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Cluster Outer Points via Proximity Spaces

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Abstract
One of the most effective mathematical concepts for developing a clear picture of topological cluster proximity spaces is the *follower points* and *the takeoff points*. These are utilized in current study to construct three sets called the *cluster outer set*, the *cluster disputed set* and the *cluster brim set* denoted by $P_{O_\sigma}, D_\delta(P), B_\delta(P)$, respectively. These three sets have divided the space into three separate pairs. On the other hand, the most important results, properties, and relationships between the sets were highlighted and studied.

Keywords: Cluster, proximity, Cluster outer set, Cluster disputed set, Cluster brim set.

النقاط الخارجية عبر فضاءات القرب

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الخلاصة
من المفاهيم الرياضية التي لها تاثيرات مهمة في عملية وضع صورة واضحة الى فضاءات القرب العنقودية التبلوجية نقاط التابع ونقاط الاقلاع ، والتي تم توظيفها في هذه الدراسة في بناء ثلاث مجاميع اطلق عليها المجموعة الخارجية العنقودية و المجموعة المتنازع عليها العنقودية ومجموعة الحافة العنقودية والتي رمز لها بالرمز $P_{O_\sigma}, D_\delta(P), B_\delta(P)$ بالتتابع . قسمت هذه المجاميع الفضاء الى ثلاث ازواج منفصلة . من جانب اخر تم ابراز ودراسة أهم النتائج والخصائص والعلاقات فيما بينهم .

1. Introduction

Metric spaces are significant in different fields of engineering and applied and pure sciences [1]. They determine the distance between points and sets. However, some problems confused scientists and researchers including when the distance between points or sets is zero. Scientists had different explanations of this case and had different views of its significance.
In 1908, the scientist Reazi solved these problems and described this situation by finding definition that goes in line with the metric spaces, which is the proximity relationship [2]. This relationship has found a wide scope in engineering and scientific applications that have a direct impact on our daily lives in various forms and types. Hence, scientists and researchers tended to study these new spaces on the one hand, and the development of mathematical concepts on

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the other hand, especially in metric spaces or topology spaces by assigning proximity spaces (see Wellman, Efromovic, Lodato, Leader, and Smirnove) [3].

It is important here to mention that some colleague researchers used these spaces to define and build new sets, which they called *center set* and linked them to the topological spaces to create new concepts [4-7]. Other researchers linked them to ideal spaces and soft spaces [8-14]. In the current study, another method is used. The concept of proximity is linked to the concept of clusters presented by Lodato in 1957 and its basics are formalized [15]. In the second part, the concepts of *follower set*, *take-off set*, *bushy set*, and *dismountable space* are introduced. In the third part, the concept of complementary follower set is introduced, which is called the *cluster outer*. The intersection points between the follower set and the takeoff set are studied, and represented by the cluster disputed set. The difference between the follower set points and the takeoff set points is explained and called the *cluster brim set*. The relationship between these three sets and their relationships within the proximity space is finally investigated.

1.1 Basic Definitions and Concepts

The basic relevant definitions are mentioned below. For further understanding of these concepts, readers can refer to the sources. Notation, $A\delta B$ means $A\ is\ near\ B$ and negation $A\bar{\delta}B$ means $A\ is\ far$ from B or $A\ is\ not\ near\ B$.

Definition 1 [3] The relation δ on the family $\mathcal{P}(X)$ of all subsets of set X is called a proximity on X if δ meets the following conditions:
1) If $A\delta B$, then $B\delta A$.
2) $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$.
3) $X\bar{\delta}\emptyset$.
4) $\{x\}\delta\{x\}$ for each $x \in X$.
5) If $A\bar{\delta}B$, then there exists $E \in \mathcal{P}(X)$ such that $A\bar{\delta}E$ and $X - E\bar{\delta}B$.

Definition 2 [15] Subfamily of the proximity space is called a cluster denoted by σ if and only if it satisfies the following three conditions:
1. For all $A, B \in \sigma \Longrightarrow A\delta B$.
2. $(A \cup B) \in \sigma \Leftrightarrow A \in \sigma$ or $B \in \sigma$.
3. $A\delta B$ for each $B \in \sigma \Rightarrow A \in \sigma$.

Therefore, the pair (X, δ) is called proximity space. Also, if (X, δ) is proximity space, and τ is any topology defined on X and σ is cluster defined on (X, δ), then $(X, \delta, \tau, \sigma)$ is denoted by the cluster topological proximity space.

Definition 3 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space. A point $x \in X$ is said to be *follower point* of a subset P of topological space (X, τ) , if and only if for every $\mathcal{U} \in \tau(x)$, and every $C \in \sigma$ such that $(\mathcal{U} \cap P)\delta C$, where $\tau(x)$ is the set of all open neighborhood of point x. All the follower point of set P is denoted by P_{f_σ}.

Definition 4 Let $(X, \delta, \tau, \sigma)$ be cluster topological proximity space. A point $x \in X$ is said to be *takeoff point* of a subset P of topological space (X, τ) , if and only if there exist $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P^c)\overline{\delta}C$ for some $C \in \sigma$. All the takeoff points of a set P is denoted by P_{t_σ}.

Definition 5 Let $(X, \delta, \tau, \sigma)$ be cluster topological proximity space, A is a nonempty subset of X is called *bushy set* if and only if every $x \in X$ and every $\mathcal{U} \in \tau(x), \mathcal{U} \cap A\delta c$ for every $c \in \sigma$.

Definition 6 Let $(X, \delta, \tau, \sigma)$ be cluster topological proximity space, a subset A of X is called $scant\ set$ if and only if $(A_f)_t = \emptyset$.

Definition 7 $(X, \delta, \tau, \sigma)$ is non-dismountable space if and only if X cannot contain two disjoint bushy subsets. Otherwise X is a dismountable space.

Example 1 Let $X = \{1,2,3\}$, $\tau = \{X, \emptyset, \{3\}, \{1,2\}\}$, δ is indiscrete proximity (Where $A\ \delta B \Leftrightarrow A \neq \emptyset, B \neq \emptyset$ see [3]), hence $\sigma = \{A \subseteq X; A \neq \emptyset\}$. If $P_1 = \{2,3\}$, $P_2 = \{2\}, P_3 = \{3\}, P_4 = \emptyset, then$
$P_{1_{f_\sigma}} = X$, $P_{2_{f_\sigma}} = \{1,2\}, P_{3_{f_\sigma}} = \{3\}, P_{4_{f_\sigma}} = \emptyset$, and $P_{1_{t_\sigma}} = \{3\}, P_{2_{t_\sigma}} = \emptyset, P_{3_{t_\sigma}} = \{3\} P_{4_{t_\sigma}} = \emptyset$. So that P_1 is bushy set. Empty set is only scant set, X is non-dismountable space since $\{1,3\}$ and $\{2,3\}$ are bushy set but $\{1,3\} \cap \{2,3\} = \{3\}$.

1.2 Main Results

This part introduces the definition of three disjoint sets and studies the relationship between them, as well as their relationship to the non-dismountable space.

Definition 8 Let $(X, \delta, \tau, \sigma)$ be cluster topological proximity space. A point $x \in X$ is said to be *cluster outer point* of subset P of topological space (X, τ), if and only if there exists $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma$. All the cluster outer points of set P is denoted by P_{O_σ}. Hence $P_{O_\sigma}(\tau, \sigma) = \{x \in X; \exists\ \mathcal{U} \in \tau(x)\ s.t\ (\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma\}$.

Example 2 Let $X = \{1,2,3\}, \tau = \{X, \emptyset, \{3\}, \{1\}, \{1,3\}\}, \delta$ is discrete proximity (Where $A\ \delta B \Leftrightarrow A \cap B \neq \emptyset$, see [3]), hence $\sigma = \{\{2\}, \{1,2\}, \{2,3\}, X\}$. If $P_1 = \{2,3\}, P_2 = \{3\}, P_3 = X, then\ P_{1_{O_\sigma}} = \{1,3\}, P_{2_{O_\sigma}} = X, P_{3_{O_\sigma}} = \{1,3\}$.
Moreover, the following statements are equivalent:

- For every $x \in P_{O_\sigma}$.
- $x \in (X - P)_{t_\sigma}$.
- $x \in X - P_{f_\sigma}$.

Hence, $P_{O_\sigma} = (X - P)_{t_\sigma}$ and $P_{O_\sigma} = X - P_{f_\sigma}$ are equivalent definitions of the cluster outer set. The cluster outer set is complement of the follower set. Since P_{f_σ} is closed set this mean the cluster outer set is an open set. The collection of all cluster outer set denoted by $O_\sigma(X)$.

Proposition 1 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space. P_1, P_2 are non-empty subsets of X, then:

1. If $P_1 \subseteq P_2$, $then$ $P_{2_{O_\sigma}} \subseteq P_{1_{O_\sigma}}$.
2. If $P_1 \subseteq P_2$, $then (P_{1_{O_\sigma}})_{O_\sigma} \subseteq (P_{2_{O_\sigma}})_{O_\sigma}$.
3. $(P_1 \cup P_2)_{O_\sigma} = P_{1_{O_\sigma}} \cap P_{2_{O_\sigma}}$.
4. $P_{1_{O_\sigma}} \cap P_{2_{O_\sigma}} \subseteq (P_1 \cap P_2)_{O_\sigma}$.
5. $P_{O_\sigma} = int(P_{O_\sigma}) \supseteq int(X - P)$.
6. If $P \notin \sigma$, then $P_{O_\sigma} = X$.
7. $P_{O_\sigma} \subseteq \left(X - P_{O_\sigma}\right)_{O_\sigma} = (P_{f_\sigma})_{O_\sigma}$.
8. $(\emptyset)_{O_\sigma} = X$.
9. $X_{O_\sigma} = X - X_{f_\sigma}$.
10. $(P_{O_\sigma})_{O_\sigma} = (P_{f_\sigma})_{t_\sigma}$.

Proof:

1) Let $x \in P_{2_{O_\sigma}}$, there exists $\mathcal{U} \in \tau(x)$, such that $(\mathcal{U} \cap P_2)\overline{\delta}C$ for some $C \in \sigma$, so $P_1 \subset P_2$ then by property of proximity space (see[3]) $(\mathcal{U} \cap P_1)\overline{\delta}C$ for some $C \in \sigma$, hence $x \in P_{1_{O_\sigma}}$.
2) Evident through part 1.
3) $(P_1 \cup P_2)_{O_\sigma} = X - (P_1 \cup P_2)_{f_\sigma} = X - (P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}}) = X - (P_{1_{f_\sigma}}) \cap X - (P_{2_{f_\sigma}}) = P_{1_{O_\sigma}} \cap P_{2_{O_\sigma}}$.
4) By part 1.
5) $P_{O_\sigma} = X - P_{f_\sigma} = X - cl(P_{f_\sigma}) = int(x - P_{f_\sigma}) = int(P_{O_\sigma})$ and $X - cl(P_{f_\sigma}) \supseteq X - cl(P) = int(X - P)$.
6) If $P \notin \sigma$, then for every $x \in X$ there exists $\mathcal{U} \in \tau(x)$ such that $(\mathcal{U} \cap P)\overline{\delta}C$ for some $C \in \sigma$ hence $x \in P_{O_\sigma}$.
7) $P_{O_\sigma} = X - P_{f_\sigma} \subseteq X - (X - X - P_{f_\sigma})_{f_\sigma} = X - (X - P_{O_\sigma})_{f_\sigma} = (X - P_{O_\sigma})_{O_\sigma} = (P_{f_\sigma})_{O_\sigma}$.
8) $(\emptyset)_{O_\sigma} = X - \emptyset_{f_\sigma} = X - \emptyset = \mathrm{X}$.
10) $(P_{O_\sigma})_{O_\sigma} = (X - P_{f_\sigma})_{O_\sigma} = X - (X - P_{f_\sigma})_{f_\sigma} = (X - X - P_{f_\sigma})_{t_\sigma} = (P_{f_\sigma})_{t_\sigma}$.
From the above proposition, it can be concluded that, if $(P_{O_\sigma})_{O_\sigma} = \emptyset$, then P is a scant set, and the opposite is true. Also, if $P_{O_\sigma} = \emptyset$, then P is a member of the cluster family. Moreover, every subset of the scant set is a scant set. It can be noted that if $X = X_{f_\sigma}$, then$(P_{O_\sigma})_{O_\sigma} \subset P_{f_\sigma}$. But if P is closed set, then $(P_{O_\sigma})_{O_\sigma} \subset P_{t_\sigma}$. Also, $(X - P)_{O_\sigma} = P_{t_\sigma}$

Remark 1 If $(\mathrm{X}, \delta, \tau, \sigma)$ is dismountable space, then there exists $\subseteq X$, such that $A_{O_\sigma} = \emptyset$.
Proof:
Suppose X is dismountable, there exist two sets $A, B \subseteq X$, such that $A_{f_\sigma} = B_{f_\sigma} = X$, $A \cup B = X$, and $A \cap B = \emptyset$ this mean $X - A_{f_\sigma} = \emptyset$, hence $A_{O_\sigma} = \emptyset$.

It can be concluded by Remark 3.4 the space is dismountable if and only if there exist at least two disjoint cluster outer sets.

Definition 9 Let $(\mathrm{X}, \delta, \tau, \sigma)$ be a cluster topological proximity space. A is said to be *cluster disputed set* denoted by $D_\delta(P)$ if and only if $D_\delta(P) = P_{f_\sigma} \cap P_{t_\sigma}$.

By example 2.8 $D_\delta(\{2,3\}) = \{2,3\}_{f_\sigma} \cap \{2,3\}_{t_\sigma} = X \cap \{3\} = \{3\}$ and $D_\delta(X) = X$.
It is evident in the case of $P \notin \sigma$, that $D_\delta(P) = \emptyset$ because the follower set is equal to empty set for every $P \notin \sigma$. And in the case $P \subseteq P_{f_\sigma}$then the cluster disputed of university set is itself.

Proposition 2 Let $(\mathrm{X}, \delta, \tau, \sigma)$ be cluster topological proximity space. P_1, P_2 non-empty subsets of X, then:
1. $D_\delta(\emptyset) = \emptyset$.
2. $D_\delta(X) = X_{f_\sigma}$.
3. If $P_1 \subset P_2$ *then* $D_\delta(P_1) \subset D_\delta(P_2)$.
4. $D_\delta(P_1 \cup P_2) \supseteq D_\delta(P_1) \cup D_\delta(P_2)$.
5. $D_\delta(P_1 \cap P_2) \subseteq D_\delta(P_1) \cap D_\delta(P_2)$.
6. If $C \notin \sigma$,then $D_\delta(C) = \emptyset$,and $D_\delta(X - C) = (X - C)_{f_\sigma}$.
7. If P is closed set , then $D_\delta(P) \subseteq int_{f_\sigma}(P) \subseteq P$. $(int_{f_\sigma}(P) = P \cap P_{t_\sigma})$.
8. If P is a both open and closed, then $D_\delta(P) = P_{f_\sigma}$.
9. $D_\delta(P) = P_{f_\sigma} - (X - P)_{f_\sigma}$.
10. $D_\delta(P) = \emptyset$ if and only if $P_{f_\sigma} \subseteq (X - P)_{f_\sigma}$or $P_{t_\sigma} \subseteq (X - P)_{t_\sigma}$.

11. $D_\delta\big(D_\delta(P)\big) \subseteq D_\delta\big(P_{f_\sigma}\big) \cap D_\delta\big(P_{t_\sigma}\big)$.
12. $D_\delta\Big(D_\delta\big(D_\delta(P)\big)\Big) \subseteq D_\delta\big((P_{f_\sigma})_{f_\sigma}\big) \cap D_\delta\big((P_{t_\sigma})_{t_\sigma}\big) \cap D_\delta\big((P_{f_\sigma})_{t_\sigma}\big) \cap D_\delta\big((P_{t_\sigma})_{f_\sigma}\big)$.

Proof:

1) Evident, since $\emptyset_{f_\sigma} = \emptyset$.

3) $D_\delta(P_1) = P_{1_{f_\sigma}} \cap P_{1_{t_\sigma}} \subseteq P_{2_{f_\sigma}} \cap P_{2_{t_\sigma}} = D_\delta(P_2)$.

4) Clear by part 3.

6) Let $C \notin \sigma$, this means $C_{f_\sigma} = \emptyset$ and $(X-C)_{t_\sigma} = X$. Hence $D_\delta(C) = \emptyset$, and $D_\delta(X-C) = (X-C)_{f_\sigma}$.

7) $D_\delta(P) = P_{f_\sigma} \cap P_{t_\sigma}$ since P is closed we have $P_{f_\sigma} \subseteq P$, hence $P_{f_\sigma} \cap P_{t_\sigma} \subseteq P \cap P_{t_\sigma} = int_{f_\sigma}(P) \subseteq P$.

8) By fact, if P is a both open and closed, then $P_{f_\sigma} \subseteq P_{t_\sigma}$.

11) $D_\delta\big(D_\delta(P)\big) = D_\delta\big(P_{f_\sigma} \cap P_{t_\sigma}\big) = \big(P_{f_\sigma} \cap P_{t_\sigma}\big)_{f_\sigma} \cap \big(P_{f_\sigma} \cap P_{t_\sigma}\big)_{t_\sigma} \subseteq \big(P_{f_\sigma}\big)_{f_\sigma} \cap \big(P_{t_\sigma}\big)_{f_\sigma} \cap \big(P_{f_\sigma}\big)_{t_\sigma} \cap \big(P_{t_\sigma}\big)_{t_\sigma} = (\big(P_{f_\sigma}\big)_{f_\sigma} \cap \big(P_{f_\sigma}\big)_{t_\sigma}) \cap (\big(P_{t_\sigma}\big)_{f_\sigma} \cap \big(P_{t_\sigma}\big)_{t_\sigma}) = D_\delta\big(P_{f_\sigma}\big) \cap D_\delta\big(P_{t_\sigma}\big)$.

Definition 10 Let $(X, \delta, \tau, \sigma)$ be cluster topological proximity space. A is said to be *cluster brim set* denoted by $B_\delta(P)$ if and only if $B_\delta(P) = P_{f_\sigma} \cap (X-P)_{f_\sigma}$.

By Example 3.2 $B_\delta(P) = \emptyset$ for every subset of X, hence $B_\delta(P) \neq \emptyset$ if and only if P and $X-P$ belong to cluster family. It is noted that the cluster brim set depends on the proximity relationship defined on the cluster. So if we assume that the proximity relationship is indiscrete proximity on any topology, then the cluster brim set is non-empty, but in the case of the discrete proximity relationship defined on the cluster, the cluster brim set is always equal to the empty set. This does not mean that the topology has no effect, but the effect of the proximity relationship is stronger than the effect of the topology.

Proposition 3 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space, $x \in B_\delta(P)$ if and only if $x \in P_{f_\sigma} - P_{t_\sigma}$.

Proof:

Let $x \in B_\delta(P)$. Then, $x \in P_{f_\sigma}$and $x \in (X-P)_{f_\sigma}$, $x \notin X-(X-P)_{f_\sigma}$, so that $x \notin P_{t_\sigma}$ hence $x \in P_{f_\sigma} - P_{t_\sigma}$.

Conversely, let $x \in P_{f_\sigma} - P_{t_\sigma}$. This means $x \in P_{f_\sigma}$and $x \notin P_{t_\sigma}$ so, $x \in X - P_{t_\sigma}$ but $-P_{t_\sigma} = (X-P)_{f_\sigma}$, hence $x \in B_\delta(P)$.

The above proposition is an equivalent definition to the cluster brim. This could be used to prove some elements of the following proposition, where the most important properties of cluster brim are mentioned.

Proposition 4 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space, then
1. $B_\delta(\emptyset) = \emptyset$, and $B_\delta(X) = \emptyset$.
2. $B_\delta(X) \subseteq B_\delta(P)$, for every nonempty subset on X.
3. $B_\delta(C) = \emptyset$, for every $C \notin \sigma$.
4. $B_\delta(P)$ is closed set.
5. $B_\delta(P_1 \cup P_2) \subseteq B_\delta(P_1) \cup B_\delta(P_2)$.
6. $B_\delta\big(B_\delta(P)\big) \subseteq B_\delta(P)$.
7. $B_\delta(X-P) = B_\delta(P)$.
8. $cl_{f_\sigma}(P) = B_\delta(P) \cup D_\delta(P) \cup P$. (Where $cl_{f_\sigma}(P) = P \cup P_{f_\sigma}$).

Proof:

1) $B_\delta(X) = X_{f_\sigma} \cap (X - X)_{f_\sigma} = \emptyset$.
4) Evident, since P_{f_σ} is closed set.
5)$B_\delta(P_1 \cup P_2) = (P_1 \cup P_2)_{f_\sigma} \cap \big(X - (P_1 \cup P_2)\big)_{f_\sigma} \subseteq (P_{1_{f_\sigma}} \cup P_{2_{f_\sigma}}) \cap ((X - P_1)_{f_\sigma} \cap (X - P_2)_{f_\sigma}) = (P_{1_{f_\sigma}} \cap (X - P_1)_{f_\sigma} \cap (X - P_2)_{f_\sigma}) \cup (P_{2_{f_\sigma}} \cap (X - P_1)_{f_\sigma} \cap (X - P_2)_{f_\sigma})) \subseteq ((P_{1_{f_\sigma}} \cap ((X - P_1)_{f_\sigma}) \cup \left(P_{2_{f_\sigma}} \cap ((X - P_2)_{f_\sigma}\right) = B_\delta(P_1) \cup B_\delta(P_2)$.
7) $B_\delta(X - P) = (X - P)_{f_\sigma} \cap (X - X - P)_{f_\sigma} = P_{f_\sigma} \cap (X - P)_{f_\sigma} = B_\delta(P)$.
8) $B_\delta(P) \cup D_\delta(P) \cup P = (P_{f_\sigma} \cap (X - P)_{f_\sigma}) \cup (P_{f_\sigma} \cap P_{t_\sigma}) \cup P = (P_{f_\sigma} \cup (P_{f_\sigma} \cap P_{t_\sigma}) \cap ((X - P_{t_\sigma}) \cup (P_{f_\sigma} \cap P_{t_\sigma})) \cup P = (P_{f_\sigma} \cup P_{f_\sigma}) \cap (P_{f_\sigma} \cup P_{t_\sigma}) \cap ((X - P_{t_\sigma}) \cup P_{f_\sigma}) \cap ((X - P_{t_\sigma}) \cup P_{t_\sigma}) \cup P = P_{f_\sigma} \cap (P_{f_\sigma} \cup ((X - P_{t_\sigma}) \cap P_{t_\sigma})) \cup P = P_{f_\sigma} \cup P = cl_{f_\sigma}(P)$.

Proposition 5 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space, $B_\delta(P) = \emptyset$ if and only if $P_{f_\sigma} \subseteq P_{t_\sigma}$.
Proof:
If $P_{f_\sigma} = \emptyset$ the proof is done. Suppose $B_\delta(P) = \emptyset$ and $P_{f_\sigma} \neq \emptyset$, then for every $x \in P_{f_\sigma}$ implies that $x \notin (X - P)_{f_\sigma}$this mean $x \in X - (X - P)_{f_\sigma}$, hence $x \in P_{t_\sigma}$. Conversely, suppose $P_{f_\sigma} \subseteq P_{t_\sigma}$ this mean $P_{f_\sigma} \cap X - P_{t_\sigma} = \emptyset$ but $X - P_{t_\sigma} = (X - P)_{f_\sigma}$hence $B_\delta(P) = \emptyset$.

By Definition 10 and the proposition above, as well as by relying on some properties of the follower and takeoff sets, we have some results mentioned in the following proposition:

Proposition 6 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space the statement is hold,
1. If P is open set, then $B_\delta(P) \subseteq P_{f_\sigma} - P$.
2. If P is closed set, then $B_\delta(P) \subseteq P - P_{t_\sigma}$.
3. If P is open and closed set, then $B_\delta(P) = \emptyset$.

Proof:
1) Let P be open. Then $P \subseteq P_{t_\sigma}$. So that $B_\delta(P) = P_{f_\sigma} - P_{t_\sigma} \subseteq P_{f_\sigma} - P$.
2) Let P be closed. Then $P_{f_\sigma} \subseteq P$. So that $B_\delta(P) = P_{f_\sigma} - P_{t_\sigma} \subseteq P - P_{t_\sigma}$.
3) Let P be open and closed set. Then, $P_{f_\sigma} \subseteq P_{t_\sigma}$. So that $B_\delta(P) = P_{f_\sigma} - P_{t_\sigma} = \emptyset$.

Proposition 7 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space. $X = X_{f_\sigma}$ if and only if for every $G \in \tau$, $G_{f_\sigma} = cl(G)$.
Proof:
Suppose $X = X_{f_\sigma}$, then for every $x \in X$, $x \in X_{f_\sigma}$this means that for every $G \in \tau(x)$, $G \cap X \delta C$ for every $C \in \sigma$. If possible there exists $x \in cl(G)$, and $x \notin G_{f_\sigma}$, there exists $\mathcal{U} \in \tau(x)$, such that $G \cap \mathcal{U} \bar{\delta} C$ for some $C \in \sigma$. This mean $G \cap \mathcal{U} \notin \sigma$ but $G \cap \mathcal{U} \in \tau(x)$, then $(G \cap \mathcal{U}) \cap X \bar{\delta} C$ for some $C \in \sigma$, this contradicts $X = X_{f_\sigma}$. Hence $G_{f_\sigma} = cl(G)$.

Proposition 8 Let $(X, \delta, \tau, \sigma)$ be a cluster topological proximity space, then the following statements are equivalent:
1. $G \in \tau$, then $G \subseteq G_{f_\sigma}$;
2. $X = X_{f_\sigma}$;
3. $G \in \tau$, then $G_{f_\sigma} = cl(G)$.

Proof:
1 ⇒ 2 Suppose $G \in \tau$ and $G \subseteq G_{f_\sigma}$. Since $X \in \tau$, then $X \subseteq X_{f_\sigma}$,hence $X = X_{f_\sigma}$.
2 ⇒ 3 Suppose $X = X_{f_\sigma}$, by the above proposition $G_{f_\sigma} = cl(G)$.

$3 \Rightarrow 1$ Suppose $cl(G) = G_{f_\sigma}$, but $G \subseteq cl(G) = G_{f_\sigma}$, hence $G \subseteq G_{f_\sigma}$.
After examining the properties of the boundary sets, it is important to study the relationship among them and their effect on the proximity space. The following proposition explains the most important relationships between these sets.

Proposition 9 Let $(\mathrm{X}, \delta, \tau, \sigma)$ be a cluster topological proximity space, then the statement are holds:
1. $B_\delta(P) = P_{f_\sigma} \cap X - P_{t_\sigma}$.
2. $B_\delta(P) \cap D_\delta(P) = \emptyset$.
3. $P_{f_\sigma} = B_\delta(P) \cup D_\delta(P)$.

Proof:
2) $B_\delta(P) \cap D_\delta(P) = (P_{f_\sigma} \cap \left(X - P\right)_{f_\sigma}) \cap (P_{f_\sigma} \cap P_{t_\sigma}) = (P_{f_\sigma} \cap X - P_{t_\sigma}) \cap (P_{f_\sigma} \cap P_{t_\sigma}) = P_{f_\sigma} \cap (P_{t_\sigma} \cap X - P_{t_\sigma}) = \emptyset$.
3) The proof is similar to the proof of Proposition 3.9 part 8.

It is concluded that X is divided into three pairwise disjoint sets: cluster outer, cluster disputed and cluster brim, hence $X = B_\delta(P) \cup D_\delta(P) \cup P_{O_\sigma}$. Also, $B_\delta(P) \cap D_\delta(P) \cap P_{O_\sigma} = \emptyset$. By above proposition $B_\delta(P) \cap D_\delta(P) = \emptyset$, to prove $B_\delta(P) \cap P_{O_\sigma} = \emptyset$ and $D_\delta(P) \cap P_{O_\sigma} = \emptyset$. $B_\delta(P) \cap P_{O_\sigma} = (P_{f_\sigma} \cap (X - P)_{f_\sigma}) \cap X - P_{f_\sigma} = (P_{f_\sigma} \cap X - P_{f_\sigma}) \cap (X - P)_{f_\sigma} = \emptyset$. Also, $D_\delta(P) \cap P_{O_\sigma} = \left(P_{f_\sigma} \cap P_{t_\sigma}\right) \cap X - P_{f_\sigma} = P_{t_\sigma} \cap \left(P_{f_\sigma} \cap X - P_{f_\sigma}\right) = \emptyset$.

Note that if P is bushy set then P divides the space into two disjoint sets $B_\delta(P)$ and $D_\delta(P)$, hence if P is bushy set, then:
- $P_{O_\sigma} = \emptyset$.
- $B_\delta(P) = X - P_{t_\sigma}$.
- $D_\delta(P) = P_{t_\sigma}$.
-

Proposition 10 Let $(\mathrm{X}, \delta, \tau, \sigma)$ be a cluster topological proximity space. If X is dismountable space, then the following are held for P subset of X:
1. $B_\delta(P) = X$.
2. $D_\delta(P) = \emptyset$
3. $P_{O_\sigma} = \emptyset$.

Proof:
1) Suppose X is dismountable space, then there exist nonempty P_1, P_2 disjoint subsets of X, such that $P_{1_{f_\sigma}} = P_{2_{f_\sigma}} = X$ and so $P_2 = X - P_1$. $B_\delta(P_1) = P_{1_{f_\sigma}} \cap (X - P_1)_{f_\sigma} = (X - P_1)_{f_\sigma} = P_{2_{f_\sigma}} = X$.
2) $D_\delta(P_1) = P_{1_{f_\sigma}} \cap P_{1_{t_\sigma}} = P_{1_{t_\sigma}} = X - (X - P_1)_{f_\sigma} = X - X = \emptyset$.

Proposition 11 Let $(\mathrm{X}, \delta, \tau, \sigma)$ be a cluster topological proximity space. If $\mathcal{U} \in \sigma$ for every non-empty $\mathcal{U} \in \tau$, then $\mathcal{U}$ is a bushy set.

Proof:
There could exist $x \notin \mathcal{U}_{f_\sigma}$, then there exists $\mathcal{V} \in \tau(x)$ such that $(\mathcal{V} \cap \mathcal{U})\overline{\delta}C$ for some $C \in \sigma$ this mean $\mathcal{V} \cap \mathcal{U} \notin \sigma$ ($if\ A, B \in \sigma, then\ A\delta B$) but $\mathcal{V} \cap \mathcal{U} \in \tau$ this is an contradiction with hypothesis.

Therefore, it can be concluded from the above proposition, if every non-empty open subset of X is a member on cluster, then $\mathcal{U} \subseteq \mathcal{U}_{f_\sigma}$.

Proposition 12 Let $(\mathrm{X}, \delta, \tau, \sigma)$ be a cluster topological proximity space. If A or B is open set then
$int(A_{f_\sigma}) \cap \ int(B_{f_\sigma}) = int(A \cap B)_{f_\sigma}$.

Proof:

Suppose $A \in \tau$ we have $A \cap B_{f_\sigma} \subseteq (A \cap B)_{f_\sigma}$, and $int\big(A \cap B_{f_\sigma}\big) \subseteq int(A \cap B)_{f_\sigma}$ which implies that $intA \cap intB_{f_\sigma} \subseteq int(A \cap B)_{f_\sigma}$, but $intA = A$,hence $A \cap intB_{f_\sigma} \subseteq int(A \cap B)_{f_\sigma}$, and this implies $int\big(A \cap intB_{f_\sigma}\big)_{f_\sigma} \subseteq int\big(int(A \cap B)_{f_\sigma}\big)_{f_\sigma}$------(1)

But $intB_{f_\sigma} \in \tau$ which means that $intB_{f_\sigma} \cap A_{f_\sigma} \subseteq (intB_{f_\sigma} \cap A)_{f_\sigma}$. It implies that $int\big(intB_{f_\sigma}\big) \cap intA_{f_\sigma} = intB_{f_\sigma} \cap intA_{f_\sigma} \subseteq int(\big(intB_{f_\sigma} \cap A\big)_{f_\sigma} = int\big(A \cap intB_{f_\sigma}\big)_{f_\sigma}$ hence by (1) we have $intB_{f_\sigma} \cap intA_{f_\sigma} \subseteq int\big(int(A \cap B)_{f_\sigma}\big)_{f_\sigma}$ ------ (2).

But $\big(int(A \cap B)_{f_\sigma}\big)_{f_\sigma} \subseteq \big((A \cap B)_{f_\sigma}\big)_{f_\sigma} \subseteq (A \cap B)_{f_\sigma}$ implying that $int\big(int(A \cap B)_{f_\sigma}\big)_{f_\sigma} \subseteq int(A \cap B)_{f_\sigma}$-------- (3). By (2) and (3) we have $intB_{f_\sigma} \cap intA_{f_\sigma} \subseteq int(A \cap B)_{f_\sigma}$.

Corollary 1: Let $(\mathrm{X}, \delta, \tau, \sigma)$ be a cluster topological proximity space. If A or B is open set, then
1. $int(A_{t_\sigma} \cap B)_{f_\sigma} = int(A_{t_\sigma})_{f_\sigma} \cap int\ B_{f_\sigma}$.
2. $int(A_{t_\sigma})_{f_\sigma} \cap int(B_{t_\sigma})_{f_\sigma} = int(A \cap B)_{t_\sigma})_{f_\sigma}$.

The proof is directly through the above proposition.

2. Conclusions

In the current study, three types of families were built. The first family, given the symbol $\mathrm{O}_\sigma(X)$, represents all cluster outer sets. The second family, $\mathrm{D}_\sigma(X)$, represents all cluster disputed sets. Finally, the third family, $B_\sigma(X)$, represents all cluster brim sets. It was noticed through the Definitions 3.1, 3.5, and 3.7, and Propositions 3.3, 3.6, and 3.9 that these sets represent a classification into X. That is the space is decomposition to X, meaning that it is pairwise disjoint and union of X. From another axis, it appeared that all families are sub base on the topology defined on X, but in the case of bushy set condition, the second family $\mathrm{D}_\sigma(\mathrm{X})$ forms the base. These three topologies are left to researchers for further investigation.

3. Future works

For future studies, the researcher will try to study the bushy set in the proximity space more comprehensively and examine its effect. Also, spaces that contain at least one bushy set, can be studied and its impact on the relationships between the takeoff set and follower set and its effect on the dismountable and non-dismountable spaces.

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