes \mathfrak{p}) \divideontimes (r \divideontimes \mathfrak{q}) \in S$. The following example explains the definition above.

Example (3.24):
Let $\mho = \{0, \mathcal{L}, \hbar, g, ð, \alpha, \ell\}$ be a set with the following table:

Table 8: S is $\delta -$ normal

$\divideontimes$	0	$\mathcal{L}$	$\hbar$	g	$ð$	α	ℓ
0	0	0	0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\hbar$	$\hbar$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
g	$ð$	$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$ð$	0	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$	$\mathcal{L}$
α	α	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	0	$\mathcal{L}$
ℓ	ℓ	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$	0

It is clear $(\mho, \divideontimes, 0)$ is $\delta -$ algebra and $S = \{0, \mathcal{L}\}$ is $\delta -$ normal

Definition (3.25):
Assume that $(\mho, \divideontimes, 0)$ is a $\delta -$algebra. The system $(\mho, \divideontimes, 0)$ is called a regular $\delta -$ algebra if $t \divideontimes 0 = t$, for any $t \in \mho$.

Example (3.26):
Suppose that $\mho = \{0, \mathcal{L}, \hbar, g, ð, \alpha\}$ and define $\divideontimes$ as Table (9)

Table 9: $(\mho, \circledast, 0)$ is a regular $\delta -$ algebra.

$\circledast$	0	$\mathcal{L}$	$\hbar$	$\mathcal{g}$	ð	α
0	0	0	0	0	0	0
$\mathcal{L}$	$\mathcal{L}$	0	$\hbar$	$\mathcal{g}$	ð	α
$\hbar$	$\hbar$	$\hbar$	0	$\mathcal{g}$	ð	α
$\mathcal{g}$	$\mathcal{g}$	$\mathcal{g}$	$\mathcal{g}$	0	ð	α
ð	ð	ð	ð	ð	0	α
α	α	α	α	α	α	0

Hence $(\mho, \circledast, 0)$ is a regular $\delta -$ algebra.

Lemma (3.27):

Every $\delta -$ normal in regular $\delta -$ algebra is $\delta -$ subalgebra.

Proof:

Let $(\mho, \circledast, 0)$ be a regular $\delta -$ algebra and S be $\delta -$ normal of $\mho$. For any $\hbar, \mathcal{g} \in S \subseteq \mho$, we have $\hbar = \hbar \circledast 0$ and $\mathcal{g} = \mathcal{g} \circledast 0$ (since $\mho$ is regular $\delta -$ algebra). So, $\hbar \circledast 0 \in S$ and $\mathcal{g} \circledast 0 \in S$ and hence $(\hbar \circledast \mathcal{g}) \circledast (0 \circledast 0) \in S$, but $(\hbar \circledast \mathcal{g}) \circledast (0 \circledast 0) = (\hbar \circledast \mathcal{g}) \circledast 0 = (\hbar \circledast \mathcal{g})$. Thus $\hbar \circledast \mathcal{g} \in S$, then S is $\delta -$ subalgebra.

Conclusion:

To expand on our results and opinions from this study on $\delta -$filter, complete $\delta -$filter, complete $\delta -$ideal and $\delta -$norma in $\delta -$algebra and in regular $\delta -$algebra. In a future study, instead of utilizing classical sets to investigate and explain fresh ideas and conclusions in algebra, to study their expansions in fuzzy settings, fuzzy sets will be used.

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