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Quaternary Continuous Classical Boundary Optimal Control Problem Dominating by Parabolic System

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Abstract

In this paper, our purpose is to study the quaternary continuous classical boundary optimal control vector problem dominated by a quaternary linear parabolic boundary value problem. Under suitable assumptions and with a given quaternary continuous classical boundary control vector, the existence theorem for a unique quaternary state vector solution to the weak form is stated and demonstrated via the method of Galerkin. Furthermore, the continuity of the Lipschitz operator between the state vector solution to the weak form for the dominating equations and the corresponding are proved. The existence of a quaternary continuous classical boundary optimal control vector is stated and demonstrated under suitable Assumptions. The mathematical formulation for the quaternary adjoint boundary value problem associated with each considered boundary value problem is obtained and the Frèchet derivative for the objective function is derived. Finally, the necessary conditions for the optimality theorem of the problem are stated and demonstrated.

Keywords: Quaternary continuous classical boundary optimal control vector problem, Quaternary linear parabolic boundary value problem, Method of Galerkin, Lipschitz continuity operator.

مسألة السيطرة الامثلية الحدودية المستمرة التقليدية الرباعية المسيطرة بنظام مكافيء رباعي

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قسم الرياضيات , كلية العلوم , الجامعة المستنصرية , بغداد, العراق

الخلاصة

الغرض من هذا البحث هو دراسة مسالة السيطرة الامثلية الحدودية المستمرة التقليدية الرباعية المسيطرة بمسالة قيم حدودية رباعية خطية من النوع المكافئ. بوجود فرضيات مناسبة وعندما يكون متجه السيطرة الامثلية الحدودية الرباعية معلوما تم ذكر نص وبرهان مبرهنة وجود الحل للصيغة الضعيفة لمسالة القيم الحدودية الرباعية الخطية من النوع المكافيء باستخدام طريقة كاليركن. واكثر من هذا تم برهنة استمرارية مؤثر ليبشتز بين متجه الحل الرباعي للصيغة الضعيفة ومتجه السيطرة الحدودية الرباعية . تم ذكر نص وبرهان مبرهنة وجود سيطرة امثلية حدودية رباعية بوجد شروط مناسبة .تم ايجاد صياغة رياضية لمسالة القيم الحدودية المصاحبة لمسالة القيم الحدودية المكافئة ومن ثم ايجاد مشتقة فريشيه لدالة الهدف واخيرا" تم ذكر نص وبرهان مبرهة الشروط الضرورية للامثلية .

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1. Introduction

Optimal control problems play an important role in many practical applications, such as in medicine [1], aircraft [2], economics [3], robotics [4], weather conditions [5] and many other scientific fields. They are two types of optimal control problems; the classical and the relax type, each one of these two types is dominated either by ODEqs [6] or PDEqs [7]. The Continuous Classical boundary optimal control problem (CCBOCP) dominated by a couple of parabolic, elliptic, or hyperbolic PDEqs was studied in [8-10]. Later on, these studies for these three types were generalized to deal with CCBOCP dominated by triple linear PDEqs (TLPDEqs) of them so as [11-13]. In each type of these mentioned CCBOCP, the authors introduced a new mathematical model and they also studied and proved the following theorems; the existence theorem for a unique state vector solution for where the classical continuous control vector CCBCV is given, The continuity of the Lipschitz between the state vector solution to the weak form for the dominating equations and the corresponding CCBCV is proved. The existence theorem of a classical continuous optimal control vector(CCBOCV)under suitable conditions is stated and proved, of course, a new mathematical formulation for the adjoint boundary value problem associated with each considered boundary value problem (according to each problem) is obtained so as the Frèchet derivative of the objective function, and the necessary conditions for the optimality theorem.

All of the above mentioned studies encouraged us to consider generalizing the study of the CCBOCP dominated by TLPDEqs of parabolic type to a quaternary continuous classical boundary optimal control vector problem (QCCBOCVP) dominating by quaternary inear parabolic boundary value problem (QLPBVP). According to this idea of generalization, the mathematical model for the dominating equation is needed to be discovered, as well as the objective function. The study of the QCCBOCVP dominating by the QLPBVP that is proposed in this paper starts with the state and prove of the existence theorem of the quaternary state vector solution of the weak form for the QLPBVP using the method of Galerkin, under suitable assumptions and in case the quaternary CCBCV (QCCBCV) is known. The continuity of the Lipschitz between the quaternary state vector solution of the weak form and the corresponding QCCBCV is proved. The existence theorem of a QCCBOCV is stated and demonstrated under suitable Assumptions. The mathematical formulation for the quaternary adjoint boundary value problem (QABVP) associated with the QLPBVP is obtained and the Frèchet derivative for the objective function is derived. The optimality theorem of the problem is stated and demonstrated.

2. Problem Description:

Let $\Omega \subset \mathbb{R}^2$be a bounded open region with boundary $\Gamma, Q = I \times \Omega$, $\Sigma = \Gamma \times I$ and $x = (x_1, x_2)$. The QCCBOCVP consists of the state quaternary equations, which are considered as follows(in Q):

$$y_{1t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{1ij}\frac{\partial y_1}{\partial x_j}\right) + b_1 y_1 - b_5 y_2 + b_6 y_3 + b_7 y_4 = f_1(x,t), \tag{1}$$

$$y_{2t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{2ij}\frac{\partial y_2}{\partial x_j}\right) + b_2 y_2 + b_5 y_1 - b_9 y_3 - b_{11} y_4 = f_2(x,t), \tag{2}$$

$$y_{3t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{3ij}\frac{\partial y_3}{\partial x_j}\right) + b_3 y_3 + b_9 y_2 - b_6 y_1 + b_{15} y_4 = f_3(x,t), \tag{3}$$

$$y_{4t} - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{4ij}\frac{\partial y_1}{\partial x_j}\right) + b_4 y_4 - b_7 y_1 + b_{11} y_2 - b_{15} y_3 = f_4(x,t), \tag{4}$$

with the following boundary conditions and initial conditions :

$$\frac{\partial y_r}{\partial n_r} = \sum_{l,j=1}^{2} a_{rij}\frac{\partial y_r}{\partial x_j}\cos(n_r, x_i) = u_r(x,t), \text{ on } \Sigma \tag{5}$$

$$y_r(x,0) = y_r^0(x), \text{ on } \Omega\ . \tag{6}$$

Where $\vec{f}=(f_1,f_2,f_3,f_4)\in\left(L^2(Q)\right)^4=L^2(Q)$ is a vector of functions for each $(x,t)\in\Omega$, $\vec{u}=(u_1,u_2,u_3,u_4)\in\left(L^2(\Sigma)\right)^4=\boldsymbol{L^2(\Sigma)}$ is a QCCBCV and $\vec{y}=(y_1,y_2,y_3,y_4)=(y_{u1},y_{u2},y_{u3},y_{u4})\in(H^2(\Omega))^4=\boldsymbol{H^2(\Omega)}$is the quaternary state vector solution corresponding to the QCCBCV $\vec{u}$, and n_r (for $r=1,2,3,4$)is a normal vector on the boundary Σ from out and (n_r,x_j) is the angle between n_r and x_j-axis.

The set of admissible QCCBCV is defined by:

$\overrightarrow{W_A}=\{\vec{u}\in L^2(\Sigma)|\,\vec{u}\in\vec{U}=U_1\times U_2\times U_3\times U_4\subset\mathbb{R}^4\ a.e\ \ in\,\Sigma\}$, $\vec{U}$ is a convex set.

The objective function is defined by:

$$\min G_0(\vec{u})=\frac{1}{2}\sum_{r=1}^{4}[\,\|y_r-y_{rd}\|_{L^2(Q)}^2+\beta\|u_r\|_{L^2(\Sigma)}^2]\,,\quad \beta\in\mathbb{R}^+ \tag{7}$$

Let $\vec{V}=V_1\times V_2\times V_3\times V_4=(H^1(\Omega))^4=H^1(\Omega)$, s.t. $\vec{V}=\{\vec{v}:\ \vec{v}=(v_1,v_2,v_3,v_4)\in H^1(\Omega), v_1=v_2=v_3=v_4=0 \text{ on } \partial\Omega\}$.

The weak form of QLPBVP, with $\vec{y}\in\left(H^2(\Omega)\right)^4=H^2(\Omega)$ is:

$$(y_{1t},v_1)+a_1(t,y_1,v_1)+(b_1(t)y_1,v_1)_\Omega-(b_5(t)y_2,v_1)_\Omega+(b_6(t)y_3,v_1)_\Omega+(b_7(t)y_4,v_1)_\Omega=(f_1,v_1)_\Omega+(u_1,v_1)_\Gamma \tag{8a}$$

$$(y_1(0),v_1)_\Omega=(y_1^0,v_1)_\Omega \tag{8b}$$

$$(y_{2t},v_2)+a_2(t,y_2,v_2)+(b_2(t)y_2,v_2)_\Omega+(b_5(t)y_1,v_2)_\Omega-(b_9(t)y_3,v_2)_\Omega-(b_{11}(t)y_4,v_2)_\Omega=(f_2,v_2)_\Omega+(u_2,v_2)_\Gamma \tag{9a}$$

$$(y_2(0),v_2)_\Omega=(y_2^0,v_2)_\Omega \tag{9b}$$

$$(y_{3t},v_3)+a_3(t,y_3,v_3)+(b_3(t)y_3,v_3)_\Omega+(b_9(t)y_2,v_3)_\Omega-(b_6(t)y_1,v_3)_\Omega+(b_{15}(t)y_4,v_3)_\Omega=(f_3,v_3)_\Omega+(u_3,v_3)_\Gamma\ , \tag{10a}$$

$$(y_3(0),v_3)_\Omega=(y_3^0,v_3)_\Omega \tag{10b}$$

$$(y_{4t},v_4)+a_4(t,y_4,v_4)+(b_4(t)y_4,v_4)_\Omega-(b_7(t)y_1,v_4)_\Omega+(b_{11}(t)y_2,v_4)_\Omega-(b_{15}(t)y_3,v_4)_\Omega=(f_4,v_4)_\Omega+(u_4,v_4)_\Gamma \tag{11a}$$

$$(y_4(0),v_4)_\Omega=(y_4^0,v_4)_\Omega\ . \tag{11b}$$

2.1 Assumptions (A):

(i) $f_r\ \forall r=1,2,3,4$ satisfy $|\,f_r(x,t)\,|\le\ \eta_r(x,t),\ \eta_r\in L^2(Q,\mathbb{R})$

(ii)$|\ a_r(t,\ y_r,\ v_r)\ |\le\ \alpha_r\|\ y_r\|_1\|\ v_r\|_1$,$|\ (b_r(\text{t})\ y_r,\ v_r)_\Omega\ |\le\ \beta_r\|\ y_r\|_0\|\ v_r\|_0$

$a_r(t,\ y_r,\ y_r)\ge\ \overline{\alpha}_r\|\ y_r\|_1^2,\ (b_r(\text{t})\ y_r,\ y_r)_\Omega\ge\ \overline{\beta}_r\|\ y_r\|_0^2,\forall r=1,2,3,4.$

$|\ (b_{r+3}(\text{t})\ y_r,\ v_1)_\Omega\ |\le\ \epsilon_r\|\ y_r\|_0\|\ v_1\|_0,\forall r=2,3,4.$

$|\ (b_{2r+3}(\text{t})\ y_r,\ v_2)_\Omega\ |\le\ \bar{\epsilon}_r\|\ y_r\|_0\|\ v_2\|_0,\forall r=1,3,4.$

$|\ (b_{3r+3}(\text{t})\ y_r,\ v_3)_\Omega\ |\le\ \hat{\epsilon}_r\|\ y_r\|_0\|\ v_3\|_0,\forall r=1,2,4.$

$|\ (b_{4r+3}(\text{t})\ y_r,\ v_4)_\Omega\ |\le\ \tilde{\epsilon}_r\|\ y_r\|_0\|\ v_4\|_0,\forall r=1,2,3.$

$c(t,\vec{y},\vec{y})=\ a_1(t,\ y_1,\ y_1)+\ (b_1(\text{t})\ y_1,\ y_1)_\Omega+\ a_2(t,\ y_2,\ y_2)+\ (b_2(\text{t})\ y_2,\ y_2)_\Omega+\ a_3(t,\ y_3,\ y_3)+\ (b_3(\text{t})\ y_3,\ y_3)_\Omega+\ a_4(t,\ y_4,\ y_4)+(b_4(\text{t})\ y_4,\ y_4)_\Omega$

and $(t,\vec{y},\vec{y})\ge\bar{\alpha}\|\vec{y}\|_1^2\ ,\ \|\vec{y}\|_1^2=\sum_{r=1}^{4}\|\ y_r\|_1^2,$

where $\alpha_r,\ \beta_r,\ \epsilon_r,\bar{\alpha}\ \in\mathbb{R}^+,\ r=1,2,3,4.$

Lemma (2.1)[14]:

Let V,H and V' be three Hilbert spaces, each space is included in the following one as in $\|u\|_{L^p(\Omega)}=(\int_\Omega|u(x)|^p dx)^{\frac{1}{p}}$ or $\|u\|_{L^\infty(\Omega)}=ess.sup|u(x)|$

V' is the dual of V. If a function u belongs to $L^2(0,T;V)$,and its derivative u'belongs to $L^2(0,T;V')$, then u is almost every equal to a function continuous from [0,T] into H so one has the following equality which holds in the scalar distribution sense on (0,T) $\frac{d}{dt}|u|^2=2\langle u',u\rangle$.

Main Results

3. The Existence of a unique quaternary state vector solution for the weak form:

Theorem (3.1): With Assumptions (A), for each fixed QCCBCV $\vec{u} \in L^2(\Omega)$, the weak form of the state quaternary equations ((8) - (11)) has a unique quaternary state vector solution $\vec{y} \in \left(L^2(\mathrm{I},\mathrm{V})\right)^4 = \boldsymbol{L^2}(\mathbf{I},\mathbf{V}), \vec{y_t} \in \left(L^2(I,V^*)\right)^4 = \boldsymbol{L^2}(\boldsymbol{I},\boldsymbol{V}^*)$.

Proof: Let $\overrightarrow{V_n} \subset \vec{V}$ be the set of piecewise affine in Ω, let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be the basis of $\overrightarrow{V_n}$ with $n = 4N$, then the approximation solution to $\vec{y}$ of ((8) - (11)) is expressed by

$\vec{y}_n(x,t) = (y_{1n}, y_{2n}, y_{3n}, y_{4n}) = \sum_{j=1}^{n} c_j(t)\, \vec{v}_j(x)$, (12)

where $\vec{V_j} = ((2^{r-1} mod 2) v_{1k}, \left(2^{(r-2)^2} mod 2\right) v_{2k}, \left(2^{(r-3)^2} mod 2\right) v_{3k}, (2^{4-r} mod 2) v_{4k})$, $c_j = c_{rj}$, for $\mathrm{k} = (1,\dots,\mathrm{N})$, $\forall \mathrm{r} = 1,2,3,4$. $\mathrm{j} = \mathrm{k} + \mathrm{N}(\mathrm{r}-1)$ and $c_{rj}(t)$ is an unknown function of t.

The quaternary state vector solution of the weak form is approximated using the method of Galerkin to get

$$(y_{1nt}, v_1)_\Omega + a_1(t, y_{1n}, v_1) + (b_1(t) y_{1n}, v_1)_\Omega - (b_5(t) y_{2n}, v_1)_\Omega + (b_6(t) y_{3n}, v_1)_\Omega + (b_7(t) y_{4n}, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma \quad (13a)$$

$$(y_{1n}(0), v_1)_\Omega = (y_1^0, v_1)_\Omega, \quad \forall v_1 \in V_n \quad (13b)$$

$$(y_{2nt}, v_2)_\Omega + a_2(t, y_{2n}, v_2) + (b_2(t) y_{2n}, v_2)_\Omega + (b_5(t) y_{1n}, v_2)_\Omega - (b_9(t) y_{3n}, v_2)_\Omega - (b_{11}(t) y_{4n}, v_2)_\Omega = (f_2, v_2)_\Omega + (u_2, v_2)_\Gamma \quad (14a)$$

$$(y_{2n}(0), v_2)_\Omega = (y_2^0, v_2)_\Omega , \quad \forall v_2 \in V_n \quad (14b)$$

$$(y_{3nt}, v_3)_\Omega + a_3(t, y_{3n}, v_3) + (b_3(t) y_{3n}, v_3)_\Omega + (b_9(t) y_{2n}, v_3)_\Omega - (b_6(t) y_{1n}, v_3)_\Omega + (b_{15}(t) y_{4n}, v_3)_\Omega = (f_3, v_3)_\Omega + (u_3, v_3)_\Gamma \quad (15a)$$

$$(y_{3n}(0), v_3)_\Omega = (y_3^0, v_3)_\Omega , \forall v_3 \in V_n \quad (15b)$$

$$(y_{4nt}, v_4)_\Omega + a_4(t, y_{4n}, v_4) + (b_4(t) y_{4n}, v_4)_\Omega - (b_7(t) y_{1n}, v_4)_\Omega + (b_{11}(t) y_{2n}, v_4)_\Omega - (b_{15}(t) y_{3n}, v_4)_\Omega = (f_4, v_4)_\Omega + (u_4, v_4)_\Gamma \quad (16a)$$

$$(y_{4n}(0), v_4)_\Omega = (y_4^0, v_4)_\Omega , \forall v_4 \in V_n \quad (16b)$$

Where $y_{rn}^0 = y_{rn}^0(x) = y_{rn}(x,0) \in V_n \subset V \subset L^2(\Omega)$ is the projection of y_r^0 w.r.t. the norm$\| . \|_0$, i.e.$\forall r = 1,2,3,4$.

$(y_{rn}^0, v_r)_\Omega = (y_r^0, v_r)_\Omega \Leftrightarrow \|y_{rn}^0 - y_r^0\|_0 \le \|y_r^0 - v_r\|_0, \forall v_r \in V_n$.

Utilizing (12) in ((13) - (16)), we set $v_r = v_{ri}$, $\forall\, r = 1,2,3,4$, then eqs.((13)-(16)) are equivalent to the following linear system of ODEqs with initial conditions:

$$A_1 C'_{1j}(t) + D_1 C_1(t) - E_1\, C_2(t) + F_1 C_3(t) - H_1 C_4(t) = b_1 \quad (17a)$$

$$A_1 C_1(0) = b_1^0 \quad (17b)$$

$$A_2 C'_{2j}(t) + D_2 C_2(t) + E_2\, C_1(t) - F_2 C_3(t) - H_2 C_4(t) = b_2 \quad (18a)$$

$$A_2 C_2(0) = b_2^0 \quad (18b)$$

$$A_3 C'_{3j}(t) + D_3 C_3(t) + E_3\, C_2(t) - F_3 C_1(t) + H_3 C_4(t) = b_3 \quad (19a)$$

$$A_3 C_3(0) = b_3^0 \quad (19b)$$

$$A_4 C'_{4j}(t) + D_4 C_4(t) - E_4\, C_1(t) + F_4 C_2(t) - H_4 C_3(t) = b_4 \quad (20a)$$

$$A_4 C_4(0) = b_4^0 \quad (20b)$$

Where $A_r = \left(a_{rij}\right)_{n\times n}, a_{rij} = \left(v_{rj}, v_{ri}\right)_\Omega, D_r = \left(d_{rij}\right)_{n\times n}, d_{rij} = [a_r\left(t, v_{rj}, v_{ri}\right) + \left(b_r(t) v_{rj}, v_{ri}\right)_\Omega], E_1 = \left(e_{ij}\right)_{n\times n}, e_{ij} = \left(b_5(t) v_{2j}, v_{1i}\right)_\Omega, E_2 = \left(s_{ij}\right)_{n\times n}, s_{ij} = \left(b_5(t) v_{1j}, v_{2i}\right)_\Omega, E_3 = \left(p_{ij}\right)_{n\times n}, p_{ij} = \left(b_9(t) v_{2j}, v_{3i}\right)_\Omega, E_4 = \left(w_{ij}\right)_{n\times n}, w_{ij} = \left(b_7(t) v_{1j}, v_{4i}\right)_\Omega, F_1 = \left(f_{ij}\right)_{n\times n}, f_{ij} = \left(b_4(t) v_{3j}, v_{1i}\right)_\Omega, F_2 = \left(m_{ij}\right)_{n\times n}, m_{ij} = \left(b_9(t) v_{3j}, v_{2i}\right)_\Omega, F_3 = \left(g_{ij}\right)_{n\times n}, g_{ij} = \left(b_6(t) v_{1j}, v_{3i}\right)_\Omega, F_4 = \left(k_{ij}\right)_{n\times n}, k_{ij} = \left(b_{11}(t) v_{2j}, v_{4i}\right)_\Omega, H_1 = \left(h_{ij}\right)_{n\times n} h_{ij} = \left(b_7(t) v_{4j}, v_{1i}\right)_\Omega, H_2 = \left(l_{ij}\right)_{n\times n}, l_{ij} = , \left(b_{11}(t) v_{4j}, v_{2i}\right)_\Omega, H_3 = \left(q_{ij}\right)_{n\times n}, q_{ij} = \left(b_{15}(t) v_{4j}, v_{3i}\right)_\Omega, H_4 = \left(x_{ij}\right)_{n\times n}, x_{ij} =$

$\left(b_{15}(t)v_{3j},v_{4i}\right)_{\Omega}, C_r(t)=\left(c_{rj}(t)\right)_{n\times 1}, \ c'_r(t)=\left(c'_{rj}(t)\right)_{n\times 1}, C_r(0)=\left(c_{rj}(0)\right)_{n\times 1},$
$b_r=(b_{ri})_{n\times 1}, b_{ri}=(f_r,v_{ri})_{\Omega}+(u_r,v_{ri})_{\Gamma}, b_{ri}^0=(y_r^0,v_{ri})_{\Omega}, \forall i,j=1,2,\dots,n$.

The norm $\left\|\overrightarrow{y_n^0}\right\|_0$ is bounded :

Since $y_r^0=y_r^0(x)\in L^2(\Omega)$, then there exists $\{v_{rn}^0\}$ with $v_{rn}^0\in V_n\subset V\subset L^2(\Omega)$, such that $v_{rn}^0 \xrightarrow{s} y_r^0$ in $L^2(\Omega)$, $\forall r=1,2,3,4$. By projection theorem, we have
$\|y_{rn}^0-y_r^0\|_0\le\|y_r^0-v_r\|_0 \ ,\forall v_r\in V_n$.
Then for $v_r=v_{rn}^0$, $\forall n=1,2,\dots$,
$\|y_{rn}^0-y_r^0\|_0\le\|y_r^0-v_{rn}^0\|_0 \ ,\forall v_{rn}^0\in V_n\subset \mathrm{V}\subset L^2(\Omega)$
Thus, $\|y_{rn}^0-y_r^0\|_0\to 0$, so $y_{rn}^0\xrightarrow{s} y_r^0$ in $L^2(\Omega)$ with $\|y_{rn}^0\|_0\le b_r$.
Therefore, the norm$\|\overrightarrow{y_n}(t)\|_{L^\infty\left(I,L^2(\Omega)\right)}$ and $\|\overrightarrow{y_n}(t)\|_Q$are bounded:

Set $v_r=y_{rn}$ where $\forall r$=1,2,3,4, in Eqs. ((13a) - (16a)) and integrating both sides with respect to t from 0 to T**,** then adding the four equations using Assumption (A-ii), we get:
$\int_0^T(\overrightarrow{y_{nt}},\overrightarrow{y_n})dt+\bar{\alpha}\int_0^T\|\overrightarrow{y_n}\|_1^2\,dt\le\int_0^T(f_1,y_{1n})_{\Omega}dt+\int_0^T(u_1,y_{1n})_{\Gamma}dt+\int_0^T(f_2,y_{2n})_{\Omega}\,dt+$
$\int_0^T(u_2,y_{2n})_{\Gamma}\,dt+\int_0^T(f_3,y_{3n})_{\Omega}dt+\int_0^T(u_3,y_{3n})_{\Gamma}dt+\int_0^T(f_4,y_{4n})_{\Omega}dt+\int_0^T(u_4,y_{4n})_{\Gamma}\,dt$ (21)
Now, Lemma (2.1) can be used in the first term of the L.H.S. in Eq.(21), since $\overrightarrow{y_{nt}}\in$ $\boldsymbol{L^2(I,v^*)=L^2(I,v)}$ and $\overrightarrow{y_n}\in\boldsymbol{L^2(I,v)}$. On the other hand, since the second term is nonnegative, we take $T=t\in[0,T]$ and then by the Cauchy-Schwarz inequality and using Assumption (A-i), we get:
$\frac{1}{2}\int_0^t\frac{\mathrm{d}}{\mathrm{dt}}\|\overrightarrow{y_n}(t)\|_0^2\,\mathrm{dt}\le\frac{1}{2}[\int_0^t\int_{\Omega}(\eta_1^2+|y_{1n}|^2)\,dx\,dt+\int_0^t\int_{\Gamma}(|u_1|^2+|y_{1n}|^2)\,d\gamma\,dt+$
$\int_0^t\int_{\Omega}(\eta_2^2+|y_{2n}|^2)\,dx\,dt+\int_0^t\int_{\Gamma}(|u_2|^2+|y_{2n}|^2)\,d\gamma dt+\int_0^t\int_{\Omega}(\eta_3^2+|y_{3n}|^2)\,dx\,dt+$
$\int_0^t\int_{\Gamma}(|u_3|^2+|y_{3n}|^2)\,d\gamma dt+\int_0^t\int_{\Omega}(\eta_4^2+|y_{4n}|^2)\,dx\,dt+\int_0^t\int_{\Gamma}(|u_4|^2+|y_{4n}|^2)\,d\gamma dt]$.
Using trace theorem and Assumption (A-ii) in the R.H.S., the above inequality gives:
$\|\overrightarrow{y_n}(t)\|_0^2-\|\overrightarrow{y_n}(0)\|_0^2\le\|\eta_1\|_Q^2+(1+\beta_1)\int_0^t\|y_{1n}\|_0^2\,dt+\|u_1\|_{\Sigma}^2+\|\eta_2\|_Q^2+(1+\beta_2)$
$\int_0^t\|y_{2n}\|_0^2\,dt+\|u_2\|_{\Sigma}^2+\|\eta_3\|_Q^2+(1+\beta_3)\int_0^t\|y_{3n}\|_0^2\,dt+\|u_3\|_{\Sigma}^2+\|\eta_4\|_Q^2+(1+\beta_4)$
$\int_0^t\|y_{4n}\|_0^2\,dt+\|u_4\|_{\Sigma}^2$
since $\|\eta_r\|_Q^2\le k_r, \|u_r\|_{\Sigma}^2\le p_r$, $\forall \mathrm{r}=1,2,3,4$, $\|\overrightarrow{y_n}(0)\|_0^2\le c$, then
$\|\overrightarrow{y_n}(t)\|_0^2\le \mathrm{s}+\bar{\mathrm{s}}\int_0^t\|\overrightarrow{y_n}(t)\|_0^2\,\mathrm{dt}$,
where, $s=\sum_{r=1}^4 k_r+\sum_{r=1}^4 p_r+c$, $\bar{\mathrm{s}}=\max\,(1+\beta_1,1+\beta_2,1+\beta_3,1+\beta_4)$.
$\Rightarrow\|\overrightarrow{y_n}(t)\|_0^2\le se^{\int_0^t\bar{s}\,dt}\le se^{\int_0^T\bar{s}\,dt}=se^{\bar{s}T}=s_1\Rightarrow\|\overrightarrow{y_n}(t)\|_Q\le s_2$

The norm $\|\overrightarrow{\mathbf{y_n}}(\mathbf{t})\|_{\mathbf{L^2(I,V)}}$ is bounded:

Using Lemma (2.1) for the first terms of L.H.S. in Eq.(21), then using the same results which are obtained from the R.H.S., setting $t=T$ and using $\|\overrightarrow{y_n}(t)\|_0^2$ is positive so that Eq.(21) becomes:
$\|\overrightarrow{y_n}(t)\|_0^2+2\bar{\alpha}\int_0^T\|\overrightarrow{y_n}\|_1^2\,dt\le \mathrm{s}+\bar{\mathrm{s}}\|\overrightarrow{y_n}(t)\|_Q^2\Rightarrow\int_0^T\|\overrightarrow{y_n}\|_1^2\,dt\le\frac{\mathrm{s}+\bar{\mathrm{s}}\,s_3}{2\bar{\alpha}}=s_5$, where $s_6=\sqrt{s_5}$
$\Rightarrow\|\overrightarrow{y_n}(\mathrm{t})\|_{\mathbf{L^2(I,v)}}\le s_6$.

The convergence of the Approximation Solution:

Let $\{\overrightarrow{V_n}\}_{n=1}^{\infty}$be a sequence of subspaces of $\vec{V}$ such that $\forall\vec{v}\in\vec{V}$, then there exists a sequence $\{\overrightarrow{v_n}\}$, such that $\overrightarrow{v_n}\xrightarrow{s}\vec{v}$ in $\vec{V}$ s.t. $\overrightarrow{v_n}\xrightarrow{s}\vec{v}$ in $\boldsymbol{L^2(\Omega)}$**,** with $\overrightarrow{v_n}\in\overrightarrow{V_n}\subset\vec{V}\subset\boldsymbol{L^2(\Omega)}$. Utilizing $\vec{v}=\overrightarrow{v_n}=(v_{1n},v_{2n},v_{3n},v_{4n})$in Eqs.((13) - (16) a & b) to obtain

$(y_{1nt}, v_{1n})_\Omega + a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b_5(t)y_{2n}, v_{1n})_\Omega + (b_6(t)y_{3n}, v_{1n})_\Omega + (b_7(t)y_{4n}, v_{1n})_\Omega = (f_1, v_{1n})_\Omega + (u_1, v_{1n})_\Gamma$ (22a)

$(y_{1n}(0), v_{1n})_\Omega = (y_1^0, v_{1n})_\Omega$ (22b)

$(y_{2nt}, v_{2n})_\Omega + a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b_5(t)y_{1n}, v_{2n})_\Omega - (b_9(t)y_{3n}, v_{2n})_\Omega - (b_{11}(t)y_{4n}, v_{2n})_\Omega = (f_2, v_{2n})_\Omega + (u_2, v_{2n})_\Gamma$ (23a)

$(y_{2n}(0), v_{2n})_\Omega = (y_2^0, v_{2n})_\Omega$ (23b)

$(y_{3nt}, v_{3n})_\Omega + a_3(t, y_{3n}, v_{3n}) + (b_3(t)y_{3n}, v_{3n})_\Omega + (b_9(t)y_{2n}, v_{3n})_\Omega - (b_6(t)y_{1n}, v_{3n})_\Omega + (b_{15}(t)y_{4n}, v_{3n})_\Omega = (f_3, v_{3n})_\Omega + (u_3, v_{3n})_\Gamma$ (24a)

$(y_{3n}(0), v_{3n})_\Omega = (y_3^0, v_{3n})_\Omega$ (24b)

$(y_{4nt}, v_{4n})_\Omega + a_4(t, y_{4n}, v_{4n}) + (b_4(t)y_{4n}, v_{4n})_\Omega - (b_7(t)y_{1n}, v_{4n})_\Omega + (b_{11}(t)y_{2n}, v_{4n})_\Omega - (b_{15}(t)y_{3n}, v_{4n})_\Omega = (f_4, v_{4n})_\Omega + (u_4, v_{4n})_\Gamma$ (25a)

$(y_{4n}(0), v_{4n})_\Omega = (y_4^0, v_{4n})_\Omega$ (25b)

Which has a sequence of approximation solution $\{\overrightarrow{y_n}\}_{n=1}^{\infty}$ with $\|\overrightarrow{y_n}(t)\|_{L^2(Q)}$, $\|\overrightarrow{y_n}(t)\|_{L^2(\mathbf{I},\mathbf{V})}$ are bounded, then by Alauglu's theorem, there exists a subsequence of $\{\overrightarrow{y_n}\}_{n\in N}$, say again $\{\overrightarrow{y_n}\}_{n\in N}$ s.t. $\overrightarrow{y_n} \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(Q)}$ and $\boldsymbol{L^2(I,V)}$, multiplying both sides of Eqs.((22)-(25)) by $\varphi_r(t) \in C^1[0,T]$, $\forall r = 1,2,3,4$, S.t. $\varphi_r(T) = 0$,$\varphi_r(0) \neq 0$ and integrating both sides with respect to t from 0 to T, then integrating by parts for the first terms in the L.H.S., to obtain

$-\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)\,dt + \int_0^T [a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b_5(t)y_{2n}, v_{1n})_\Omega + (b_6(t)y_{3n}, v_{1n})_\Omega + (b_7(t)y_{4n}, v_{1n})_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_{1n})_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_{1n})_\Gamma \varphi_1(t)dt + (y_{1n}^0, v_{1n})_\Omega\, \varphi_1(0)$ (26)

$-\int_0^T (y_{2n}, v_{2n})\, \varphi_2'(t)dt + \int_0^T [a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b_5(t)y_{1n}, v_{2n})_\Omega - (b_9(t)y_{3n}, v_{2n})_\Omega - (b_{11}(t)y_{4n}, v_{2n})_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_{2n})_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_{2n})_\Gamma \varphi_2(t)dt + (y_{2n}^0, v_{2n})_\Omega\, \varphi_2(0)$ (27)

$-\int_0^T (y_{3n}, v_{3n})\varphi_3'(t)\,dt + \int_0^T [a_3(t, y_{3n}, v_{3n}) + (b_3(t)y_{3n}, v_{3n})_\Omega + (b_9(t)y_{2n}, v_{3n})_\Omega - (b_6(t)y_{1n}, v_{3n})_\Omega + (b_{15}(t)y_{4n}, v_{3n})_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_{3n})_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_{3n})_\Gamma \varphi_3(t)dt + (y_{3n}^0, v_{3n})_\Omega \varphi_3(0)$ (28)

$-\int_0^T (y_{4n}, v_{4n})\varphi_4'(t)dt + \int_0^T [a_4(t, y_{4n}, v_{4n}) + (b_4(t)y_{4n}, v_{4n})_\Omega - (b_7(t)y_{1n}, v_{4n})_\Omega + (b_{11}(t)y_{2n}, v_{4n})_\Omega - (b_{15}(t)y_{3n}, v_{4n})_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_{4n})_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_{4n})_\Gamma \varphi_4(t)dt + (y_{4n}^0, v_{4n})_\Omega \varphi_4(0)$ (29)

Since $v_{rn} \xrightarrow{s} v_r$ in V and $L^2(\Omega)$, then $v_{rn}\varphi_r'(t) \xrightarrow{s} v_r\, \varphi_r'(t)$ in $L^2(I,V)$, $v_{rn}\varphi_r(t) \xrightarrow{s} v_r\, \varphi_r(t)$ in $L^2(Q)$, and since $y_{rn} \xrightarrow{W} y_r$ in $L^2(Q)$, $y_{rn}^0 \xrightarrow{s} y_r^0$ in $L^2(\Omega)$, $\forall r = 1,2,3,4$, on the other hand since $v_{rn} \xrightarrow{s} v_r$ in $L^2(\Omega)$, then $v_{rn} \xrightarrow{w} v_r$, and $v_{rn}\varphi_r(t) \xrightarrow{w} v_r\, \varphi_r(t)$, hence:

$-\int_0^T (y_{1n}, v_{1n})\varphi_1'(t)\,dt + \int_0^T [a_1(t, y_{1n}, v_{1n}) + (b_1(t)y_{1n}, v_{1n})_\Omega - (b_5(t)y_{2n}, v_{1n})_\Omega + (b_6(t)y_{3n}, v_{1n})_\Omega + (b_7(t)y_{4n}, v_{1n})_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_{1n})_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_{1n})_\Gamma \varphi_1(t)dt + (y_{1n}^0, v_{1n})_\Omega \varphi_1(0) \rightarrow$

$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1^0, v_1)_\Omega \varphi_1(0)$ (30)

$-\int_0^T (y_{2n}, v_{2n})\, \varphi_2'(t)dt + \int_0^T [a_2(t, y_{2n}, v_{2n}) + (b_2(t)y_{2n}, v_{2n})_\Omega + (b_5(t)y_{1n}, v_{2n})_\Omega - (b_9(t)y_{3n}, v_{2n})_\Omega - (b_{11}(t)y_{4n}, v_{2n})_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_{2n})_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_{2n})_\Gamma \varphi_2(t)dt + (y_{2n}^0, v_{2n})_\Omega \varphi_2(0) \rightarrow$

$$-\int_0^T (y_2, v_2)\ \varphi_2'(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2^0, v_2)_\Omega \varphi_2(0) \qquad (31)$$

$$-\int_0^T (y_{3n}, v_{3n})\varphi_3'(t)\,dt + \int_0^T [\,a_3(t, y_{3n}, v_{3n}) + (b_3(t)y_{3n}, v_{3n})_\Omega + (b_9(t)y_{2n}, v_{3n})_\Omega - (b_6(t)y_{1n}, v_{3n})_\Omega + (b_{15}(t)y_{4n}, v_{3n})_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_{3n})_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_{3n})_\Gamma \varphi_3(t)dt + (y_{3n}^0, v_{3n})_\Omega \varphi_3(0) \to$$

$$-\int_0^T (y_3, v_3)\varphi_3'(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y_3^0, v_3)_\Omega \varphi_3(0) \qquad (32)$$

$$-\int_0^T (y_{4n}, v_{4n})\varphi_4'(t)dt + \int_0^T [\,a_4(t, y_{4n}, v_{4n}) + (b_4(t)y_{4n}, v_{4n})_\Omega - (b_7(t)y_{1n}, v_{4n})_\Omega + (b_{11}(t)y_{2n}, v_{4n})_\Omega - (b_{15}(t)y_{3n}, v_{4n})_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_{4n})_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_{4n})_\Gamma \varphi_4(t)dt + (y_{4n}^0, v_{4n})_\Omega \varphi_4(0) \to$$

$$-\int_0^T (y_4, v_4)\varphi_4'(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma \varphi_4(t)dt + (y_4^0, v_4)_\Omega \varphi_4(0) \qquad (33)$$

which means:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1^0, v_1)_\Omega \varphi_1(0) \qquad (34)$$

$$-\int_0^T (y_2, v_2)\ \varphi_2'(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt + (y_2^0, v_2)_\Omega \varphi_2(0) \qquad (35)$$

$$-\int_0^T (y_3, v_3)\varphi_3'(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt + (y_3^0, v_3)_\Omega \varphi_3(0) \qquad (36)$$

$$-\int_0^T (y_4, v_4)\varphi_4'(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma \varphi_4(t)dt + (y_4^0, v_4)_\Omega \varphi_4(0) \qquad (37)$$

Case 1: Choose $\varphi_r \in D[0,T]$, i.e. $\varphi_r(T) = \varphi_r(0) = 0$,$\forall r = 1,2,3,4$, So, ((34) - (37)), then Using integrating by parts for the first terms in the L.H.S. of the above equations:

$$\int_0^T (y_{1t}, v_1)\varphi_1(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt \qquad (38)$$

$$\int_0^T (y_{2t}, v_2)\ \varphi_2(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_2)_\Omega \varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma \varphi_2(t)dt \qquad (39)$$

$$\int_0^T (y_{3t}, v_3)\varphi_3(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_3)_\Omega \varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma \varphi_3(t)dt \qquad (40)$$

$$\int_0^T (y_{4t}, v_4)\varphi_4(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_4)_\Omega \varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma \varphi_4(t)dt \qquad (41)$$

i.e. $\vec{y}$ is the quaternary state vector solution of the weak form

Case 2: Choose $\varphi_r \in C^1[0,T]$, i.e. $\varphi_r(T) = 0$ & $\varphi_r(0) \neq 0, \forall r = 1,2,3,4$. Using integrating by parts for the first term in the L.H.S. of (38) to get:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega \varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma \varphi_1(t)dt + (y_1(0), v_1)_\Omega \varphi_1(0). \quad (42)$$

By subtracting (34) from (42), we get:

$(y_1^0, v_1)_\Omega \varphi_1(0) = (y_1(0), v_1)_\Omega \varphi_1(0)$, $\varphi_1(0) \neq 0, \forall \varphi_1(0) \in [0,T]$

$(y_1^0, v_1)_\Omega = (y_1(0), v_1)_\Omega$.

That means the first initial condition holds. The same manner can be utilized to get that the initial conditions hold, which means that $(y_r^0, v_r)_\Omega = (y_r(0), v_r)_\Omega$, $\forall r = 1,2,3,4$.

The strong convergence for the Approximation Solution:

Utilizing $v_r = y_{rn}$ and $v_r = y_r, \forall r = 1,2,3,4$. In Eqs.((8a)-(11a)) and Eqs.((13a)-(16a)), respectively. By integrating both sides of these Eqs. from 0 to T, then we add all equations together using Assumption (A-ii) to get:

$$\int_0^T (\overrightarrow{y_{nt}}, \overrightarrow{y_n})dt + \int_0^T c(t, \overrightarrow{y_n}, \overrightarrow{y_n})\,dt = \int_0^T (f_1, y_{1n})_\Omega dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (f_2, y_{2n})_\Omega\,dt + \int_0^T (u_2, y_{2n})_\Gamma\,dt + \int_0^T (f_3, y_{3n})_\Omega dt + \int_0^T (u_3, y_{3n})_\Gamma dt + \int_0^T (f_4, y_{4n})_\Omega dt + \int_0^T (u_4, y_{4n})_\Gamma\,d \quad (43a)$$

$$\int_0^T (\vec{y_t}, \vec{y})dt + \int_0^T c(t, \vec{y}, \vec{y})\,dt = \int_0^T (f_1, y_1)_\Omega dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (f_2, y_2)_\Omega\,dt + \int_0^T (u_2, y_2)_\Gamma\,dt + \int_0^T (f_3, y_3)_\Omega dt + \int_0^T (u_3, y_3)_\Gamma dt + \int_0^T (f_4, y_4)_\Omega dt + \int_0^T (u_4, y_4)_\Gamma\,dt \quad 43b)$$

Using Lemma (2.1) for the first terms in the L.H.S. of Eq.((43) a and b), it becomes:

$$\frac{1}{2}\|\overrightarrow{y_n}(T)\|_0^2 - \frac{1}{2}\|\overrightarrow{y_n}(0)\|_0^2 + \int_0^T c(t, \overrightarrow{y_n}, \overrightarrow{y_n})\,dt = \int_0^T (f_1, y_{1n})_\Omega dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (f_2, y_{2n})_\Omega\,dt + \int_0^T (u_2, y_{2n})_\Gamma\,dt + \int_0^T (f_3, y_{3n})_\Omega dt + \int_0^T (u_3, y_{3n})_\Gamma dt + \int_0^T (f_4, y_{4n})_\Omega dt + \int_0^T (u_4, y_{4n})_\Gamma\,dt \quad (44a)$$

$$\frac{1}{2}\|\vec{y}(T)\|_0^2 - \frac{1}{2}\|\vec{y}(0)\|_0^2 + \int_0^T c(t, \vec{y}, \vec{y})\,dt = \int_0^T (f_1, y_1)_\Omega dt + \int_0^T (u_1, y_1)_\Gamma dt + \int_0^T (f_2, y_2)_\Omega\,dt + \int_0^T (u_2, y_2)_\Gamma\,dt + \int_0^T (f_3, y_3)_\Omega dt + \int_0^T (u_3, y_3)_\Gamma dt + \int_0^T (f_4, y_4)_\Omega dt + \int_0^T (u_4, y_4)_\Gamma\,dt \quad (44b)$$

Now, consider the following equality:

$$\frac{1}{2}\|\overrightarrow{y_n}(T) - \vec{y}(T)\|_0^2 - \frac{1}{2}\|\overrightarrow{y_n}(0) - \vec{y}(0)\|_0^2 + \int_0^T c(t, \overrightarrow{y_n} - \vec{y}, \overrightarrow{y_n} - \vec{y})\,dt = P_1 - P_2 - P_3 \quad (45)$$

Where

$$P_1 = \frac{1}{2}\|\overrightarrow{y_n}(T)\|_0^2 - \frac{1}{2}\|\overrightarrow{y_n}(0)\|_0^2 + \int_0^T c(t, \overrightarrow{y_n}, \overrightarrow{y_n})\,dt$$

$$P_2 = \frac{1}{2}(\overrightarrow{y_n}(T), \vec{y}(T))_\Omega - \frac{1}{2}(\overrightarrow{y_n}(0), \vec{y}(0))_\Omega + \int_0^T c(t, \overrightarrow{y_n}(t), \vec{y}(t))\,dt$$

$$P_3 = \frac{1}{2}(\vec{y}(T), \overrightarrow{y_n}(T) - \vec{y}(T))_\Omega - \frac{1}{2}\left(\vec{y}(0), \overrightarrow{y_n}(0) - \vec{y}(0)\right)_\Omega + \int_0^T c(t, \vec{y}(t), \overrightarrow{y_n}(t) - \vec{y}(t))dt$$

Since

$$\overrightarrow{y_n^0} = \overrightarrow{y_n}(0) \xrightarrow{s} \overrightarrow{y^0} = \vec{y}(0), \text{ in } \boldsymbol{L^2(\Omega)}, \overrightarrow{y_n}(T) \xrightarrow{s} \vec{y}(T) \text{ in } \boldsymbol{L^2(\Omega)} \quad (46)$$

Then

$$(\vec{y}(0), \overrightarrow{y_n}(0) - \vec{y}(0))_\Omega \to 0 \text{ and } (\vec{y}(T), \overrightarrow{y_n}(T) - \vec{y}(T))_\Omega \to 0 \quad (47)$$

$$\|\overrightarrow{y_n}(0) - \vec{y}(0)\|_0^2 \to 0 \text{ and } \|\overrightarrow{y_n}(T) - \vec{y}(T)\|_0^2 \to 0 \quad (48)$$

and since $\overrightarrow{y_n} \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(I, V)} \Rightarrow \overrightarrow{y_n} \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(\Sigma)}$, then

$$\int_0^T c(t, \vec{y}(t), \overrightarrow{y_n}(t) - \vec{y}(t))dt \to 0 \quad (49)$$

Since $\overrightarrow{y_n} \xrightarrow{w} \vec{y}$ in $\boldsymbol{L^2(Q)}$**,** hence:

$$\int_0^T (f_1, y_{1n})_\Omega dt + \int_0^T (u_1, y_{1n})_\Gamma dt + \int_0^T (f_2, y_{2n})_\Omega\,dt + \int_0^T (u_2, y_{2n})_\Gamma\,dt + \int_0^T (f_3, y_{3n})_\Omega dt + \int_0^T (u_3, y_{3n})_\Gamma dt + \int_0^T (f_4, y_{4n})_\Omega dt + \int_0^T (u_4, y_{4n})_\Gamma\,dt \to \int_0^T (f_1, y_1)_\Omega dt + \int_0^T (u_1, y_1)_\Gamma dt +$$

$$\int_0^T (f_2, y_2)_\Omega\, dt + \int_0^T (u_2, y_2)_\Gamma\, dt + \int_0^T (f_3, y_3)_\Omega dt + \int_0^T (u_3, y_3)_\Gamma dt + \int_0^T (f_4, y_4)_\Omega dt + \int_0^T (u_4, y_4)_\Gamma\, dt \quad (50)$$

Now, when $n \to \infty$ in both sides of Eq.(45), we have the following results:
1-The first two terms in L.H.S. of Eq.(45) are tending to zero from Eq.(48).
2- L.H.S. of Eq. (P_1) →L.H.S. of Eq.(44b).
3- Eq. (P_2) → L.H.S.Eq.(44b)
4- The three terms in P_3 are tending to zero from Eq.(47) and Eq.(49), from the convergence of the above. The above sides of Eq.(45) give:
$\int_0^T c(t, \overrightarrow{y_n} - \vec{y}, \overrightarrow{y_n} - \vec{y})\, dt \to 0$, as $n \to \infty$.
By Assumption (A-ii), one gets
$\bar{\alpha} \int_0^T \|\overrightarrow{y_n} - \vec{y}\|_1^2\, dt \le \int_0^T c(t, \overrightarrow{y_n} - \vec{y}, \overrightarrow{y_n} - \vec{y})\, dt \to 0$.
$\bar{\alpha} \int_0^T \|\overrightarrow{y_n} - \vec{y}\|_1^2\, dt \to 0 \Rightarrow$So, $\overrightarrow{y_n} \xrightarrow{s} \vec{y}$ in $\boldsymbol{L^2(I, V)}$.

Uniqueness of the quaternary state vector solution: Let $\vec{y}$ and $\vec{\hat{y}}$ be two quaternary state vector solutions for the weak form ((8a)-(11a)), consider the first equation we have:

$$(y_{1t}, v_1) + a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma \quad (51)$$

$$(\hat{y}_{1t}, v_1) + a_1(t, \hat{y}_1, v_1) + (b_1(t)\hat{y}_1, v_1)_\Omega - (b_5(t)\hat{y}_2, v_1)_\Omega + (b_6(t)\hat{y}_3, v_1)_\Omega + (b_7(t)\hat{y}_4, v_1)_\Omega = (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma \quad (52)$$

By subtracting the Eq.(51) from Eq.(52), then utilizing $v_1 = y_1 - \hat{y}_1$ to get:

$$((y_1 - \hat{y}_1)_t, y_1 - \hat{y}_1) + a_1(t, y_1 - \hat{y}_1, y_1 - \hat{y}_1) + (b_1(t)y_1 - \hat{y}_1, y_1 - \hat{y}_1)_\Omega - (b_5(t)y_2 - \hat{y}_2, y_1 - \hat{y}_1)_\Omega + (b_6(t)y_3 - \hat{y}_3, y_1 - \hat{y}_1)_\Omega + (b_7(t)y_4 - \hat{y}_4, y_1 - \hat{y}_1)_\Omega = 0 \quad (53)$$

The same manner can be used to get that:

$$((y_2 - \hat{y}_2)_t, y_2 - \hat{y}_2) + a_2(t, y_2 - \hat{y}_2, y_2 - \hat{y}_2) + (b_2(t)y_2 - \hat{y}_2, y_2 - \hat{y}_2)_\Omega + (b_5(t)y_1 - \hat{y}_1, y_2 - \hat{y}_2)_\Omega - (b_9(t)y_3 - \hat{y}_3, y_2 - \hat{y}_2)_\Omega - (b_{11}(t)y_4 - \hat{y}_4, y_2 - \hat{y}_2)_\Omega = 0 \quad (54)$$

$$((y_3 - \hat{y}_3)_t, y_3 - \hat{y}_3) + a_3(t, y_3 - \hat{y}_3, y_3 - \hat{y}_3) + (b_3(t)y_3 - \hat{y}_3, y_3 - \hat{y}_3)_\Omega + (b_9(t)y_2 - \hat{y}_2, y_3 - \hat{y}_3)_\Omega - (b_6(t)y_1 - \hat{y}_1, y_3 - \hat{y}_3)_\Omega + (b_{15}(t)y_4 - \hat{y}_4, y_3 - \hat{y}_3)_\Omega = 0 \quad (55)$$

$$((y_4 - \hat{y}_4)_t, y_4 - \hat{y}_4) + a_4(t, y_4 - \hat{y}_4, y_4 - \hat{y}_4) + (b_3(t)y_4 - \hat{y}_4, y_4 - \hat{y}_4)_\Omega - (b_7(t)y_1 - \hat{y}_1, y_4 - \hat{y}_4)_\Omega + (b_{11}(t)y_2 - \hat{y}_2, y_4 - \hat{y}_4)_\Omega - (b_{15}(t)y_3 - \hat{y}_3, y_4 - \hat{y}_4)_\Omega = 0 \quad (56)$$

Adding Eqs.((53)-(56)) and using Lemma (2.1) and Assumption (A-ii) to obtain:

$$\frac{1}{2}\frac{d}{dt}\left\|\vec{y} - \vec{\hat{y}}\right\|_0^2 + \bar{\alpha}\left\|\vec{y} - \vec{\hat{y}}\right\|_1^2 \le 0 . \quad (57)$$

Since the second term of L.H.S. of (57) is positive, and by integrating both sides from 0 to T,
$\int_0^T \frac{d}{dt}\left\|\vec{y} - \vec{\hat{y}}\right\|_0^2 dt \le 0 \Rightarrow \left\|(\vec{y} - \vec{\hat{y}})(t)\right\|_0^2 \le 0 \Rightarrow \left\|\vec{y} - \vec{\hat{y}}\right\|_0^2 = 0$, $\forall t \in I$
Now, integrating both sides (57) from 0 to T, using the initial conditions and the above result, we get:
$\int_0^T \frac{d}{dt}\left\|\vec{y} - \vec{\hat{y}}\right\|_0^2 dt + 2\bar{\alpha}\int_0^T \left\|\vec{y} - \vec{\hat{y}}\right\|_1^2 dt \le 0 \Rightarrow \int_0^T \left\|\vec{y} - \vec{\hat{y}}\right\|_1^2 dt = 0 \Rightarrow \vec{y} = \vec{\hat{y}}$.

4. Existence of a QCCBOCV:

Theorem (4.1):
a-Consider Assumptions (A) hold, $\vec{y}$ and $\vec{y} + \Delta\vec{y}$ are the QSVS corresponding to the QCCBCV $\vec{u} \in \boldsymbol{L^2(\Sigma)}$and $\vec{u} + \Delta\vec{u} \in \boldsymbol{L^2(\Sigma)}$, respectively. Then:
$\|\Delta\vec{\boldsymbol{y}}\|_{L^\infty(I, L^2(\Omega))} \le \boldsymbol{k}\|\Delta\vec{\boldsymbol{u}}\|_\Sigma$,$\|\Delta\vec{\boldsymbol{y}}\|_{L^2(Q)} \le \boldsymbol{k}\|\Delta\vec{\boldsymbol{u}}\|_\Sigma$,$\|\Delta\vec{\boldsymbol{y}}\|_{L^2(I,V)} \le \boldsymbol{k}\|\Delta\vec{\boldsymbol{u}}\|_\Sigma$
b- With Assumptions (A), the LIO $\vec{u} \to \vec{y}_{\vec{u}}$ from $\boldsymbol{L^2(\Sigma)}$ into $\boldsymbol{L^\infty(I, L^2(\Omega))}$or into$\boldsymbol{L^2(Q)}$is continuous.

Proof : Let $\vec{u},\vec{\hat{u}} \in \boldsymbol{L^2(\Sigma)}$ and let $\Delta\vec{u} = \vec{\hat{u}} - \vec{u}$, hence by Theorem (3.1), there exists a unique QSVS $\vec{y}$ &$\vec{\hat{y}}$ of ((2.14) - (2.17)) satisfies the following equations:

$$(\hat{y}_{1t}, v_1) + a_1(t,\hat{y}_1,v_1) + (b_1(t)\hat{y}_1,v_1)_\Omega - (b_5(t)\hat{y}_2,v_1)_\Omega + (b_6(t)\hat{y}_3,v_1)_\Omega + (b_7(t)\hat{y}_4,v_1)_\Omega = (f_1,v_1)_\Omega + (\Delta u_1 + u_1, v_1)_\Gamma \quad (58a)$$
$$(\hat{y}_1(0),v_1)_\Omega = (\hat{y}_1^0,v_1)_\Omega \quad (58b)$$
$$(\hat{y}_{2t}, v_2) + a_2(t,\hat{y}_2,v_2) + (b_2(t)\hat{y}_2,v_2)_\Omega + (b_5(t)\hat{y}_1,v_2)_\Omega - (b_9(t)\hat{y}_3,v_2)_\Omega - (b_{11}(t)\hat{y}_4,v_2)_\Omega = (f_2,v_2)_\Omega + (\Delta u_2 + u_2, v_2)_\Gamma \quad (59a)$$
$$(\hat{y}_2(0),v_2)_\Omega = (\hat{y}_2^0,v_2)_\Omega \quad (59b)$$
$$(\hat{y}_{3t}, v_3) + a_3(t,\hat{y}_3,v_3) + (b_3(t)\hat{y}_3,v_3)_\Omega + (b_9(t)\hat{y}_2,v_3)_\Omega - (b_6(t)\hat{y}_1,v_3)_\Omega + (b_{15}(t)\hat{y}_4,v_3)_\Omega = (f_3,v_3)_\Omega + (\Delta u_3 + u_3, v_3)_\Gamma \quad (60a)$$
$$(\hat{y}_3(0),v_3)_\Omega = (\hat{y}_3^0,v_3)_\Omega \quad (60b)$$
$$(\hat{y}_{4t}, v_4) + a_4(t,\hat{y}_4,v_4) + (b_4(t)\hat{y}_4,v_4)_\Omega - (b_7(t)\hat{y}_1,v_4)_\Omega + (b_{11}(t)\hat{y}_2,v_4)_\Omega - (b_{15}(t)\hat{y}_3,v_4)_\Omega = (f_4,v_4)_\Omega + (\Delta u_4 + u_4, v_4)_\Gamma \quad (61a)$$
$$(\hat{y}_4(0),v_4)_\Omega = (\hat{y}_4^0,v_4)_\Omega \quad (61b)$$

Subtracting Eqs.((8) a & b) from Eqs.((58) a & b), Eqs.((9) a & b) from Eqs.((59) a & b), Eqs.((10) a & b) from Eqs.((60) a & b) and Eqs.((11) a & b) from Eqs.((61) a & b), then set $\Delta y_r = \hat{y}_r - y_r$, $\forall r = 1,2,3,4$. We get:

$$(\Delta y_{1t}, v_1) + a_1(t,\Delta y_1,v_1) + (b_1(t)\Delta y_1,v_1)_\Omega - (b_5(t)\Delta y_2,v_1)_\Omega + (b_6(t)\Delta y_3,v_1)_\Omega + (b_7(t)\Delta y_4,v_1)_\Omega = (\Delta u_1, v_1)_\Gamma \quad (62a)$$
$$(\Delta y_1(0),v_1)_\Omega = 0 \quad (62b)$$
$$(\Delta y_{2t}, v_2) + a_2(t,\Delta y_2,v_2) + (b_2(t)\Delta y_2,v_2)_\Omega + (b_5(t)\Delta y_1,v_2)_\Omega - (b_9(t)\Delta y_3,v_2)_\Omega - (b_{11}(t)\Delta y_4,v_2)_\Omega = (\Delta u_2, v_2)_\Gamma \quad (63a)$$
$$(\Delta y_2(0),v_2)_\Omega = 0 \quad (63b)$$
$$(\Delta y_{3t}, v_3) + a_3(t,\Delta y_3,v_3) + (b_3(t)\Delta y_3,v_3)_\Omega + (b_9(t)\Delta y_2,v_3)_\Omega - (b_6(t)\Delta y_1,v_3)_\Omega + (b_{15}(t)\Delta y_4,v_3)_\Omega = (\Delta u_3, v_3)_\Gamma \quad (64a)$$
$$(\Delta y_3(0),v_3)_\Omega = 0 \quad (64b)$$
$$(\Delta y_{4t}, v_4) + a_4(t,\Delta y_4,v_4) + (b_4(t)\Delta y_4,v_4)_\Omega - (b_7(t)\Delta y_1,v_4)_\Omega + (b_{11}(t)\Delta y_2,v_4)_\Omega - (b_{15}(t)\Delta y_3,v_4)_\Omega = (\Delta u_4, v_4)_\Gamma \quad (65a)$$
$$(\Delta y_4(0),v_4)_\Omega = 0 \quad (65b)$$

By utilizing $v_r = \Delta y_r$, $\forall r = 1,2,3,4$,Into Eqs. ((62a) - (65a)) respectively, then adding the obtained four equations together, then using Lemma (2.1) for the first term in L.H.S. Finally, using Assumption (A-ii) to get:

$$\frac{1}{2}\frac{d}{dt}\|\Delta\vec{y}\|_0^2 + \bar{\alpha}\|\Delta\vec{y}\|_1^2 \le (\Delta u_1,\Delta y_1)_\Gamma + (\Delta u_2,\Delta y_2)_\Gamma + (\Delta u_3,\Delta y_3)_\Gamma + (\Delta u_4,\Delta y_4)_\Gamma \quad (66)$$

Since the second term in the L.H.S. of (66) is positive, then integrating both sides for t from 0 to t, we obtain:

$$\int_0^t \frac{\mathrm{d}}{\mathrm{dt}}\|\Delta\vec{y}\|_0^2\,\mathrm{dt} \le \int_0^t \|\Delta u_1\|_\Gamma^2\,dt + \int_0^t \|\Delta y_1\|_\Gamma^2\,dt + \int_0^t \|\Delta u_2\|_\Gamma^2\,dt + \int_0^t \|\Delta y_2\|_\Gamma^2\,dt + \int_0^t \|\Delta u_3\|_\Gamma^2\,dt + \int_0^t \|\Delta y_3\|_\Gamma^2\,dt + \int_0^t \|\Delta u_4\|_\Gamma^2\,dt + \int_0^t \|\Delta y_4\|_\Gamma^2\,dt.$$

Now, by using the trace theorem to get:

$$\|\Delta\vec{y}\|_0^2 \le \|\Delta\vec{u}\|_\Sigma^2 + s\int_0^t \|\Delta\vec{y}\|_0^2\,dt.$$

Applying the Gromwell- Bellman inequality gives

$\|\Delta\vec{y}\|_0^2 \le K^2\|\Delta\vec{u}\|_\Sigma^2$, $K > 0$ for each $t \in [0,T] \Longrightarrow \|\Delta\vec{y}\|_{L^\infty\left(I,L^2(\Omega)\right)} \le K\|\Delta\vec{u}\|_\Sigma$

And $\|\Delta\vec{y}\|_{\mathrm{L}^2(\mathrm{Q})}^2 = \int_0^t \|\Delta\vec{y}\|_{\mathrm{L}^2(\Omega)}^2\,dt \le \mathrm{T}K^2\|\Delta\vec{u}\|_\Sigma$Then:

$\|\Delta\vec{y}\|_{L^2(Q)} \le K\|\Delta\vec{u}\|_\Sigma$, where $TK^2 = K^2$

Using the same way which is used in the above steps for R.H.S. of Eq.(66), then integrating both sides for t from 0 to T to get:

$$\int_0^T \frac{d}{dt}\|\Delta\vec{y}\|_0^2\,dt + 2\bar{\alpha}\int_0^T \|\Delta\vec{y}\|_1^2\,dt \le \|\Delta\vec{u}\|_\Sigma^2 + s\int_0^t \|\Delta\vec{y}\|_0^2\,dt \Longrightarrow$$

$2\bar{\alpha}\int_0^T \|\Delta\vec{y}\|_1^2\,dt \le (1+sK^2)\|\Delta\vec{u}\|_\Sigma^2 = K^2\|\Delta\vec{u}\|_\Sigma^2 \Longrightarrow$
$\|\Delta\vec{y}\|_{L^2(I,V)}^2 \le K^2\|\Delta\vec{u}\|_\Sigma^2$, where $K^2 = (1+sK^2)/2\bar{\alpha}$
$\|\Delta\vec{y}\|_{L^2(I,V)} \le K\|\Delta\vec{u}\|_\Sigma$.

b- Let $\Delta\vec{u} = \vec{\bar{u}} - \vec{u}$ and $\Delta\vec{y} = \vec{\bar{y}} - \vec{y}$ where $\vec{\bar{y}}$and $\vec{y}$ are the corresponding the quaternary state vector solution to the QCCBCV $\overrightarrow{\bar{u}}$and $\vec{u}$, then using part (a) of this theorem to get:
$\left\|\vec{\bar{y}} - \vec{y}\right\|_{L^\infty\left(I,L^2(\Omega)\right)} \le K\left\|\vec{\bar{u}} - \vec{u}\right\|_\Sigma$.

This means the LIO $\vec{u} \to \vec{y}$ is continuous from $\boldsymbol{L^2(\Sigma)}$into $\boldsymbol{L^\infty\left(I, L^2(\Omega)\right)}$, then the other results are easily obtained.

Lemma (4.1)[15]:
The norm $\|.\|_0$ is weakly lower semi continuous.

Lemma (4.2):
The objective function in Eq(10) is a weakly lower semi continuous.

Proof: From Lemma (4.1), the norm $\|\vec{u}\|_{L^2(Q)}$ is a weakly lower semi continuous on the other hand when $\vec{u}_r \overset{W}{\to} \vec{u}$ in $L^2(Q)$, then (by theorem(3.1)) $\vec{y}_r \overset{W}{\to} \vec{y} = \vec{y}_{\vec{u}}$ in $L^2(Q)$, which gives $\|\vec{y} - \vec{y}_d\|_{L^2(Q)}$ is a weakly lower semi continuous by Lemma (4.1), hence $G_0(\vec{u})$ is weakly lower semi continuous.

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Theorem (4.2):
In addition to assumptions (A), if the objective function of Eq. (7) is coercive, then there exists a CCBOQCV**.**

Proof: Since $G_0(\vec{u})$ is nonnegative and coercive. Then there exists a minimizing sequence $\vec{u}_k\} = \{(u_{1k}, u_{2k}, u_{3k}, u_{4k})\} \in \overrightarrow{W}_A$,$\forall k$ s.t.: $\lim_{n\to\infty} G_0(\vec{u}_k) = inf_{\vec{\bar{u}}\in\overrightarrow{W}_A} G_0(\vec{\bar{u}})$, i.e.
$\|\vec{u}_k\|_\Sigma \le c$,$\forall k$. Then by the theorem of Alauglu, there exists a subsequence of $\{\vec{u}_k\}$ say again $\{\vec{u}_k\}$ s.t. $\vec{u}_k \overset{w}{\to} \vec{u}$ in $L^2(\Sigma)$. Then by Theorem (3.1), there exists a sequence of quaternary state vector solution $\{\vec{y}_k\}$, corresponding to the sequence of the CCBQCV $\{\vec{u}_k\}$ and that $\|\vec{y}_k\|_{L^\infty(I,L^2(\Omega))}$, $\|\vec{y}_k\|_{L^2(Q)}$ and$\|\vec{y}_k\|_{L^2(I,V)}$ are bounded. Then by Alauglu's theorem, there exists a subsequence of $\{\vec{y}_k\}$, say again $\{\vec{y}_k\}$ s.t. $\vec{y}_k \overset{w}{\to} \vec{y}$ in $L^\infty(I, L^2(\Omega))$, $L^2(Q)$, and $L^2(I,V)$.
To show the norm $\|\vec{y}_{kt}\|_{L^2(I,V^*)}$ is bounded, the weak form of the state quaternary equations can be written as:

$$(y_{1kt}, v_1) = -a_1(t, y_{1k}, v_1) - (b_1(t)y_{1k}, v_1)_\Omega + (b_5(t)y_{2k}, v_1)_\Omega - (b_6(t)y_{3k}, v_1)_\Omega - (b_7(t)y_{4k}, v_1)_\Omega + (f_1, v_1)_\Omega + (u_1, v_1)_\Gamma \quad (67)$$

$$(y_{2kt}, v_2) = -a_2(t, y_{2k}, v_2) - (b_2(t)y_{2k}, v_2)_\Omega - (b_5(t)y_{1k}, v_2)_\Omega + (b_9(t)y_{3k}, v_2)_\Omega + (b_{11}(t)y_{4k}, v_2)_\Omega + (f_2, v_2)_\Omega + (u_2, v_2)_\Gamma \quad (68)$$

$$(y_{3kt}, v_3) = -a_3(t, y_{3k}, v_3) - (b_3(t)y_{3k}, v_3)_\Omega - (b_9(t)y_{2k}, v_3)_\Omega + (b_6(t)y_{1k}, v_3)_\Omega - (b_{15}(t)y_{4k}, v_3)_\Omega + (f_3, v_3)_\Omega + (u_3, v_3)_\Gamma \quad (69)$$

$$(y_{4kt}, v_4) = -a_4(t, y_{4k}, v_4) - (b_4(t)y_{4k}, v_4)_\Omega + (b_7(t)y_{1k}, v_4)_\Omega - (b_{11}(t)y_{2k}, v_4)_\Omega + (b_{15}(t)y_{3k}, v_4)_\Omega + (f_4, v_4)_\Omega + (u_4, v_4)_\Gamma \quad (70)$$

By adding Eqs.((67)-(70) and then integrating both sides with respect to t from 0 to T. Finally, taking the absolute value for both sides, then using the Cauchy-Schwarz inequality for the terms which contains the QCCBCV and the function f_r $\forall r = 1,2,3,4$, after using Assumptions (A-I and ii) to get:

$\left|\int_0^T(\vec{y}_{kt},\vec{v})dt\right| \leq \int_0^T \sum_{r=1}^4(\alpha_r \|y_{rk}\|_1\|v_r\|_1 + \beta_r\|y_{rk}\|_Q\|v_r\|_Q)dt + \epsilon_2\int_0^T\|y_{2k}\|_Q\|v_1\|_Q dt +$
$\epsilon_3\int_0^T\|y_{3k}\|_Q\|v_1\|_Q dt + \epsilon_4\int_0^T\|y_{4k}\|_Q\|v_1\|_Q dt + \bar{\epsilon}_1\int_0^T\|y_{1k}\|_Q\|v_2\|_Q dt +$
$\bar{\epsilon}_3\int_0^T\|y_{3k}\|_Q\|v_2\|_Q dt + \bar{\epsilon}_4\int_0^T\|y_{4k}\|_Q\|v_2\|_Q dt + \hat{\epsilon}_1\int_0^T\|y_{1k}\|_Q\|v_3\|_Q dt +$
$\hat{\epsilon}_2\int_0^T\|y_{2k}\|_Q\|v_3\|_Q dt + \hat{\epsilon}_4\int_0^T\|y_{4k}\|_Q\|v_3\|_Q dt + \tilde{\epsilon}_1\int_0^T\|y_{1k}\|_Q\|v_4\|_Q dt +$
$\tilde{\epsilon}_2\int_0^T\|y_{2k}\|_Q\|v_4\|_Q dt + \tilde{\epsilon}_3\int_0^T\|y_{3k}\|_Q\|v_4\|_Q dt \quad \int_0^T\sum_{r=1}^4(\|\eta_r\|_Q\|v_r\|_Q + \|u_r\|_\Sigma\|v_r\|_\Sigma)dt$
where $\hat{\alpha} = \sum_{r=1}^4 \alpha_r$, $\hat{\beta} = \sum_{r=1}^4 \alpha_r$
since $\|y_{rk}\|_Q \leq \|\vec{y}_k\|_Q \leq \|\vec{y}_k\|_1$, $\|v_r\|_Q \leq \|\vec{v}\|_Q \leq \|\vec{v}\|_1$, $\|u_{rk}\|_\Sigma \leq \|\vec{u}_k\|_\Sigma$,
$\|\vec{v}\|_\Sigma \leq c\|\vec{v}\|_1$ and $\|\eta_r\|_Q \leq \|\vec{\eta}\|_Q$, $\forall r = 1,2,3,4$,
But $\|\vec{u}_k\|_\Sigma \leq h$, $\|\vec{y}_k\|_1 \leq s_6$ and $\|\vec{\eta}\|_Q \leq l$.then:
$\left|\int_0^T(\vec{y}_{kt},\vec{v})dt\right| \leq (s_7)\|\vec{v}\|_{L^2(I,V)}$
Where $s_7 = (\hat{\alpha} + \hat{\beta} + \epsilon_2 + \epsilon_3 + \epsilon_4 + \bar{\epsilon}_1 + \bar{\epsilon}_3 + \bar{\epsilon}_4 + \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_4 + \tilde{\epsilon}_1$
$+\tilde{\epsilon}_2 + \tilde{\epsilon}_3)s_6 + 4l + 4ch$.
$\|\vec{y}_{kt}\|_{L^2(I,V^*)} = \frac{\left|\int_0^T(\vec{y}_{kt},\vec{v})dt\right|}{\|\vec{v}\|_{L^2(I,V)}} \leq s_7,\ \forall \vec{y}_{kt} \in V^*$

Since $\vec{y}_k$ is an approximation solution of the weak form of the state quaternary equations, then:
$(y_{1kt},v_1) + a_1(t,y_{1k},v_1) + (b_1(t)y_{1k},v_1)_\Omega - (b_5(t)y_{2k},v_1)_\Omega + (b_6(t)y_{3k},v_1)_\Omega +$
$(b_7(t)y_{4k},v_1)_\Omega = (f_1,v_1)_\Omega + (u_{1k},v_1)_\Gamma$ (71)
$(y_{2kt},v_2) + a_2(t,y_{2k},v_2) + (b_2(t)y_{2k},v_2)_\Omega + (b_5(t)y_{1k},v_2)_\Omega - (b_9(t)y_{3k},v_2)_\Omega -$
$(b_{11}(t)y_{4k},v_2)_\Omega = (f_2,v_2)_\Omega + (u_{2k},v_2)_\Gamma$ (72)
$(y_{3kt},v_3) + a_3(t,y_{3k},v_3) + (b_3(t)y_{3k},v_3)_\Omega + (b_9(t)y_{2k},v_3)_\Omega - (b_6(t)y_{1k},v_3)_\Omega +$
$(b_{15}(t)y_{4k},v_3)_\Omega = (f_3,v_3)_\Omega + (u_{3k},v_3)_\Gamma$ (73)
$(y_{4kt},v_4) + a_4(t,y_{4k},v_4) + (b_4(t)y_{4k},v_4)_\Omega - (b_7(t)y_{1k},v_4)_\Omega + (b_{11}(t)y_{2k},v_4)_\Omega -$
$(b_{15}(t)y_{3k},v_4)_\Omega = (f_4,v_4)_\Omega + (u_{4k},v_4)_\Gamma$ (74)
Let $\varphi_r \in C^1[0,T]$, s.t. $\varphi_r(T) = 0, \forall r = 1,2,3,4$. Now, rewriting the first terms in the L.H.S. of ((71) - (74)), multiplying both sides by $\varphi_1(t), \varphi_2(T), \varphi_3(T)$ and $\varphi_4(T)$ respectively, then multiplying both sides of each obtained equation with respect to t from 0 to T. Finally, integrating by parts for the first terms in L.H.S. to get:

$$-\int_0^T(y_{1k},v_1)\varphi_1'(t)\,dt + \int_0^T[a_1(t,y_{1k},v_1) + (b_1(t)y_{1k},v_1)_\Omega - (b_5(t)y_{2k},v_1)_\Omega + (b_6(t)y_{3k},v_1)_\Omega + (b_7(t)y_{4k},v_1)_\Omega]\varphi_1(t)dt = \int_0^T(f_1,v_1)_\Omega\varphi_1(t)dt + \int_0^T(u_{1k},v_1)_\Gamma\varphi_1(t)dt + (y_{1k}(0),v_1)_\Omega\varphi_1(0) \quad (75)$$

$$-\int_0^T(y_{2k},v_2)\,\varphi_2'(t)dt + \int_0^T[a_2(t,y_{2k},v_2) + (b_2(t)y_{2k},v_2)_\Omega + (b_5(t)y_{1k},v_2)_\Omega - (b_9(t)y_{3k},v_2)_\Omega - (b_{11}(t)y_{4k},v_2)_\Omega]\varphi_2(t)dt = \int_0^T(f_2,v_2)_\Omega\varphi_2(t)dt + \int_0^T(u_{2k},v_2)_\Gamma\varphi_2(t)dt + (y_{2k}(0),v_2)_\Omega\,\varphi_2(0) \quad (76)$$

$$-\int_0^T(y_{3k},v_3)\varphi_3'(t)\,dt + \int_0^T[a_3(t,y_{3k},v_3) + (b_3(t)y_{3k},v_3)_\Omega + (b_9(t)y_{2k},v_3)_\Omega - (b_6(t)y_{1k},v_3)_\Omega + (b_{15}(t)y_{4k},v_3)_\Omega]\varphi_3(t)dt = \int_0^T(f_3,v_3)_\Omega\varphi_3(t)dt + \int_0^T(u_{3k},v_3)_\Gamma\varphi_3(t)dt + (y_{3k}(0),v_3)_\Omega\varphi_3(0) \quad (77)$$

$$-\int_0^T(y_{4k},v_4)\varphi_4'(t)dt + \int_0^T[a_4(t,y_{4k},v_4) + (b_4(t)y_{4k},v_4)_\Omega - (b_7(t)y_{1k},v_4)_\Omega + (b_{11}(t)y_{2k},v_4)_\Omega - (b_{15}(t)y_{3k},v_4)_\Omega]\varphi_4(t)dt = \int_0^T(f_4,v_4)_\Omega\varphi_4(t)dt + \int_0^T(u_{4k},v_4)_\Gamma\varphi_4(t)dt + (y_{4k}(0),v_4)_\Omega\varphi_4(0) \quad (78)$$

Since $\vec{y}_k \overset{w}{\to} \vec{y}$ in $L^2(Q)$and $L^2(I,V)$, $\vec{y}_k(0) \overset{S}{\to} \vec{y}(0)$ in $L^2(\Omega)$ and $\vec{u}_k \overset{w}{\to} \vec{u}$ in $L^2(\Sigma)$. Then using the similar steps to those which are used in the proof of Theorem (3.1) in Eqs.((30) - (33)) to get:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma\varphi_1(t)dt + (y_1(0), v_1)_\Omega\varphi_1(0) \quad (79)$$

$$-\int_0^T (y_2, v_2)\,\varphi_2'(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_2)_\Omega\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma\varphi_2(t)dt + (y_2(0), v_2)_\Omega\,\varphi_2(0) \quad (80)$$

$$-\int_0^T (y_3, v_3)\varphi_3'(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_3)_\Omega\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma\varphi_3(t)dt + (y_3(0), v_3)_\Omega\varphi_3(0) \quad (81)$$

$$-\int_0^T (y_4, v_4)\varphi_4'(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_4)_\Omega\varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma\varphi_4(t)dt + (y_4(0), v_4)_\Omega\varphi_4(0) \quad (82)$$

Now, one has the following two cases:

Case1: Choose $\varphi_r \in D[0, T]$, i.e. $\varphi_r(T) = \varphi_r(0) = 0$,$\forall r = 1,2,3,4,$

Now, by using integrating by parts for the first terms in the L.H.S. of Eqs.((79) – (82)), to get:

$$\int_0^T (y_{1t}, v_1)\,\varphi_1(t)dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma\varphi_1(t)dt \quad (83)$$

$$\int_0^T (y_{2t}, v_2)\,\varphi_2(t)dt + \int_0^T [\,a_2(t, y_2, v_2) + (b_2(t)y_2, v_2)_\Omega + (b_5(t)y_1, v_2)_\Omega - (b_9(t)y_3, v_2)_\Omega - (b_{11}(t)y_4, v_2)_\Omega]\varphi_2(t)dt = \int_0^T (f_2, v_2)_\Omega\varphi_2(t)dt + \int_0^T (u_2, v_2)_\Gamma\varphi_2(t)dt \quad (84)$$

$$\int_0^T (y_{3t}, v_3)\varphi_3(t)\,dt + \int_0^T [\,a_3(t, y_3, v_3) + (b_3(t)y_3, v_3)_\Omega + (b_9(t)y_2, v_3)_\Omega - (b_6(t)y_1, v_3)_\Omega + (b_{15}(t)y_4, v_3)_\Omega]\varphi_3(t)dt = \int_0^T (f_3, v_3)_\Omega\varphi_3(t)dt + \int_0^T (u_3, v_3)_\Gamma\varphi_3(t)dt \quad (85)$$

$$\int_0^T (y_{4t}, v_4)\varphi_4(t)dt + \int_0^T [\,a_4(t, y_4, v_4) + (b_4(t)y_4, v_4)_\Omega - (b_7(t)y_1, v_4)_\Omega + (b_{11}(t)y_2, v_4)_\Omega - (b_{15}(t)y_3, v_4)_\Omega]\varphi_4(t)dt = \int_0^T (f_4, v_4)_\Omega\varphi_4(t)dt + \int_0^T (u_4, v_4)_\Gamma\varphi_4(t)dt. \quad (86)$$

This means $y_r = y_{ur} \forall r = 1,2,3,4,$. That satisfies the weak form of the state quaternary equations.

Case 2: Choose $\varphi_r \in C^1[0, T]$, i.e. $\varphi_r(T) = 0$ & $\varphi_r(0) \neq 0, \forall r = 1,2,3,4,$ using integrating by parts of the first term in the L.H.S. of Eq.(83) to obtain:

$$-\int_0^T (y_1, v_1)\varphi_1'(t)\,dt + \int_0^T [\,a_1(t, y_1, v_1) + (b_1(t)y_1, v_1)_\Omega - (b_5(t)y_2, v_1)_\Omega + (b_6(t)y_3, v_1)_\Omega + (b_7(t)y_4, v_1)_\Omega]\varphi_1(t)dt = \int_0^T (f_1, v_1)_\Omega\varphi_1(t)dt + \int_0^T (u_1, v_1)_\Gamma\varphi_1(t)dt + (y_1^0, v_1)_\Omega\varphi_1(0) \quad (87)$$

By subtracting Eq.(87) from Eq.(79) to get:

$(y_1^0, v_1)_\Omega\varphi_1(0) = (y_1(0), v_1)_\Omega\varphi_1(0), \forall\varphi_1 \in [0, T] \Longrightarrow y_1^0 = y_1(0) = y_1^0(x)$.

That means the first initial condition holds. The same manner can be utilized to get that:

$y_r^0 = y_r(0) = y_r^0(x), \forall r = 1,2,3,4.$

Then$y_r = y_{ur}, \forall r = 1,2,3,4,$ are the quaternary state vector solution of the weak form of the state quaternary equations.

From Lemma 4.2,

$G_0(\vec{u}) \leq \lim_{n\to\infty} inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k)$

and since $\vec{u}_k \overset{w}{\rightarrow} \vec{\vec{u}}$ in $\boldsymbol{L^2(\Sigma)}$, then:

$G_0(\vec{u}) \leq \lim_{n\to\infty} inf_{\vec{u}_k \in \vec{W}_A} G_0(\vec{u}_k) = \lim_{n\to\infty} G_0(\vec{u}_k) = inf_{\vec{\bar{u}} \in \vec{W}_A} G_0(\vec{\bar{u}})$

$\Longrightarrow G_0(\vec{u}) = min_{\vec{\bar{u}} \in \vec{W}_A} G_0(\vec{\bar{u}})$.

Therefore, is a QCCBOCV.

5. The optimality theorem:

In this part, we find the mathematical formulation for the QABVP and the derivation of Frèchet, then we study the necessary conditions for CCBOQCVP.

Theorem (5.1): Consider the objective function $G_0(\vec{u})$, and the following adjoint $(z_1, z_2, z_3, z_4) = (z_{1u_1}, z_{2u_2}, z_{3u_3}, z_{4u_4})$ QABVP, of the QABVP ((1) - (6)) are given in Q by:

$$-z_{1t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z_1}{\partial x_j}\right) + b_1 z_1 - b_5 z_2 + b_6 z_3 + b_7 z_4 = (y_1 - y_{1d}), \quad (88)$$

$$-z_{2t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(b_{ij}\frac{\partial z_2}{\partial x_j}\right) + b_2 z_2 + b_5 z_1 - b_9 z_3 - b_{11} z_4 = (y_2 - y_{2d}), \quad (89)$$

$$-z_{3t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(c_{ij}\frac{\partial z_3}{\partial x_j}\right) + b_3 z_3 + b_9 z_2 - b_6 z_1 + b_{15} z_4 = (y_3 - y_{3d}), \quad (90)$$

$$-z_{4t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(d_{ij}\frac{\partial z_1}{\partial x_j}\right) + b_4 z_4 - b_7 z_1 + b_{11} z_2 - b_{15} z_3 = (y_4 - y_{4d}), \quad (91)$$

with the following boundary conditions and initial conditions:

$$\frac{\partial z_r}{\partial n_r} = 0, r = 1,2,3,4, \text{ on } \Sigma \quad . \quad (92)$$

$$z_r(x,0) = 0, \quad r = 1,2,3,4 \text{ , on } \Omega \quad . \quad (93)$$

Then the Frèchet derivative of $G_0$is $(G_0'(\vec{u}), \overrightarrow{\Delta u}) = (\vec{z} + \beta\vec{u}, \overrightarrow{\Delta u})$

Proof: As in the state quaternary equations, the weak form of the QABVP for each $v_r \in V$, ,$\forall r = 1,2,3,4$, is:

$$-(z_{1t}, v_1) + a_1(t, z_1, v_1) + (b_1(t)z_1, v_1)_\Omega - (b_5(t)z_2, v_1)_\Omega + (b_6(t)z_3, v_1)_\Omega + (b_7(t)z_4, v_1)_\Omega = (y_1 - y_{1d}, v_1)_\Omega \quad (94)$$

$$-(z_{2t}, v_2) + a_2(t, z_2, v_2) + (b_2(t)z_2, v_2)_\Omega + (b_5(t)z_1, v_2)_\Omega - (b_9(t)z_3, v_2)_\Omega - (b_{11}(t)z_4, v_2)_\Omega = (y_2 - y_{2d}, v_2)_\Omega \quad (95)$$

$$-(z_{3t}, v_3) + a_3(t, z_3, v_3) + (b_3(t)z_3, v_3)_\Omega + (b_9(t)z_2, v_3)_\Omega - (b_6(t)z_1, v_3)_\Omega + (b_{15}(t)z_4, v_3)_\Omega = (y_3 - y_{3d}, v_3)_\Omega \quad (96)$$

$$-(z_{4t}, v_4) + a_4(t, z_4, v_4) + (b_4(t)z_4, v_4)_\Omega - (b_7(t)z_1, v_4)_\Omega + (b_{11}(t)z_2, v_4)_\Omega - (b_{15}(t)z_3, v_4)_\Omega = (y_4 - y_{4d}, v_4)_\Omega \quad (97)$$

Now, utilizing $v_r = z_r, \forall r = 1,2,3,4$, in ((62a) - (65a)) respe, to get:

$$(\Delta y_{1t}, z_1) + a_1(t, \Delta y_1, z_1) + (b_1(t)\Delta y_1, z_1)_\Omega - (b_5(t)\Delta y_2, z_1)_\Omega + (b_6(t)\Delta y_3, z_1)_\Omega + (b_7(t)\Delta y_4, z_1)_\Omega = (\Delta u_1, z_1)_\Gamma \quad (98)$$

$$(\Delta y_{2t}, z_2) + a_2(t, \Delta y_2, z_2) + (b_2(t)\Delta y_2, z_2)_\Omega + (b_5(t)\Delta y_1, z_2)_\Omega - (b_9(t)\Delta y_3, z_2)_\Omega - (b_{11}(t)\Delta y_4, z_2)_\Omega = (\Delta u_2, z_2)_\Gamma \quad (99)$$

$$(\Delta y_{3t}, z_3) + a_3(t, \Delta y_3, z_3) + (b_3(t)\Delta y_3, z_3)_\Omega + (b_9(t)\Delta y_2, z_3)_\Omega - (b_6(t)\Delta y_1, z_3)_\Omega + (b_{15}(t)\Delta y_4, z_3)_\Omega = (\Delta u_3, z_3)_\Gamma \quad (100)$$

$$(\Delta y_{4t}, z_4) + a_4(t, \Delta y_4, z_4) + (b_4(t)\Delta y_4, z_4)_\Omega - (b_7(t)\Delta y_1, z_4)_\Omega + (b_{11}(t)\Delta y_2, z_4)_\Omega - (b_{15}(t)\Delta y_3, z_4)_\Omega = (\Delta u_4, z_4)_\Gamma \quad (101)$$

Also, utilizing $v_r = \Delta y_r, \forall r = 1,2,3,4$. In Eqs.((94) - (97)) respectively to get:

$$-(z_{1t}, \Delta y_1) + a_1(t, z_1, \Delta y_1) + (b_1(t)z_1, \Delta y_1)_\Omega - (b_5(t)z_2, \Delta y_1)_\Omega + (b_6(t)z_3, \Delta y_1)_\Omega + (b_7(t)z_4, \Delta y_1)_\Omega = (y_1 - y_{1d}, \Delta y_1)_\Omega \quad (102)$$

$$-(z_{2t}, \Delta y_2) + a_2(t, z_2, \Delta y_2) + (b_2(t)z_2, \Delta y_2)_\Omega + (b_5(t)z_1, \Delta y_2)_\Omega - (b_9(t)z_3, \Delta y_2)_\Omega - (b_{11}(t)z_4, \Delta y_2)_\Omega = (y_2 - y_{2d}, \Delta y_2)_\Omega \quad (103)$$

$$-(z_{3t}, \Delta y_3) + a_3(t, z_3, \Delta y_3) + (b_3(t)z_3, \Delta y_3)_\Omega + (b_9(t)z_2, \Delta y_3)_\Omega - (b_6(t)z_1, \Delta y_3)_\Omega + (b_{15}(t)z_4, \Delta y_3)_\Omega = (y_3 - y_{3d}, \Delta y_3)_\Omega \quad (104)$$

$-(z_{4t},\Delta y_4)+a_4(t,z_4,\Delta y_4)+(b_4(t)z_4,\Delta y_4)_\Omega-(b_7(t)z_1,\Delta y_4)_\Omega+(b_{11}(t)z_2,\Delta y_4)_\Omega-$
$(b_{15}(t)z_3,\Delta y_4)_\Omega=(y_4-y_{4d},\Delta y_4)_\Omega$ (105)

Now, integrating both sides of Eqs.(98) - (105)) with respect to t from 0 to T. By using integrating by parts for the first terms of the L.H.S. of Eqs ((102) - (105)), then subtracting each one of the obtained equations from its corresponding equations and then adding all the obtained equations to get:

$$\int_0^T[(\Delta u_1,z_1)_\Gamma+(\Delta u_2,z_2)_\Gamma+(\Delta u_3,z_3)_\Gamma+(\Delta u_4,z_4)_\Gamma]\,dt=\int_0^T[(y_1-y_{1d},\Delta y_1)_\Omega+(y_2-y_{2d},\Delta y_1)_\Omega+(y_3-y_{3d},\Delta y_1)_\Omega+(y_4-y_{4d},\Delta y_4)_\Omega]\,dt \quad (106)$$

Now, adding (8a) and (62a), (9a) and (63a), (10a) and (64a), and (11a) and (65a), to get:

$$((y_1+\Delta y_1)_t,v_1)+a_1(t,(y_1+\Delta y_1),v_1)+(b_1(t)(y_1+\Delta y_1),v_1)_\Omega-(b_5(t)(y_2+\Delta y_2),v_1)_\Omega+(b_6(t)(y_3+\Delta y_3),v_1)_\Omega+(b_7(t)(y_4+\Delta y_4),v_1)_\Omega=(f_1,v_1)_\Omega+(u_1+\Delta u_1,v_1)_\Gamma \quad (107)$$

$$((y_2+\Delta y_2)_t,v_2)+a_2(t,(y_2+\Delta y_2),v_2)+(b_2(t)(y_2+\Delta y_2),v_2)_\Omega+(b_5(t)(y_1+\Delta y_1),v_2)_\Omega-(b_9(t)(y_3+\Delta y_3),v_2)_\Omega-(b_{11}(t)(y_4+\Delta y_4),v_2)_\Omega=(f_2,v_2)_\Omega+(u_2+\Delta u_2,v_2)_\Gamma \quad (108)$$

$$((y_3+\Delta y_3)_t,v_3)+a_3(t,(y_3+\Delta y_3),v_3)+(b_3(t)(y_3+\Delta y_3),v_3)_\Omega+(b_9(t)(y_2+\Delta y_2),v_3)_\Omega-(b_6(t)(y_1+\Delta y_1),v_3)_\Omega+(b_{15}(t)(y_4+\Delta y_4),v_3)_\Omega=(f_3,v_3)_\Omega+(u_3+\Delta u_3,v_3)_\Gamma \quad (109)$$

$$((y_4+\Delta y_4)_t,v_4)+a_4(t,(y_4+\Delta y_4),v_4)+(b_4(t)(y_4+\Delta y_4),v_4)_\Omega-(b_7(t)(y_1+\Delta y_1),v_4)_\Omega+(b_{11}(t)(y_2+\Delta y_2),v_4)_\Omega-(b_{15}(t)(y_3+\Delta y_3),v_4)_\Omega=(f_4,v_4)_\Omega+(u_4+\Delta u_4,v_4)_\Gamma \quad (110)$$

Which means that, the QCCBCV $\vec{u}+\overrightarrow{\Delta u}$ is given the quaternary state vector solution $\vec{y}+\overrightarrow{\Delta y}$ of Eqs.((107) - (110)), respectively. Hence, the objective function:

$$G_0(\vec{u}+\overrightarrow{\Delta u})=\frac{1}{2}\sum_{r=1}^4[\int_0^T\int_\Omega(y_r+\Delta y_r-y_{rd})^2\,dx\,dt+\beta\int_0^T\int_\Gamma(u_r+\Delta u_r)^2\,d\gamma\,dt$$

Then,

$$G_0(\vec{u}+\overrightarrow{\Delta u})-G_0(\vec{u})=\sum_{r=1}^4\int_0^T[\int_\Omega(y_r-y_{rd})\Delta y_r\,dx+\int_\Gamma\beta u_r\Delta u_r\,d\gamma]dt+\frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|_Q^2++\frac{\beta}{2}\left\|\overrightarrow{\Delta u}\right\|_\Sigma^2$$

Using Eq.(106) in the R.H.S. of the above equation to get:

$$G_0(\vec{u}+\overrightarrow{\Delta u})-G_0(\vec{u})=\sum_{r=1}^4(\Delta u_r,z_r)_\Sigma+\beta\sum_{r=1}^4(u_r,\Delta u_r)_\Sigma+\frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|_Q^2+\frac{\beta}{2}\left\|\overrightarrow{\Delta u}\right\|_\Sigma^2$$

or

$$G_0(\vec{u}+\overrightarrow{\Delta u})-G_0(\vec{u})=(\vec{z}+\beta\vec{u},\overrightarrow{\Delta u})_\Sigma+\frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|_Q^2+\frac{\beta}{2}\left\|\overrightarrow{\Delta u}\right\|_\Sigma^2 \quad (111)$$

Then from the result of Theorem (4.1), part a, we can see that:

$$\frac{1}{2}\left\|\overrightarrow{\Delta y}\right\|_Q^2=\varepsilon_1(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma \ \&\ \frac{\beta}{2}\left\|\overrightarrow{\Delta u}\right\|_\Sigma^2\le\varepsilon_2(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma \ . \quad (112)$$

Such that. $\varepsilon_1(\overrightarrow{\Delta u})=\frac{1}{2}k^2\left\|\overrightarrow{\Delta u}\right\|_\Sigma$, $\varepsilon_2(\overrightarrow{\Delta u})=\frac{\beta}{2}\left\|\overrightarrow{\Delta u}\right\|_\Sigma$, with $\varepsilon_1(\overrightarrow{\Delta u}),\varepsilon_2(\overrightarrow{\Delta u})\to 0$, as $\left\|\overrightarrow{\Delta u}\right\|_\Sigma\to 0$

Utilizing Eq.(112) in Eq(111),to get:

$$G_0(\vec{u}+\overrightarrow{\Delta u})-G_0(\vec{u})=(\vec{z}+\beta\vec{u},\overrightarrow{\Delta u})_\Sigma+\varepsilon(\overrightarrow{\Delta u})\left\|\overrightarrow{\Delta u}\right\|_\Sigma \quad (114)$$

where $\varepsilon(\overrightarrow{\Delta u})=\varepsilon_1(\overrightarrow{\Delta u})+\varepsilon_2(\overrightarrow{\Delta u})\to 0$,as $\left\|\overrightarrow{\Delta u}\right\|_\Sigma\to 0$.

From the Frèchet derivative one gets: $(G_0'(\vec{u}),\overrightarrow{\Delta u})_\Sigma=(\vec{z}+\beta\vec{u},\overrightarrow{\Delta u})_\Sigma$.

Theorem (5.2): The QCCBOCV is $\vec{z}=-\beta\vec{\vec{u}}$,
The optimality theorem of the above problem is,
$G_0'(\vec{u})=\vec{z}+\beta\vec{u}=0$ with $\vec{y}=\vec{y}_{\vec{u}}$ &$\vec{z}=\vec{z}_{\vec{u}}$.

Proof: if $\vec{\vec{u}}$ is QCCBOCV of the problem, then
$G_0(\vec{\vec{u}}) = min_{\vec{\vec{u}}\in\vec{W}_A} G_0(\vec{u}), \forall\vec{u} \in \boldsymbol{L^2(\Sigma)}$, $\Rightarrow G_0'(\vec{\vec{u}}) = 0 \;\; \Rightarrow \vec{z} + \beta\vec{\vec{u}} = 0 \Rightarrow \vec{z} = -\beta\vec{\vec{u}}$
The necessary conditions for optimality with $\overrightarrow{\Delta u} = \vec{m} - \vec{u}$ is:
$(G_0'(\vec{\vec{u}}), \overrightarrow{\Delta u})_\Sigma \geq 0 \Rightarrow (\vec{z} + \beta\vec{\vec{u}}, \overrightarrow{\Delta u})_\Sigma \geq 0 \Rightarrow (\vec{z} + \beta\vec{\vec{u}}, \vec{u})_\Sigma \leq (\vec{z} + \beta\vec{\vec{u}}, \vec{m})_\Sigma$, $\forall\vec{m} \in L^2(\Sigma)$.

6. Conclusions:

One of the first important things that shall be ensured is the weak form of the QLPBVP when the QCCBCV is fixed and has a quaternary state vector solution. This is successfully stated and proved by employing the method of Galerkin under suitable assumptions. The continuity of the Lipchitz between the quaternary state vector solution of the weak form and the QCCBCV is proved. In fact, this point played an important role in proving the existence theorem of a QCCBOCV governed by the QLPBVP which is developed and proved in this paper under suitable assumptions. The existence and uniqueness of a solution for the QABVP associated with the QLPBVP are studied. The Frèchet derivative for the objective function is obtained depending on the existence of the QCCBOCV and the continuity Lipchitz. In the end, the optimality theorem of the QCCBOCVP is stated and proved.

References

[1] M.G. Cojocaru, A.S. Jaber, " Optimal Control of a Vaccinating Game Toward Increasing Overall Coverage," *J. appl. math. phys.*,Vol.6, pp. 754-769, 2018.

[2] E. Staffetti E, X. Li,Y. Matsuno, M. Soler, "Optimal Control Techniques in Aircraft Guidance and Control,," *Int. J. Aerosp. Eng*. 2 pages, 2019.

[3] I. Syahrini, R. Masabar, A. Aliasuddin, S. Munzir, Y. Hazim, "The Application of Optimal Control Through Fiscal Policy on Indonesian Economy," *J. Asian Finance Econ. Bus*. Vol.8, No. 3, pp. 0741-0750, 2021.

[4] G. Rigatos, M. Abbaszadeh, "Nonlinear Optimal Control for Multi-DOF Robotic Manipulators with Flexible Joints,," *Optim. Control Appl. Methods*, Vol.42, no. 6,pp. 1708-1733, 2021

[5] D. Derome D, H. Razali, A. Fazlizan, A. Jedi, K.P. Roberts. Determination of Optimal Time - Average Wind Speed Data in the Southern Part of Malaysia. *Baghdad Sci. J*. Vol.19, no.5 , pp.1111-1122, 2022.

[6] P. Lin, W.Wang, "Optimal Control Problems for Some Ordinary Differential Equations with Behavior of Blowup or Quenching," *Math. Control Relat. Fields*.Vol.8, no. 4,pp. 809-828, 2018.

[7] A. Manzoni, A. Quarteroni, S. Salsa, Optimal Control of Partial Differential Equations: Analysis, Approximation, and Applications (Applied Mathematical Sciences, 207). 1st edition. New York: Spriger; 2021, p. 515.

[8] Gh. M. Kadhem, A. A. H. Naeif, J. A. A. Al-Hawasy, "The Classical Continuous Mixed Optimal Control of Couple Nonlinear Parabolic Partial Differential Equations with State Constraints,"*Iraqi Journal of Science,* Vol. 62, No. 12, pp: 4859-4874, 2021.

[9] S. J. Al-Qaisi, Gh. M. Kadhem, J. A. Al-Hawasy," Mixed Optimal Control Vector for a Boundary Value Problem of Couple Nonlinear Elliptic Equations," *Iraqi Journal of Science,* Vol. 63, No. 9, pp: 3861-3866, 2022.

[10] J. A. Ali Al-Hawasy ," The Continuous Classical Boundary Optimal Control of a Couple Nonlinear Hyperbolic Boundary Value Problem with Equality and Inequality Constraints, *Baghdad Sci. J*. Vol.16, no.4, Pp.1064-1074, Supplement 2019.

[11] J. A. Ali Al-Hawasy, M. A. K. Jaber," The Continuous Classical Boundary Optimal Control Vector Governing by Triple Linear Partial Differential Equations of Parabolic Type," Ibn Al *Haitham Journal for Pure and Applied science*, Vol. 33, No. 3, pp. 113-126,2020.

[12] J. A. Ali Al-Hawasy, D. K. Jasim," The Continuous Classical Optimal Control Problems for Triple Elliptic Partial Differential Equations," *Ibn Al Haitham Jour. For Pure & Appl. Sci*. vol.33 ,no. 1, 2020.

[13] J. A. Ali Al-Hawasy, "Solvability for Classical Continuous Boundary Optimal Control Problem dominating by Triple Hyperbolic equations," *IOP Conf. Series, Materials Science and Engineering* 871,pp. 1-17 , 2020.

[14] R. Temam, Navier-Stokes equations. Amsterdam-New York: North-Holand Publishing Company; 1977. p. 470.

[15] Al-Rawdhance, E.H. "The Continuous Classical Optimal Control of Couple Elliptic Partial Differential Equations. Master thesis; Al- Mustansiriyah University, 2015.