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On Integrability of Christou's Sixth Order Solitary Wave Equations

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Abstract
We examine the integrability in terms of Painlevè analysis for several models of higher order nonlinear solitary wave equations which were recently derived by Christou. Our results point out that these equations do not possess Painlevè property and fail the Painlevè test for some special values of the coefficients; and that indicates a non-integrability criteria of the equations by means of the Painlevè integrability.

Keywords: Painlevè analysis, Integrability, Sixth order solitary wave equations, Painlevè property, Wiess, Tabor and Carnevale approach, Kruskal's simplification.

تكاملية معادلات جيرستو الموجية المنعزلة من الرتب السادسة

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الخلاصة
بحثنا بالتفصيل التكاملية من مفهوم تحليل بينليفا لمجموعة من المعادلات غير الخطية الموجية المنعزلة من الرتب العليا التي قدمت مؤخرا من قبل جريستو. نتائجنا بينت ان هذه المعادلات لا تملك خاصية بينليفا وتفشل في اختبار بينلفا لبعض القيم الخاصة للمعاملات وهذا يشير إلى معايير عدم القدرة على التكاملية للمعادلات من مفهوم تكاملية بينليفا.

1 Introduction

A variety of new nonlinear partial differential equations were recently introduced in the work of [1] from applying a different type of techniques; The author managed to exploit fundamental physics laws, Taylor series expansion and Hirota's bilinear operator to derive some higher order solitary wave equations. The first model, the sixth order solitary wave equation using Ohm's law is given by

$$u_{tt} - c_1 u_{xx} - 2c_2 u_x^2 - 2c_2 u u_{xx} + 6c_3 u u_x^2 + 3c_3 u^2 u_{xx} - c_4 u_{xxxx} - c_5 u_{xxxxxx} = 0, \quad (1)$$

where u is a function of x and t, the subscripts denote to partial derivatives with respect to the independent variables, and $c_1 = \frac{h^2}{C_0 L}$, $c_2 = \frac{1}{2F_0}$, $c_3 = \frac{1}{3F_0^2}$, $c_4 = \frac{(12\delta+1)h^4}{12C_0 L}$, $c_5 = \frac{(60\delta+1)h^6}{360C_0 L}$ and $C_0 = \frac{Q_0}{F_0}$. The Q_0 is the charge on the capacitor, F_0 is Faraday's constant, L is the length on the C_0 capacitor, h is a small parameter of the Taylor expansion and δ controls the triple interactions between sections and must be non-negative.

The travelling wave solutions for the equation (1) was obtained in [2] by using the improved generalized tanh-coth method. The second model, the sixth order solitary wave equation using Hirota's bilinear operator is given by

$$-u_{tt} + 2u_{xx} - 15u_{xx}^2 - 30u_x u_{xxx} - 15u u_{xxxx} + 90u u_x^2 + 45u^2 u_{xx} - u_{xxxx} + u_{xxxxxx} = 0. \quad (2)$$

The third model, the sixth order Sine-Gordon equation is written as

$$\theta_{tt} - c_0^2 \theta_{xx} - c_1^2 \theta_{xxxx} - c_2^2 \theta_{xxxxxx} = c_3^2 \sin(\theta), \quad (3)$$

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where θ is a function of x and t, and c_j^2 , $j = 0,1,2,3$ are some physical quantities.

We shall inspect Painlevè integrability for the equations (1), (2) and (3); and the integrability means here is that the differential equation does have Painlevè property. For a given partial differential equation $\mathcal{F}(u, u_{z_1}, u_{z_2}, u_{z_1 z_1}, \dots) = 0$, where u is a function of $z_1, z_2, \dots, z_n$, is said to have the Painlevè property if solutions are single valued about non-characteristic movable singularity manifolds, and these manifolds are determined by the condition of the form $\Phi(z_1, z_2, \dots, z_n) = 0$, where Φ is an analytic function. In other words, if $u(z_1, z_2, \dots, z_n)$ is a solution for Partial differential equation, then it takes Laurent type expansion

$$u(z_1, z_2, \dots, z_n) = \Phi(z_1, z_2, \dots, z_n)^{\lambda} \sum_{i=0}^{\infty} u_i \, \Phi(z_1, z_2, \dots, z_n)^i, \tag{4}$$

where Φ and u are both analytic functions, λ is an integer number, and the number of arbitrary functions u_i is equal to the order of the differential equation. Wiess, Tabor and Carnevale (WTC) [3] introduced an approach that one can examine singularity structure of partial differential equations directly. In addition, Wiess [4-7] investigated the Painlevè property for several partial differential equations and he showed how to construct their Bäcklund transformations and Lax pairs.

The WTC approach is basically built on three steps. Firstly, the leading order analysis, obtaining the dominant behavior of all possible singularities of the equation. Secondly, finding the resonances where arbitrary constants may occur in the Laurent expansion. Thirdly, verifying the resonance conditions in each Laurent expansion explicitly. The equation survives Painlevè test if all the three steps are satisfied. A concise review of many methods of Painlevè tests can be found in [8], and for recent applications see [9-12].

The rest of the paper is organized as follows, in section two the Painlevè analysis for the sixth order solitary wave equations using Ohm's law is considered, section three is dedicated to apply the Painlevè test for the nonlinear sixth order equation using Hirota's bilinear operator. We move to section four where the test is performed for solitary wave Sine-Gordon equation. The last section is conclusions.

2 Painlevè analysis for the sixth order solitary wave equations using Ohm's law

We consider the case when the coefficients of the equation (1) are taken to be $c_1 = c_5 = 1, c_2 = \frac{1}{2}, c_3 = \frac{1}{3}$ and $c_4 = 3$. The equation then becomes

$$u_{tt} - u_{xx} - u_x^2 - uu_{xx} + 2uu_x^2 + u^2 u_{xx} - 3u_{xxxx} - u_{xxxxxx} = 0. \tag{5}$$

We search for solution of equation (5) expressed in an infinite series of Laurent type expansion

$$u(x,t) = \Phi^{\lambda} \sum_{i=0}^{\infty} u_i(x,t) \Phi^i(x,t), \tag{6}$$

in the neighborhood of a non-characteristic movable singular manifold $\Phi(x,t) = 0$, and the number of arbitrary functions u_i should be the same as the order of the equation. To determine the leading dominant behavior, let $u = \kappa \Phi^{\lambda}$,where κ is a constant, into the equation (5) to obtain

$$\begin{aligned}
&-\kappa\lambda\{3\Phi^{\lambda-1}\Phi_{xxxx} - \Phi^{\lambda-2}\Phi_t^2 - \Phi^{\lambda-1}\Phi_{tt} - 18\Phi^{\lambda-4}\Phi_x^4 + 30\Phi^{\lambda-3}\Phi_{xx}^3 - 10\Phi^{\lambda-2}\Phi_{xxx}^2 \\
&-\Phi^{\lambda-2}\Phi_x^2 - 9\Phi^{\lambda-2}\Phi_{xx}^2 + \Phi^{\lambda-1}\Phi_{xx} + \Phi^{\lambda-1}\Phi_{xxxxxx} - 120\Phi^{\lambda-6}\Phi_x^6 + \lambda^5\Phi^{\lambda-6}\Phi_x^6 - \\
&15\lambda^5\Phi^{\lambda-4}\Phi_x^6 + 85\lambda^3\Phi^{\lambda-6}\Phi_x^6 - 225\lambda^2\Phi^{\lambda-6}\Phi_x^6 - 3\lambda^3\Phi^{\lambda-4}\Phi_x^4 + 274\lambda\Phi^{\lambda-6}\Phi_x^6 \\
&-18\lambda^2\Phi^{\lambda-4}\Phi_x^4 + 33\lambda\Phi^{\lambda-4}\Phi_x^4 + 15\lambda^2\Phi^{\lambda-3}\Phi_{xx}^3 + 360\Phi_x^4\Phi_{xx}\Phi^{\lambda-5} - 120\Phi^{\lambda-4}\Phi_{xxx}\Phi_x^3 \\
&-270\Phi^{\lambda-4}\Phi_x^2\Phi_{xx}^2 - 45\lambda\Phi^{\lambda-3}\Phi_{xx}^3 + \kappa^2\Phi_x^2\Phi^{3\lambda-2} + 36\Phi^{\lambda-3}\Phi_x^2\Phi_{xx} + 30\Phi^{\lambda-3}\Phi_x^2\Phi_{xxxx} \\
&+10\lambda\Phi^{\lambda-2}\Phi_{xxx}^2 + \lambda\Phi^{\lambda-2}\Phi_x^2 + 9\lambda\Phi^{\lambda-2}\Phi_{xx}^2 - 15\Phi^{\lambda-2}\Phi_{xx}\Phi_{xxxx} + \kappa\Phi^{2\lambda-1}\Phi_{xx} \\
&+15\lambda^4\Phi^{\lambda-5}\Phi_x^4\Phi_{xx} - 150\lambda^3\Phi^{\lambda-5}\Phi_x^4\Phi_{xx} + 20\lambda^3\Phi^{\lambda-4}\Phi_x^3\Phi_{xxx} + 45\lambda^3\Phi^{\lambda-4}\Phi_x^2\Phi_{xx}^2 \\
&+525\lambda^2\Phi^{\lambda-5}\Phi_x^4\Phi_{xx} - 120\lambda^2\Phi^{\lambda-4}\Phi_x^3\Phi_{xxx} - 270\lambda^2\Phi^{\lambda-4}\Phi_x^2\Phi_{xx}^2 - 750\lambda\Phi^{\lambda-5}\Phi_x^4\Phi_{xx} \\
&+220\lambda\Phi^{\lambda-4}\Phi_x^3\Phi_{xxx} + 490\lambda\Phi^{\lambda-4}\Phi_x^2\Phi_{xx}^2 + 18\lambda^2\Phi^{\lambda-3}\Phi_x^2\Phi_{xx} + 15\lambda^2\Phi^{\lambda-3}\Phi_x^2\Phi_{xxxx} \\
&-3\kappa^2\lambda\Phi^{3\lambda-2}\Phi_x^2 - 54\lambda\Phi^{\lambda-3}\Phi_x^2\Phi_{xx} - 45\lambda\Phi^{\lambda-3}\Phi_x^2\Phi_{xxxx} + 120\Phi^{\lambda-3}\Phi_x\Phi_{xx}\Phi_{xxx} \\
&+12\lambda\Phi^{\lambda-2}\Phi_x\Phi_{xxx} + 6\lambda\Phi^{\lambda-2}\Phi_x\Phi_{xxxxx} + 15\lambda\Phi^{\lambda-2}\Phi_{xx}\Phi_{xxxx} + 2\kappa\lambda\Phi^{2\lambda-2}\Phi_x^2 \\
&+\lambda\Phi^{\lambda-2}\Phi_t^2 + 60\lambda^2\Phi^{\lambda-3}\Phi_x\Phi_{xx}\Phi_{xxx} - 180\lambda\Phi^{\lambda-3}\Phi_x\Phi_{xx}\Phi_{xxx}\} = 0.
\end{aligned}$$

The lowest exponents of Φ are $\{2\lambda - 2, 3\lambda - 2, \lambda - 6\}$ and all the possible balances give the singularity orders $\lambda = -2$ and $\lambda = -4$, so the most singular terms in the equation are

$$2uu_x^2 + u^2 u_{xx} - u_{xxxxxx} \sim 0, \tag{7}$$

and

$$-u_x^2 - uu_{xx} - u_{xxxxxx} \sim 0. \tag{8}$$

There exist two families (7) and (8) of Painèlve expansions that needs to be discussed separately. For first family (7), inserting $u(x,t)=u_0\Phi^{\lambda}$ into equation (7) yields

$$2u_0^3\lambda^2\Phi_x^2\Phi^{2\lambda-2}-u_0^3\lambda(\lambda-1)\Phi_x^2\Phi^{2\lambda-2}-u_0\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)\Phi_x^6\Phi^{\lambda-6}\sim 0,$$

when $\lambda=-2$ that leads to the two branches $u_0=+6\sqrt{10}\Phi_x^2$ and $u_0=-6\sqrt{10}\Phi_x^2$.
In order to find the resonances, where the arbitrary constants may occur in the series, take a linear perturbation of the leading order

$$u(x,t)\sim u_0\Phi^{-2}(1+\zeta\Phi^r)\,, \tag{9}$$

where ζ is a small parameter correction to the leading order. Substituting (9) into (7) yields

$$\begin{aligned}&2\{4+12\zeta\Phi^r-4\mathrm{r}\zeta\Phi^r\}\Phi^{-8}u_0^3\Phi_x^2-18\zeta\Phi^r u_0^3\Phi_x^2\Phi^{-8}-6\zeta\Phi^r u_0^3\Phi_{xx}\Phi^{-7}+6\zeta u_0^3\Phi_x^2\Phi^{-8}\\&-5\mathrm{r}\zeta\Phi^r u_0^3\Phi_x^2\Phi^{-8}-2u_0^3\Phi_{xx}\Phi^{-7}+\mathrm{r}^2\zeta\Phi^r u_0^3\Phi_x^2\Phi^{-8}-r\zeta u_0^3\Phi^r\Phi_{xx}\Phi^{-7}-1560u_0r\zeta\\&\Phi^r\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5}-540u_0r^2\zeta\Phi^r\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5}+60u_0r^3\zeta\Phi^r\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5}\\&-30u_0r\zeta\Phi^r\Phi_x\Phi_{xxxxx}\Phi^{-4}+1420u_0r^2\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6}+390u_0r\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5}\\&-75u_0r\zeta\Phi^r\Phi_{xx}\Phi_{xxxx}\Phi^{-4}+15u_0r^3\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5}+15u_0r^2\zeta\Phi^r\Phi_{xx}\Phi_{xxxx}\Phi^{-4}\\&+15660u_0r\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7}+u_0r\zeta\Phi^r\Phi_{xxxxx}\Phi^{-3}-1440u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5}\\&+15u_0r^3\zeta\Phi^r\Phi_{xx}^3\Phi^{-5}-135u_0r^2\zeta\Phi^r\Phi_{xx}^3\Phi^{-5}-50u_0r\zeta\Phi^r\Phi_{xxx}^2\Phi^{-4}+390u_0r\zeta\Phi^r\\&\Phi_{xx}^3\Phi^{-5}+10u_0r^2\zeta\Phi^r\Phi_{xxx}^2\Phi^{-4}-8028u_0r\zeta\Phi^r\Phi_x^6\Phi^{-8}-1665u_0r^3\zeta\Phi^r\Phi_x^6\Phi^{-8}+\\&295u_0r^4\zeta\Phi^r\Phi_x^6\Phi^{-8}-27u_0r^5\zeta\Phi^r\Phi_x^6\Phi^{-8}+u_0r^6\zeta\Phi^r\Phi_x^6\Phi^{-8}+5104u_0r^2\zeta\Phi^r\Phi_x^6\Phi^{-8}\\&+36u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xxxxx}\Phi^{-4}-10800u_0(1+\zeta\Phi^r)\Phi_x^4\Phi_{xx}\Phi^{-7}+2400u_0(1+\\&\zeta\Phi^r)\Phi_x^3\Phi_{xxx}\Phi^{-6}-360u_0(1+\zeta\Phi^r)\Phi_x^2\Phi_{xxxx}\Phi^{-5}+90u_0(1+\zeta\Phi^r)\Phi_{xx}\Phi_{xxxx}\Phi^{-4}\\&+5400u_0(1+\zeta\Phi^r)\Phi_x^2\Phi_{xx}^2\Phi^{-6}-360u_0(1+\zeta\Phi^r)\Phi_{xx}^3\Phi^{-5}+5040u_0(1+\zeta\Phi^r)\Phi_x^6\Phi^{-8}\\&+5040u_0(1+\zeta\Phi^r)\Phi_x^6\Phi^{-8}\ ++60u_0(1+\zeta\Phi^r)\Phi_{xxx}^2\Phi^{-4}-2u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xxxxxx}\Phi^{-3}\\&-30u_0r\zeta\Phi^r\Phi_x\Phi_{xxxxx}\Phi^{-4}-75u_0r\zeta\Phi^r\Phi_{xx}\Phi_{xxxx}\Phi^{-4}+15u_0r^3\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5}\\&+15u_0r^2\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-4}+15660u_0r\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7}-300u_0r^4\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7}\\&-15u_0r^5\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7}+1420u_0r^2\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6}+390u_0r\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5}\\&+3195u_0r^2\zeta\Phi^r\Phi_x^2\Phi_{xx}^2\Phi^{-6}-820u_0r^3\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6}-3080u_0r\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6}\sim 0.\end{aligned}$$

Now, collecting the terms that are linear in ζ, and by using Kruskal's formula [13], $\Phi(x,t)=x-\Theta(t)$, where $\Theta(t)$ is an arbitrary function, and setting $\Phi_x=1$ that gives

$$\{(4(r-3)-(r^2-5r+18))(x-\Theta)^r\}u_0^3-\{(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)(x-\Theta)^r\}u_0\sim 0,$$

applying then $u_0=\mp 6\sqrt{10}$ to obtain sixth degree resonances polynomial

$$r^6-12r^5+295r^4-1665r^3+4744r^2-3348r-10080=0.$$

Solve the algebraic equation for r to gain the resonances

$$r=-1,6,7,8,\frac{7}{2}-\frac{i\sqrt{71}}{2},\frac{7}{2}+\frac{i\sqrt{71}}{2}\,,\ i=\sqrt{-1}\ .$$

We have six resonances corresponding to sixth order partial differential equation. There exist two resonances (so-called Fuchs indices) of the first branch are non-integer numbers. The equation does not pass the Painlevè test which indicates that the equation does not have a single valued around movable singularity, but it may have a movable algebraic or logarithmic branch point. To truncate the expansion at zero order, we substitute the expression

$$u=\frac{u_0}{\Phi^2}+\frac{u_1}{\Phi}+u_2,$$

into the equation (5) and then grouping the terms with the same power of Φ to get a set of equations, here we write down a few of these equations

$$-5040u_0\Phi_x^6+14u_0^3\Phi_x^2=0, \tag{10}$$

$$-720u_1\Phi_x^6+10800u_0\Phi_x^4\Phi_{xx}+4320\Phi_x^5u_{0,x}+30u_0^2u_1\Phi_x^2-2u_0^3\Phi_{xx}-12u_0^2u_{0,x}\Phi_x=0, \tag{11}$$

$$720\Phi_x^5u_{1,x}+1800\Phi_x^4u_0\Phi_{xx}-360u_0\Phi_x^4-1800u_{0,xx}\Phi_x^4-7200u_{0,x}\Phi_x^3\Phi_{xx}-2400u_0\Phi_x^3\Phi_{xxx}+20u_0u_1^2\Phi_x^2-5400u_0\Phi_x^2\Phi_{xx}^2+20u_2u_0^2\Phi_x^2-10u_0^2\Phi_x^2-10u_0^2\Phi_xu_{1,x}-20u_0u_1u_{0,x}\Phi_x-5u_1u_0^2\Phi_{xx}+(u_0^2u_{0,x})_x=0, \tag{12}$$

solving equation (10) for u_0 to obtain $u_0=\pm 6\sqrt{10}\Phi_x^2$, plugging the value of u_0 into equation (11) and solving it for u_1 yields $u_1=\mp 6\sqrt{10}\Phi_{xx}$. Using the values of u_0 and u_1 in equation (12) and solve it for u_2 to get

$$u_2 = \mp\frac{20\sqrt{10}\Phi_x\Phi_{xx}+3\sqrt{10}\Phi_x^2-15\sqrt{10}\Phi_{xx}^2\mp 5\Phi_x^2}{10\Phi_x^2}.$$

Hence, the truncated equation of the expansion (6) at zero order is

$$\mathrm{u}= \mp\frac{6\sqrt{10}\Phi_x^2}{\Phi^2} \pm \frac{6\sqrt{10}\Phi_{xx}}{\Phi} \mp \frac{20\sqrt{10}\Phi_x\Phi_{xx}+3\sqrt{10}\Phi_x^2-15\sqrt{10}\Phi_{xx}^2\mp 5\Phi_x^2}{10\Phi_x^2}.$$

A finite number of terms here represent a local solution of the equation (5). Now, we move to deal with the second family (8). Substituting $u = u_0\Phi^\lambda$ into equation (8) to have

$$-u_0^2\lambda^2\Phi_x^2\Phi^{2\lambda-2} - u_0^2(\lambda^2-\lambda)\Phi_x^2\Phi^{2\lambda-2} - u_0\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)\Phi_x^6\Phi^{\lambda-6}\sim 0.$$

The dominant balancing at the singular order $\lambda = -4$ leads to the branch $u_0 = -1680\Phi_x^4$. For the sake of finding the resonances, taking a linear perturbation of the leading order

$$u(x,t)\sim u_0\Phi^{-4}(1+\zeta\Phi^r)\,, \tag{13}$$

Substituting (13) into (8) to get

$$-[(16-8\zeta(r-4)\Phi^r+(r-4)^2\zeta^2\Phi^{2r})u_0^2\Phi_x^2\Phi^{-10}] - [(\zeta(r-8)\Phi^{r+1}\Phi_{xx}+(r-4)(r-5)\zeta^2\Phi_x^2\Phi^{2r}+20\Phi_x^2+\zeta(r^2-9r+40)\Phi^r\Phi_x^2+\Phi_{xx}(-4\Phi+(r-4)\zeta^2\Phi^{2r+1}))u_0^2\Phi^{-10}] - [(u_0((\zeta(r-4)\Phi^{r+5}-4\Phi^5)\Phi_{xxxxxx}+6(\zeta(r-4)(\mathrm{r}-5)\Phi^{r+4}+20\Phi^4)\Phi_x\Phi_{xxxxx}) +(15\zeta(r-4)(r-5)(\mathrm{r}-6)\Phi_x^2\Phi^{r+3}+15\zeta(r-4)(r-5)\Phi_{xx}\Phi^{r+4}+300\Phi_{xx}\Phi^4-1800\Phi_x^2\Phi^3)\Phi_{xxxx}+(10\zeta(r-4)(r-5)\Phi^{r+4}+200\Phi^4)\Phi_{xxx}^2+20\Phi_x(3\Phi_{xx}\zeta(r-4)(r-5)(\mathrm{r}-6)\Phi^{r+3}+\zeta(r-4)(r-5)(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^{r+2}-360\Phi_{xx}\Phi^3+840\Phi^2\Phi_x^2)\Phi_x\Phi_{xxx} +15\zeta(r-4)(r-5)(\mathrm{r}-6)(\mathrm{r}-7)\Phi_{xx}^3\Phi^{r+3}+45\zeta(r-4)(r-5)(r-6)(r-7)\Phi_x^2\Phi_{xx}^2\Phi^{r+2} +15\zeta(r-4)(r-5)\zeta(r-6)(r-7)\Phi_x^4\Phi_{xx}\Phi^{r+1}-1800\Phi_{xx}^3\Phi^3+34800\Phi_x^2\Phi_{xx}^2\Phi^2-100800\Phi_x^4\Phi_{xx}\Phi+(60480+\zeta(r-4)(r-5)\zeta(r-6)(r-7)(r-8)(r-9)\Phi^r)\Phi_x^6))\Phi^{-10}]\sim 0.$$

Using Kruskal's formula [13], that is $\Phi(x,t) = x - \Theta(t)$, and grouping the terms linear in ζ , and also setting $\Phi_x = 1$ which comes from Kruskal's formula, to gain

$$[(8(r-4)-(r^2-9r+40))u_0^2-(r-4)(r-5)(r-6)(r-7)(r-8)(r-9)u_0]\zeta\sim 0,$$

with the benefit of $u_0 = -1680\Phi_x^4$, the resonances polynomial is

$$r^6-39r^5+625r^4-5265r^3+22894r^2-31656r-60480=0.$$

Solving the algebraic equation for r to gain the Fuchs indices

$$r=-1,8,9,12,\frac{11}{2}+\frac{1}{2}i\sqrt{159}\ ,\ \frac{11}{2}-\frac{1}{2}i\sqrt{159}\,,i=\sqrt{-1}\ .$$

Two of the resonances are non-integer numbers, thus, the equation (5) fails the test and that due to occurring of algebraic or logarithmic branch points.

3 Painlevè analysis for the Sixth order equation using Hirota's bilinear operator

To perform Painlevè test for equation (2), the equation is given by

$$-u_{tt}+2u_{xx}-15u_{xx}^2-30u_xu_{xxx}-15uu_{xxxx}+90uu_x^2+45u^2u_{xx}-u_{xxxx}+u_{xxxxxx}=0. \tag{14}$$

One can deduce that the most singular terms in the equation (14) are

$$-15u_{xx}^2-30u_xu_{xxx}-15uu_{xxxx}+90uu_x^2+45u^2u_{xx}+u_{xxxxxx}\sim 0. \tag{15}$$

Substituting $u(x,t) = u_0\Phi^\lambda$ into the equation (15) yields

$$-15\lambda^2(\lambda-1)^2u_0^2\Phi_x^4\Phi^{2\lambda-4}-30\lambda^2(\lambda-1)(\lambda-2)u_0^2\Phi_x^4\Phi^{2\lambda-4}-15\lambda(\lambda-1)(\lambda-2)(\lambda-3)u_0^2\Phi_x^4\Phi^{2\lambda-4}+90\lambda^2u_0^3\Phi_x^2\Phi^{3\lambda-2}+45\lambda(\lambda-1)u_0^3\Phi_x^2\Phi^{3\lambda-2}+\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)u_0\Phi_x^6\Phi^{\lambda-6}\sim 0.$$

From dominant balancing, we have two branches $u_0 = 2\Phi_x^2$ and $u_0 = 4\Phi_x^2$ at singular order $\lambda = -2$. In order to get the resonances, Take a linear perturbation of the leading order

$$u(x,t)\sim u_0\Phi^{-2}(1+\zeta\Phi^r)\,, \tag{16}$$

where ζ is a small correction of the leading order. Substituting (16) into equation (15) to have

$$-15\,[6u_0(1+\zeta\Phi^r)\Phi_x^2\Phi^{-4}-5u_0r\zeta\Phi^r\Phi_x^2\Phi^{-4}-2u_0(1+\zeta\Phi^r)\Phi_{xx}\Phi^{-3}-u_0r^2\zeta\Phi^r\Phi_x^2\Phi^{-4}+u_0r\zeta\Phi^r\Phi_{xx}\Phi^{-3}]^2-30(-2u_0(1+\zeta\Phi^r)\Phi_x\Phi^{-3}+u_0r\zeta\Phi^r\Phi_x\Phi^{-3})(u_0r\zeta\Phi^r\Phi_{xxx}\Phi^{-3}-24u_0(1+\zeta\Phi^r)\Phi_x^3\Phi^{-5}+18u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xx}\Phi^{-4}+26u_0r\zeta\Phi^r\Phi_x^3\Phi^{-5}-9u_0r^2\zeta\Phi^r\Phi_x^3\Phi^{-5}+u_0r^3\zeta\Phi^r\Phi_x^3\Phi^{-5}-2u_0(1+\zeta\Phi^r)\Phi_{xxx}\Phi^{-3}-15u_0r\zeta\Phi^r\Phi_x\Phi_{xx}\Phi^{-4}+3u_0r^2\zeta\Phi^r\Phi_x\Phi_{xx}\Phi^{-4})-15(u_0\Phi^{-2}(1+\zeta\Phi^r))(u_0r\zeta\Phi^r\Phi_{xxxx}\Phi^{-3}-20u_0r\zeta\Phi^r\Phi_x\Phi_{xxx}\Phi^{-4}-54u_0r^2\zeta\Phi^r\Phi_x^2\Phi_{xx}\Phi^{-5}+6u_0r^3\zeta\Phi^r\Phi_x^2\Phi_{xx}\Phi^{-5}-15u_0r\zeta\Phi^r\Phi_{xx}^3\Phi^{-4}+3u_0r^2\zeta\Phi^r\Phi_{xx}^2\Phi^{-4}-154u_0r\zeta\Phi^r\Phi_x^4\Phi^{-6}-120u_0(1+\zeta\Phi^r)$$

$$\Phi_x^4\Phi^{-6} - 144u_0(1+\zeta\Phi^r)\Phi_x^2\Phi_{xx}\Phi^{-5} + 18u_0(1+\zeta\Phi^r)\Phi_{xx}^2\Phi^{-4} + 24u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xxx}\Phi^{-4} + 4u_0r^2\zeta\Phi^r\Phi_x\Phi_{xxx}\Phi^{-4} + 156u_0r\zeta\Phi^r\Phi_x^2\Phi_{xx}\Phi^{-5} + 71u_0r^2\zeta\Phi^r\Phi_x^4\Phi^{-6} - 14u_0r^3\zeta\Phi^r\Phi_x^4\Phi^{-6} + u_0r^4\zeta\Phi^r\Phi_x^4\Phi^{-6} - 2u_0(1+\zeta\Phi^r)\Phi_{xxxx}\Phi^{-3}) + 90\,(u_0\Phi^{-2}(1+\zeta\Phi^r))\,([\zeta^2(r-2)^2\Phi^{2r} - 4\zeta(r-2)\Phi^r + 4]u_0^2\Phi_x^2\Phi^{-6}) + 45\,(u_0^2(\Phi^{2r}\zeta^2 + 2\zeta\Phi^r + 1)\Phi^{-4})(6u_0(1+\zeta\Phi^r)\Phi_x^2\Phi^{-4} - 5u_0r\zeta\Phi^r\Phi_x^2\Phi^{-4} - 2u_0(1+\zeta\Phi^r)\Phi_{xx}\Phi^{-3} - u_0r^2\zeta\Phi^r\Phi_x^2\Phi^{-4} + u_0r\zeta\Phi^r\Phi_{xx}\Phi^{-3}) + 1560u_0r\zeta\Phi^r\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5} - 540u_0r^2\zeta\Phi^r\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5} + 60u_0r^3\zeta\Phi^r\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5} - 30u_0r\zeta\Phi^r\Phi_x\Phi_{xxxxx}\Phi^{-4} + 1420u_0r^2\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6} + 390u_0r\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5} - 75u_0r\zeta\Phi^r\Phi_{xx}\Phi_{xxxx}\Phi^{-4} + 15u_0r^3\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5} + 15u_0r^2\zeta\Phi^r\Phi_{xx}\Phi_{xxxx}\Phi^{-4} + 15660u_0r\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7} + u_0r\zeta\Phi^r\Phi_{xxxxxx}\Phi^{-3} - 1440u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xx}\Phi_{xxx}\Phi^{-5} + 15u_0r^3\zeta\Phi^r\Phi_{xx}^3\Phi^{-5} - 135u_0r^2\zeta\Phi^r\Phi_{xx}^3\Phi^{-5} - 50u_0r\zeta\Phi^r\Phi_{xxx}^2\Phi^{-4} + 390u_0r\zeta\Phi^r\Phi_{xx}^3\Phi^{-5} + 10u_0r^2\zeta\Phi^r\Phi_{xxx}^2\Phi^{-4} - 8028u_0r\zeta\Phi^r\Phi_x^6\Phi^{-8} - 1665u_0r^3\zeta\Phi^r\Phi_x^6\Phi^{-8} + 295u_0r^4\zeta\Phi^r\Phi_x^6\Phi^{-8} - 27u_0r^5\zeta\Phi^r\Phi_x^6\Phi^{-8} + u_0r^6\zeta\Phi^r\Phi_x^6\Phi^{-8} + 5104u_0r^2\zeta\Phi^r\Phi_x^6\Phi^{-8} + 36u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xxxxx}\Phi^{-4} - 10800u_0(1+\zeta\Phi^r)\Phi_x^4\Phi_{xx}\Phi^{-7} + 2400u_0(1+\zeta\Phi^r)\Phi_x^3\Phi_{xxx}\Phi^{-6} - 360u_0(1+\zeta\Phi^r)\Phi_x^2\Phi_{xxxx}\Phi^{-5} + 90u_0(1+\zeta\Phi^r)\Phi_{xx}\Phi_{xxxx}\Phi^{-4} + 5400u_0(1+\zeta\Phi^r)\Phi_x^2\Phi_{xx}^2\Phi^{-6} - 360u_0(1+\zeta\Phi^r)\Phi_{xx}^3\Phi^{-5} + 5040u_0(1+\zeta\Phi^r)\Phi_x^6\Phi^{-8} + 5040u_0(1+\zeta\Phi^r)\Phi_x^6\Phi^{-8} + 60u_0(1+\zeta\Phi^r)\Phi_{xxx}^2\Phi^{-4} - 2u_0(1+\zeta\Phi^r)\Phi_x\Phi_{xxxxxx}\Phi^{-3} - 30u_0r\zeta\Phi^r\Phi_x\Phi_{xxxxx}\Phi^{-4} - 75u_0r\zeta\Phi^r\Phi_{xx}\Phi_{xxxx}\Phi^{-4} + 15u_0r^3\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5} + 15u_0r^2\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-4} + 15660u_0r\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7} - 300u_0r^4\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7} - 15u_0r^5\zeta\Phi^r\Phi_x^4\Phi_{xx}\Phi^{-7} + 1420u_0r^2\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6} + 390u_0r\zeta\Phi^r\Phi_x^2\Phi_{xxxx}\Phi^{-5} + 3195u_0r^2\zeta\Phi^r\Phi_x^2\Phi_{xx}^2\Phi^{-6} - 820u_0r^3\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6} - 3080u_0r\zeta\Phi^r\Phi_x^3\Phi_{xxx}\Phi^{-6} \sim 0.$$

Using Kruskal's formula [13], $\Phi(x,t) = x - \Theta(t)$, and keeping only the coefficients linear in ζ, with $\Phi_x = 1$, to obtain

$$-15[12(r-2)(r-3)u_0^2] - 30[-2(r-2)(r^2-7r+24)u_0^2] - 15[r^4-14r^3+71r^2-154r+240)u_0^2] + 90[-4(r-3)u_0^3] + 45[r^2-15r+18)u_0^3] + [(r-2)(r-3)(r-4)(r-5)(r-6)(r-7)u_0] = 0,$$

applying $u_0 = 2\Phi_x^2$ to have the resonances polynomial

$$2r^6 - 54r^5 + 530r^4 - 2250r^3 + 3428r^2 + 1224r - 5040 = 0,$$

solving the last equation for r to gain Fuchs indices $\mathrm{r} = -1, 2, 3, 6, 7, 10$.

Also, for the other branch, applying $u_0 = 4\Phi_x^2$ gives

$$4r^6 - 108r^5 + 940r^4 - 2340r^3 - 5264r^2 + 18288r + 20160 = 0,$$

solving the last equation to gain Fuchs indices $r = -2, -1, 5, 6, 7, 12$. The resonance $r = -1$, so called universal resonance, is corresponding to the arbitrary manifold Φ. For more details about negative resonances we refer the reader to [14,15]. We consider only the principle branch, when $u_0 = 2\Phi_x^2$, to verify the compatibility conditions. Substituting $u(x,t) = \sum_{i=0}^{10} u_i\Phi^{i-2}$ into equation (15) and collecting the coefficients of Φ^i. Here we write down a few of these equations

$$5040\Phi_x^6u_0 - 3780\Phi_x^4u_0^2 + 630\Phi_x^2u_0^3 = 0, \tag{17}$$

$$720\Phi_x^6u_1 - 4320\Phi_x^5u_{0,x} - 3600\Phi_x^4u_0u_1 - 10800\Phi_{xx}\Phi_x^4u_0 + 3960\Phi_x^3u_{0,x}u_0 + 1350\Phi_x^2u_0^2u_1 + 3600\Phi_{xx}\Phi_x^2u_0^2 - 540\Phi_xu_{0,x}u_0^2 - 90\Phi_{xx}u_0^3 = 0, \tag{18}$$

$$-720\Phi_x^5u_{1,x} - 600\Phi_x^4u_1^2 - 1800\Phi_{xx}\Phi_x^4u_1 - 1800\Phi_x^4u_0u_2 - 120\Phi_x^4u_0 + 1800u_{0,xx}\Phi_x^4 + 2400\Phi_x^3u_{0,x}u_1 + 7200\Phi_{xx}\Phi_x^3u_{0,x} + 2400\Phi_{xxx}\Phi_x^3u_0 + 1800\Phi_x^3u_{1,x}u_0 + 900\Phi_x^2u_0u_1^2 + 3900\Phi_{xx}\Phi_x^2u_0u_1 + 900\Phi_x^2u_2u_0^2 + 5400\Phi_{xx}^2\Phi_x^2u_0 - 780\Phi_x^2u_{0,x}^2 - 1080\Phi_x^2u_{0,xx}u_0 - 900\Phi_xu_{0,x}u_0u_1 - 2220\Phi_x\Phi_{xx}u_{0,x}u_0 - 480\Phi_x\Phi_{xxx}u_0^2 - 450\Phi_xu_{1,x}u_0^2 - 225\Phi_{xx}u_0^2u_1 - 330\Phi_{xx}^2u_0^2 + 90u_{0,x}^2u_0 + 45u_{0,xx}u_0^2 = 0, \tag{19}$$

$$180\Phi_{xxx}u_{0,x}u_0 + 840\Phi_x^3u_1u_{1,x} + 480\Phi_{xxx}\Phi_x^3u_1 + 720\Phi_x^3u_0u_{2,x} + 1440\Phi_x^3u_{0,x}u_2 + 540\Phi_x^2u_3u_0^2 - 2160\Phi_x\Phi_{xx}^2u_{0,x} - 420\Phi_{xx}^2u_0u_1 - 2160\Phi_{xx}\Phi_x^2u_{0,xx} + 144\Phi_{xx}\Phi_x^2u_0 - 180\Phi_{xx}u_0u_1^2 - 180\Phi_{xx}u_2u_0^2 - 540\Phi_x^2u_0u_{1,xx} - 780\Phi_x^2u_1u_{0,xx} - 960\Phi_x^2u_{0,x}u_{1,x} - 360\Phi_{xxxx}\Phi_x^2u_0 - 1440\Phi_{xxx}\Phi_x^2u_{0,x} - 360\Phi_xu_{0,x}u_1^2 - 360\Phi_xu_{2,x}u_0^2 + 240\Phi_{xx}u_{0,xx}u_0 + 300\Phi_xu_{0,x}u_{0,xx} + 180\Phi_xu_0u_{0,xxx} + 90u_{0,xx}u_0u_1 + 180u_0u_{0,x}u_{1,x} - 1080\Phi_x^4u_0u_3 - 360\Phi_x^4u_2u_1 + 1080\Phi_x^2\Phi_{xx}^2u_1 + 1440\Phi_{xx}\Phi_x^3u_{1,x} + 780\Phi_{xx}\Phi_x^2u_1^2 + 180\Phi_x^2u_1^3 - 360\Phi_{xx}^3u_0 + 96\Phi_x^3u_{0,x} - 480\Phi_x^3u_{0,xxx} + 180\Phi_{xx}u_{0,x}^2 + 45u_{1,xx}u_0^2 + 90u_{0,x}^2u_1$$

$$+30\Phi_{xxxx}u_0^2+360\Phi_x^4u_{1,xx}-24\Phi_x^2u_1+2160\Phi_{xx}\Phi_x^2u_0u_2+1080\Phi_x^2u_0u_1u_2$$
$$-1560\Phi_{xx}\Phi_xu_1u_{0,x}-1200\Phi_{xx}\Phi_xu_{1,x}u_0-1440\Phi_{xxx}\Phi_{xx}\Phi_xu_0-720\Phi_xu_0u_{1,x}u_1$$
$$-600\Phi_{xxx}\Phi_xu_1u_0-720\Phi_xu_2u_{0,x}u_0=0, \quad (20)$$

where $u_{j,x}=\frac{\partial u_j}{\partial x}$, $u_{j,xx}=\frac{\partial^2 u_j}{\partial x^2}$, $u_{j,xxx}=\frac{\partial^3 u_j}{\partial x^3}$ $j=0,1,2,3$. Solving equation (17) to obtain, by using Kruskal's simplification [13], $\Phi(x,t)=x-\Theta(t)$, the following branches $u_0=2\Phi_x^2$ and $u_0=4\Phi_x^2$. From equation (18) we have

$$-1080u_1\Phi_x^6-2160\Phi_x^6\Phi_{xx}=0 \text{ and } u_1=-2\Phi_{xx} \text{ and so } u_1=0.$$

Also, from equation (19) one can get $-240\Phi_x^6=0$, and since $\Phi_x=1$ that leads to $-240=0$ which is inconsistent. Therefore, the compatibility condition is not satisfied and the equation (14) fails the Painlevè test.

4 Painlevè analysis for sixth order Sine-Gordon equation

We discuss the case when the coefficients of the equation (3) are taken as $c_0^2=c_1^2=c_2^2=c_3^2=1$. The equation then is given by

$$\theta_{tt}-\theta_{xx}-\theta_{xxxx}-\theta_{xxxxxx}=\sin(\theta), \quad (21)$$

adapting the transformation $u=\exp(i\theta)$, $i=\sqrt{-1}$, with the benefit of the relation $\sin(\theta)=\frac{\exp(i\theta)-\exp(-i\theta)}{2i}$, to get the equation

$$u^7-2u_{2t}u^5+2u_t^2u^4-240u_x^6-12u_x^4u^2+720u_x^4u_{xx}u-240u_{xxx}u_x^3u^2-2u_x^2u^4$$
$$+60u_{xxxx}u_x^2u^3+24u_{xx}u_x^2u^3-540u_{xx}^2u_x^2u^2-8u_{xxx}u_xu^4-12u_{xxxxx}u_xu^4$$
$$+240u_{xxx}u_xu_{xx}u^3+2u_{xxxx}u^5+2u_{xx}u^5+2u_{xxxxxx}u^5-u^5-30u_{xxxx}u_{xx}u^4$$
$$-6u_{xx}^2u^4-20u_{xxx}^2u^4+60u_{xx}^3u^3=0. \quad (22)$$

To determine the dominant behavior, plug $u=u_0\Phi^\lambda$ into the equation (22) to obtain

$$-u_0^5[-u_0^2\Phi^{7\lambda}+2u_0\kappa\lambda\Phi^{6\lambda-1}\Phi_{tt}-2u_0\lambda\Phi^{6\lambda-2}\Phi_t^2+30u_0\lambda\Phi^{6\lambda-2}\Phi_{xx}\Phi_{xxxx}$$
$$+12u_0\lambda\Phi^{6\lambda-2}\Phi_x\Phi_{xxxxx}-60u_0\lambda\Phi^{6\lambda-3}\Phi_{xxxx}\Phi_x^2+\Phi^{5\lambda}+240u_0\lambda\Phi^{6\lambda-4}\Phi_{xxx}\Phi_x^3$$
$$-720u_0\lambda\Phi^{6\lambda-5}\Phi_{xx}\Phi_x^4+540u_0\lambda\Phi^{6\lambda-4}\Phi_x^2\Phi_{xx}^2+8u_0\lambda\Phi_x\Phi_{xxx}-24u_0\lambda\Phi^{6\lambda-3}\Phi_{xx}\Phi_x^2$$
$$+240u_0\lambda\Phi^{6\lambda-6}\Phi_x^6-60u_0\lambda\Phi^{6\lambda-3}\Phi_{xx}^3+20u_0\lambda\Phi^{6\lambda-2}\Phi_{xxx}^2+6u_0\lambda\Phi^{6\lambda-2}\Phi_{xx}^2$$
$$+12u_0\lambda\Phi^{6\lambda-4}\Phi_x^4\Phi^{\lambda-2}-240u_0\lambda\Phi^{6\lambda-3}\Phi_x\Phi_{xx}\Phi_{xxx}-2u_0\lambda\Phi^{6\lambda-1}\Phi_{xxxxxx}$$
$$-2u_0\lambda\Phi^{6\lambda-1}\Phi_{xx}+2u_0\lambda\Phi^{6\lambda-2}\Phi_x^2-2u_0\lambda\Phi^{6\lambda-1}\Phi_{xxxx}]\sim 0,$$

From the last equation, one can get the branch $u_0=-1440\Phi_x^6$ at singular order $\lambda=-6$.

To detect the resonances where arbitrary constants may occur in the equation, take a linear perturbation of the leading order, that is

$$u(x,t)\sim u_0\Phi^{-6}(1+\zeta\Phi^r), \quad (23)$$

inserting (23) into equation (22) to have

$$[u_0\Phi^{-6}(1+\zeta\Phi^r)]^7-240[u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^6-12[u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r\Phi^{-7}]^4[u_0(1+\zeta\Phi^r)\Phi^{-6}]^2+720[u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^4[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^r+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}][u_0(1+\zeta\Phi^r)\Phi^{-6}]-240[u_0((\zeta\Phi^{r+2}(r-6)-6\Phi^2)\Phi_{xxx}+(3\Phi_{xx}\zeta\Phi^{r+1}(r-6)(r-7)+126\Phi_{xx}\Phi+\Phi_x^2(-336+\zeta\Phi^r(r-6)(r-7)(r-8)))\Phi_x)\Phi^{-9}][u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^3[u_0(1+\zeta\Phi^r)\Phi^{-6}]^2-2[u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^2[u_0\Phi^{-6}(1+\zeta\Phi^r)]^4+60[(u_0((\zeta(\mathrm{r}-6)\Phi^{r+3}-6\Phi^3)\Phi_{xxxx}+4\Phi_x(\zeta\Phi^{r+2}(r-6)(r-7)+42\Phi^2)\Phi_{xxx}+3\Phi_{xx}^2\zeta\Phi^{r+2}(r-6)(r-7)+6\Phi_{xx}\Phi_x^2\zeta\Phi^{r+1}(r-6)(r-7)(r-8)+126\Phi_{xx}^2\Phi^2-2016\Phi_x^2\Phi_{xx}\Phi+\Phi_x^4(3024+\zeta\Phi^r(r-6)(r-7)(r-8)(-9))))\Phi^{-10}][u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^2[u_0\Phi^{-6}(1+\zeta\Phi^r)]^3+24[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^r+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}][u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^2[u_0\Phi^{-6}(1+\zeta\Phi^r)]^3-540[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^r+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}]^2[u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}]^2[u_0\Phi^{-6}(1+\zeta\Phi^r)]^2-8[u_0((\zeta\Phi^{r+2}(r-6)-6\Phi^2)\Phi_{xxx}+(3\Phi_{xx}\zeta\Phi^{r+1}(r-6)(r-7)+126\Phi_{xx}\Phi+\Phi_x^2(-336+\zeta\Phi^r(r-6)(r-7)(r-8)))\Phi_x)\Phi^{-9}][u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}][u_0\Phi^{-6}(1+\zeta\Phi^r)]^4-12[(((r\zeta\Phi^{r+4}-6\Phi^4(1+\zeta\Phi^r))\Phi_{xxxxx})+5(\mathrm{r}\zeta(r-13)\Phi^{r+3}+42\Phi^3(1+\zeta\Phi^r))\Phi_x\Phi_{xxxx}+(10\mathrm{r}\zeta\Phi_x^2(r^2-21\mathrm{r}+146)\Phi^{r+2}+10\mathrm{r}\Phi_{xx}\zeta(\mathrm{r}-13)\Phi^{r+3}+420\Phi^2(1+\zeta\Phi^r)(\Phi_{xx}\Phi-8\Phi_x^2))\Phi_{xxx}+\Phi_x(15\mathrm{r}\Phi_{xx}^2\zeta(r^2-21\mathrm{r}+146)\Phi^{r+2}+10\mathrm{r}\zeta\Phi_x^2\Phi_{xx}(\mathrm{r}-15)(r^2-15\mathrm{r}+110)\Phi^{r+1}-5040\Phi^2(1+\zeta\Phi^r)\Phi_{xx}^2+30240\Phi\Phi_x^2(1+\zeta\Phi^r)\Phi_{xx}+\Phi_x^4(-30240+\zeta\Phi^r(r-6)(r-7)(r-8)(r-9)(r-10)))u_0)\Phi^{-11}][u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^r)\Phi^{-7}][u_0\Phi^{-6}(1+\zeta\Phi^r)]^4+240[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^r+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}][u_0\Phi_x(-6+$$

$\zeta(\mathrm{r}-6)\Phi^{r})\Phi^{-7}][u_0\Phi^{-6}(1+\zeta\Phi^{r})]^3$ $[u_0((\zeta\Phi^{r+2}(r-6)-6\Phi^2)\Phi_{xxx}+(3\Phi_{xx}\zeta\Phi^{r+1}(r-6)(r-7)+126\Phi_{xx}\Phi+\Phi_x^2(-336+\zeta\Phi^{r}(r-6)(r-7)(r-8)))\Phi_x)\Phi^{-9}][u_0\Phi_x(-6+\zeta(\mathrm{r}-6)\Phi^{r})\Phi^{-7}]+2[(u_0((\zeta(\mathrm{r}-6)\Phi^{r+3}-6\Phi^3)\Phi_{xxxx}+4\Phi_x(\zeta\Phi^{r+2}(r-6)(r-7)+42\Phi^2)\Phi_{xxx}+3\Phi_{xx}^2\zeta\Phi^{r+2}(r-6)(r-7)+6\Phi_{xx}\Phi_x^2\zeta\Phi^{r+1}(r-6)(r-7)(r-8)+126\Phi_{xx}^2\Phi^2-2016\Phi_x^2\Phi_{xx}\Phi+\Phi_x^4(3024+\zeta\Phi^{r}(r-6)(r-7)(r-8)(-9))))\Phi^{-10}][u_0\Phi^{-6}(1+\zeta\Phi^{r})]^5+2\big[\big(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^{r}+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1})\big)u_0\Phi^{-8}\big][u_0\Phi^{-6}(1+\zeta\Phi^{r})]^5+2[(u_0((r\zeta\Phi^{r+5}-6(1+\zeta\Phi^{r})\Phi^5))\Phi_{xxxxxx}+6\Phi_x(r\zeta(r-13)\Phi^{r+4}+42(1+\zeta\Phi^{r})\Phi^4))\Phi_{xxxxx}+(15r\zeta\Phi_x^2(r^2-21\mathrm{r}+146)\Phi^{r+3}+15r\Phi_{xx}\zeta(r-13)\Phi^{r+4}+630(1+\zeta\Phi^{r})(\Phi_{xx}\Phi-8\Phi_x^2))\Phi_{xxxx}+(10r(1+\zeta\Phi^{r})\Phi^4)\Phi_{xxx}^2+20\Phi_x(3r\Phi_{xx}\zeta(r^2-21\mathrm{r}+146)\Phi^{r+3}$ $r\zeta\Phi_x^2(r-15)(r^2-15\mathrm{r}+110)\Phi^{r+2}-1008\Phi^2(1+\zeta\Phi^{r})(\Phi_{xx}\Phi-83))\Phi_{xxx}+15r\Phi_{xx}^3$ $\zeta(r^2-21\mathrm{r}+146)\Phi^{r+3}+45r\Phi_{xx}^2\Phi_x^2\zeta(r-15)(r^2-15\mathrm{r}+110)\Phi^{r+2}+15r\Phi_{xx}\Phi_x^4\zeta$ $(r^4-40r^3+635r^2-5000r+19524)\Phi^{r+1}-5040(1+\zeta\Phi^{r})\Phi^3)\Phi_{xx}^3+136080\Phi^2\Phi_x^2$ $(1+\zeta\Phi^{r})\Phi_{xx}^2-453600\Phi\Phi_x^4(1+\zeta\Phi^{r})\Phi_{xx}+(332640+\zeta(r-6)(r-7)(r-8)(r-9)(r-10)(r-11)\Phi^{r})\Phi_x^6))\Phi^{-12}]-2[((\zeta\Phi^{r+2}(r-6)-6\Phi^2)\Phi_{xx}+\Phi_x(3\Phi_{xx}\zeta(r-6)(r-7)\Phi^{r+1}+126\Phi\Phi_{xx}+\Phi_x^2(-336+\zeta(r-6)(r-7)(r-8)\Phi^{r})))u_0\Phi^{-9}][u_0\Phi^{-6}(1+\zeta\Phi^{r})]^5-30[(u_0((\zeta(\mathrm{r}-6)\Phi^{r+3}-6\Phi^3)\Phi_{xxxx}+4\Phi_x(\zeta\Phi^{r+2}(r-6)(r-7)+42\Phi^2)\Phi_{xxx}+3\Phi_{xx}^2\zeta\Phi^{r+2}(r-6)(r-7)+6\Phi_{xx}\Phi_x^2\zeta\Phi^{r+1}(r-6)(r-7)(r-8)+126\Phi_{xx}^2\Phi^2-2016\Phi_x^2\Phi_{xx}\Phi+\Phi_x^4(3024+\zeta\Phi^{r}(r-6)(r-7)(r-8)(r-9))))\Phi^{-10}][u_0\Phi^{-6}(1+\zeta\Phi^{r})]^4$ $[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^{r}+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}]-6[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^{r}+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}]^2[u_0\Phi^{-6}(1+\zeta\Phi^{r})]^4-20[u_0((\zeta\Phi^{r+2}(r-6)-6\Phi^2)\Phi_{xxx}+(3\Phi_{xx}\zeta\Phi^{r+1}(r-6)(r-7)+126\Phi_{xx}\Phi+\Phi_x^2(-336+\zeta\Phi^{r}(r-6)(r-7)(r-8)))\Phi_x)\Phi^{-9}]^2[u_0\Phi^{-6}(1+\zeta\Phi^{r})]^4+2[((42+\zeta\Phi^{r}(r-6)(r-7)\Phi_x^2+\Phi_{xx}(-6\Phi+(r-6)\zeta\Phi^{r+1})u_0\Phi^{-8}][u_0\Phi^{-6}(1+\zeta\Phi^{r})]^4+60[(42+\zeta(\mathrm{r}-6)(\mathrm{r}-7)\Phi_x^2\Phi^{r}+\Phi_{xx}(-6\Phi+(\mathrm{r}-6)\zeta\Phi^{r+1}))u_0\Phi^{-8}]^3[u_0\Phi^{-6}(1+\zeta\Phi^{r})]^3\sim 0.$

Taking into account Kruskal's formula [13], $\Phi(x,t)=x-\Theta(t)$, where $\Theta(t)$ is an arbitrary function, and $u_0=-1440\Phi_x^6$, and $\Phi_x=1$ to have sixth degree resonances polynomial

$$r^6-15r^5+85r^4-225r^3+274r^2-120r-720=0.$$

Solving the last equation for r to get

$$r=-1,6,\frac{5}{2}-\frac{1}{2}\sqrt{-7-8i\sqrt{14}},\frac{5}{2}+\frac{1}{2}\sqrt{-7-8i\sqrt{14}},\frac{5}{2}-\frac{1}{2}\sqrt{-7+8i\sqrt{14}},\frac{5}{2}+\frac{1}{2}\sqrt{-7+8i\sqrt{14}}$$

Obviously, the branch possesses non-integer resonances and the equation does not pass the Painlevé test.

Conclusion

After all what we have discussed in the current work, it seems to be that all examined equations do not survive the Painlevè test, and therefore they are not integrable, in the Painlevè sense, for some special values of the coefficients. The Painlevé analysis, in fact, gives us an idea about the nature of solutions of the equations. The bad positions of resonances provide an evidence on occurring of algebraic or logarithmic branch points in their solutions, and the inconsistent of the compatibility conditions making the equations do not pass the test. The test for other special values of the coefficients or even the more general cases of the equations that needs to be considered in the future work.

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