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Further properties of the fuzzy complete a-fuzzy normed algebra

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Abstract
In this paper further properties of the fuzzy complete a-fuzzy normed algebra have been introduced. Then we found the relation between the maximal ideals of fuzzy complete a-fuzzy normed algebra and the associated multiplicative linear function space. In this direction we proved that if ℓ is character on Z then kerℓ is a maximal ideal in Z. After this we introduce the notion structure of the a-fuzzy normed algebra then we prove that the structure, st(Z) is ω^*-fuzzy closed subset of fb(Z, $\mathbb{C}$) when (Z, n_Z , $\circledast$, $\odot$) is a commutative fuzzy complete a-fuzzy normed algebra with identity e.

Keywords: Character of a-fuzzy normed algebra, Structure of a-fuzzy normed algebra, ω^* −fuzzy topology on afb(Z, $\mathbb{C}$), Fuzzy Tychonoff Theorem, fuzzy Gelfand transform.

خواص إضافية للفضاء الجبري القياسي الضبابي–أ التام ضبابيا

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الخلاصة
في هذا البحث خواص إضافية للفضاء الجبري القياسي الضبابي–أ التام ضبابيا تم تقديمها. بعد ذلك وجدنا العلاقة بين المثاليات العظمى للإفضاء الجبري القياسي الضبابي–أ والتام ضبابيا وفضاء الدوال الخطية الضربية المرتبطة. وفي هذا الاتجاه برهنا إذا كان ℓ هو character على الفضاء Z عندئذ kerℓ هو مثالي أعظم في Z. بعد ذلك تم تقديم مفهوم structure للفضاء الجبري القياسي الضبابي–أ بعد ذلك برهنا st(Z) هو ω^*– مغلق ضبابي جزئي في الفضاء fb(Z, $\mathbb{C}$) عندما يكون الفضاء (Z, n_Z , $\circledast$, $\odot$) إبدالي وفضاء جبري قياسي ضبابي–أ وتام ضبابيا يحتوي على عنصر محايد e .

1. Introduction

In this paper we continue the previous study of fuzzy complete a-fuzzy normed algebra. We prove here the other important properties of fuzzy complete a-fuzzy normed algebra. The organization of this paper is as follows: we divided this research into four sections; the introduction will be in section one, after that in section two important properties of fuzzy length space and a-fuzzy normed space are recalled. Furthermore, basic important properties of a-fuzzy normed algebra that will be needed later can be found in section three. Moreover, further properties of fuzzy complete a-fuzzy normed algebra have been proved as a main results in the same section. Finally, in section four we highlight the conclusion for this research.

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2. Preliminaries about a-fuzzy normed algebra

Definition 2.1 [1]:

Let ⊛:I × I →I be a binary operation function, then it is said to be continuous t-conorm (or simply t-conorm) if it satisfies the following conditions:
(i) p ⊛ q= q ⊛ p;
(ii) p ⊛ [q ⊙ w] = [p ⊛ q] ⊛ w;
(iii) ⊛ is continuous function;
(iv)p ⊛ 0 = 0;
(v) (p ⊛ z) ≥ (q ⊛ w) whenever p ≥ q and z ≥ w.
For all p, q, z, w ∈ I = [0, 1].

Definition 2.2:

If $L_{\mathbb{C}}$:ℂ →I is a fuzzy set, and ⊛ is a t-conorm, then $L_{\mathbb{C}}$ is **a-fuzzy length on** ℂ if
(i)0< $L_{\mathbb{C}}(\sigma)$ ≤1;
(ii) $L_{\mathbb{C}}$ (σ)=0 if and only if σ=0;
(iii) $L_{\mathbb{C}}(\sigma\tau) \leq L_{\mathbb{C}}(\sigma).L_{\mathbb{C}}(\tau)$;
(iv) $L_{\mathbb{C}}(\sigma + \tau) \leq L_{\mathbb{C}}(\sigma)$ ⊛ $L_{\mathbb{C}}(\tau)$.
For all σ, τ ∈ ℂ. The (ℂ, $L_{\mathbb{C}}$, ⊛) is **a-fuzzy length space.**

Remark 2.3:

We will take ⊛ to be μ ⊛ σ= μ+ $\sigma - \mu\, \sigma$, for all μ, σ ∈ I = [0, 1].

Example 2.4:

If $L_{|.|}(\alpha) = \frac{|\alpha|}{1+|\alpha|}$ for all $\alpha \in$ ℂ ,where | . | is length value on ℂ Then (ℂ, $L_{|.|}$, ⊛) is a-fuzzy length space.

Definition 2.5 [3]:

If (ℂ, $L_{\mathbb{C}}$, ⊛) is a-fuzzy length space, Z is a vector space over ℂ, and ⊛ is a t-conorm and n_Z: Z→I is a fuzzy set. Then n_Z is **a-fuzzy norm on Z** if
(i)0< n_Z(z) ≤ 1;
(ii) n_Z(z) =0 ⟺ z=0;
(iii) $n_Z(\mu z) \leq L_{\mathbb{C}}(\mu)$ n(z) for all 0≠ μ ∈ ℂ;
(iv) n_Z(z + y) ≤ n_Z(z) ⊛ n_Z(y).

For all z, y ∈ Z. Then (Z, n_Z, ⊛) is **a-fuzzy normed space (or simply a-FNS)**.

Definition 2.7 [3]:

If (z_k) is a sequence in Z, then (z_k) is fuzzy converges to the limit z as k → ∞, if for all μ ∈ (0,1), we can find N ∈ ℕ when $n_Z(z_k - z) < \mu$, for all k≥N, if (Z, n_Z , ⊛) is a-FNS.
If (z_k) is a fuzzy converges to z we write $\lim_{k\to\infty} z_k = z$, or $z_k \to z$, or $\lim_{k\to\infty} n_Z(z_k - z) = 0$.

Definition 2.8 [3]:

If (z_k) is a sequence in Z, then (z_k) is fuzzy Cauchy sequence in Z if for all μ ∈ (0, 1), we can find N ∈ ℕ when $n_Z(z_k - z_m) < s$, for all k, m ≥N, if (Z, n_Z, ⊛) is a-FNS.

Definition 2.9 [3]:
If for any (z_k) fuzzy Cauchy in Z, there is z∈Z such that z_k →z, then the a-FNS (Z, n_Z, ⊛) is a fuzzy complete

Corollary 2.10 [3]:
The a-fuzzy length space (ℂ, $L_{\mathbb{C}}$, ⊛) is a fuzzy complete.

Theorem 2.11 [4]:
When (Z, n_Z, ⊛) and (W, n_W, ⊛) are two a-FNS. Then the operator H : Z→W is a fuzzy continuous at z∈Z if and only if whenever (z_k) is a fuzzy converges to z∈Z then $(H(z_k))$ is a fuzzy converges to H(z) ∈ W.

Definition 2.12 [4]:
When (Z, n_Z, ⊛) and (Y, n_Y, ⊛) are two a-FNS then the operator S : Z→Y is a **fuzzy bounded** if there is μ ∈(0, 1) with n_Y[S(z)] < μn_Z(z), for all z∈Z.

Notation [4]:
If (Z, n_Z, ⊛) and (Y, n_Y, ⊛) are two a-FNS then afb (Z, Y) = {S:Z→Y, S is a fuzzy bounded operator}.

Theorem 2.13 [4]:
Define $\boldsymbol{n_{afb(Z,Y)}(S) = \sup_{z\in D(S)} n_Y(Sz)}$**,** for all S∈ afb(Z, Y).Then (afb(Z, Y), $n_{afb(Z,Y)}$, ⊛) is a-FNS. If (Z, n_Z, ⊛) and (Y, n_Y, ⊛) are two a-FNS.

Theorem 2.14 [4]:
The space afb(Z, Y) is a fuzzy complete if Y is a fuzzy complete when (Z, n_Z, ⊛) and (Y, n_Y, ⊛) are two a-FNS.

Definition 2.15 [4]:
A linear functional h from a-FNS (Z, n_Z, ⊛) into the a-fuzzy length space (ℂ, $L_{\mathbb{C}}$, ⊛) is **fuzzy bounded** if there is δ ∈ (0, 1) with $L_{\mathbb{C}}$[h(u)] < $\delta . n_U$(u) for any u ∈D(h). Furthermore, the a-fuzzy norm of h is $n_{afb(Z,\mathbb{C})}$(h) = $\sup_{u\in D(h)} L_{\mathbb{C}}(hu)$, for all h∈ afb(Z, ℝ) and
$L_{\mathbb{C}}$[h(u)] < $n_{afb(Z,\mathbb{R})}(h). n_Z$(u) for any u ∈D(h).

Definition 2.16 [4]:
Let (Z, n_z, ⊛) be a-FNS. Then afb(Z, ℂ) = { h:Z→ ℂ: h is fuzzy bounded and linear } forms a-fuzzy normed space with a-fuzzy norm defined by $\boldsymbol{n_{afb(Z,\mathbb{C})}(h) = \sup_{u\in D(h)} L_{\mathbb{C}}(hu)}$ which is called the **fuzzy dual space** of Z.

Theorem 2.17 [4]:
If (Z, n_Z, ⊛) is a-FNS then fuzzy dual space afb(Z, ℂ) is a fuzzy complete.

Definition 2.18 [4]:
$\frac{Z}{D}$ ={z+D: z∈Z} is a 𝕂-space with the operations ; (v+D**)** +(z+D) = (v+z) +D and α(z+D) = (αz)+D. If Z is a vector space over the field 𝕂 and D is a closed subspace of Z.

Definition 2.19 [5]:
Define a-fuzzy norm for the quotient space $\frac{Z}{D}$ by q[u+D]=$inf_{d\in D} n_U$[z+d] for all z+D ∈ $\frac{Z}{D}$.

When (Z, n_Z , ⊛) be a-FNS and D⊂Z is a fuzzy closed in Z.

Theorem 2.20 [5]:

The quotient space ($\frac{Z}{D}$, q, ⊛) is a-FNS if (Z, n_Z , ⊛) is a-FNS and D⊂Z is a fuzzy closed in Z.

Remark 2.21 [5]:

If (Z, n_Z , ⊛) is a-FNS and D⊂Z is a fuzzy closed in Z. Then
(1)π:Z → $\frac{Z}{D}$ is a natural operator defined by π[z]=z+D;
(2)q(z+D) ≤ n_Z(z).

Theorem 2.22 [5]:

If ($\frac{Z}{D}$, q, ⊛) is a fuzzy complete then (Z, n_Z , ⊚) is a fuzzy complete when (Z, n_Z , ⊛) be a-FNS and D⊂Z is a fuzzy closed in Z.

Theorem 2.23 [5]:

If (Z, n_Z , ⊛) is a fuzzy complete then ($\frac{Z}{D}$, q, ⊚) is a fuzzy complete when (Z, n_Z , ⊛) is a-FNS and D⊂Z is a fuzzy closed in Z.

Definition 2.24 [1]:

Let ⊙:I × I →I be a binary operation function then ⊚ is said to be continuous t-norm (or simply t-norm) if it satisfies the following conditions:
(i)p ⊙ q= q ⊙ p;
(ii) p ⊙ [q ⊙ w] = [p ⊙ q] ⊙ w;
(iii) ⊙ is continuous function;
(iv)p ⊙ 1 = p;
(v) (p ⊙ z) ≤ (q ⊙ w) whenever p ≤ q and z ≥ w.
For all p, q, z, w ∈ I = [0, 1].

Definition 2.25 [6]:

The space (Z, n_Z, ⊛, ⊙) is called a-fuzzy normed algebra space (or simply a-FNAS) if
(1) (Z, +, .) is an algebra space over the field 𝕂, where 𝕂=ℝ or 𝕂=ℂ;
(2) (Z, n_Z, ⊛) is a-FNS, with ⊛ is a t-conorm;
(3) ⊙ is a t-norm;
(4)n_Z(p.q)≤ n_Z(p) ⊙ n_Z(q), for all p, q ∈Z.

Remark 2.26:

In this paper we take;
(1)$\sigma \odot \tau=\sigma.\tau$, for all σ, τ ∈[0, 1].
(2)γ ⊛ δ=γ+ δ − γδ, for all γ, δ ∈[0, 1].

Definition 2.29 [6]:

The space (Z, n_Z, ⊛, ⊙) is a fuzzy complete a-FNA if (Z, n_Z, ⊛) is a fuzzy complete a-FNS. Then (Z, n_Z, ⊛, ⊙) is a commutative fuzzy complete a-FNA.

Lemma 2.31 [6]:

If (Z, n_Z, ⊛, ⊙) is a-FNA, then multiplication is a fuzzy continuous function.

Theorem 2.32 [6]:
An a-FNA (Z, n_Z, ⊛, ⊙) without identity can be embedded into a-FNA, Z_e having the identity e, also Z is considered as an ideal in Z_e.

Proposition 2.33 [6]:
The space (Z_e , n_{Z_e}, ⊛, ⊙) is a fuzzy complete ⟺ (Z, n_Z, ⊛, ⊙) is a fuzzy complete.

Theorem 2.34 [6]:
Every a-FNA can be embedded as a closed subalgebra of afb(Z, Z).

Proposition 2.35 [6]:
If (Z, n_Z, ⊛, ⊙) is a fuzzy complete a-FNA and z∈Z, then (e−z) is invertible, the series $\sum_{k=0}^{\infty} z^k$ is fuzzy converges, and $\sum_{k=0}^{\infty} z^k = (e - z)^{-1}$.

Theorem 2.36 [6]:
The space ($\frac{Z}{D}$, q, ⊛, ⊙) is a fuzzy complete a-FNA if (Z, n_Z , ⊛, ⊙) is a fuzzy complete a-FNA and D is a fuzzy closed ideal in Z. Also $\frac{Z}{D}$ has an identity if Z has an identity. As well as the identity of $\frac{Z}{D}$ has a fuzzy norm equal to 1.

Remark 2.37 [6]:
If (Z, n_Z , ⊛, ⊙) is a fuzzy complete, then for any a≠ 0, a^{-1} exists and a^{-1} ∈Z.

Proposition 2.38 [6]:
If (Z, n_Z , ⊛, ⊙) is a fuzzy complete a-FNA, then T(z)=z^{-1} is fuzzy continuous mapping.

Lemma 2.39 [6]:
Let (Z, n_Z , ⊛, ⊙) be a fuzzy complete having an identity e. If z^{-1} and u^{-1} exists in Z then $(zu)^{-1}$ and $(uz)^{-1}$ are exist in Z.

Proposition 2.40 [6]:
Let (Z, n_Z , ⊛, ⊙) be a fuzzy complete a-FNA having an identity e. If z, u ∈ Z where $(e - zu)^{-1}$ exists. If d=$(e - zu)^{-1}$ then $(e - uz)^{-1}$ =e + udz.

Definition 2.41 [6]:
Let $\mathcal{A}$={A_j:j∈J} be a family of subsets of a space Z. The family $\mathcal{A}$ is **centered** if for any finite number of sets A_1, A_2, …,A_k ∈ $\mathcal{A}$ we have $\cap_{j=1}^{k} A_j \neq \emptyset$.

Definition 2.42 [6]:
Let Z be a non-empty set. A collection T of a subset of Z is said to be a **fuzzy topology** on Z if
(i)Z ∈ T and φ ∈ T;
(ii)If A_1, A_2,…., A_n ∈ T then $\cap_{i=1}^{n} A_i$ ∈ T;
(iii)If {A_j: j∈J}∈ T then $\cup_{j\in J} A_j$ ∈ T.
Then (Z, T) is called **a fuzzy topological space**.

Theorem 2.43:
If Z is a fuzzy topological space then the following statement are equivalent:
(1) Z is a fuzzy compact;

(2) For any centred family $\mathcal{A}$ of a fuzzy closed subset of Z we have $\cap_{A\in\mathcal{A}} \neq \emptyset$.

Proof: (2)⟹(1)

Let $\mathcal{A}$={A_j:j∈J} be a fuzzy open cover of Z. We need to show that $\mathcal{A}$ has a finite subcover. For j∈J, define G_j=Z− $A_j$this gives a family $\mathcal{G}$=={G_j:j∈J} of fuzzy closed sets in Z. We have $\cap_{j\in J}$ G_j= $\cap_{j\in J}$[Z− A_j] = Z−[$\cup_{j\in J}$ A_j]=Z−Z=∅, since Z=$\cup_{j\in J}$ A_j.This implies that $\mathcal{G}$ is not centered family, so there exists G_1, G_2, …,G_k ∈ $\mathcal{G}$ such that $\cap_{j=1}^{k}$ G_j= ∅. This gives, ∅=$\cap_{j=1}^{k}$ G_j= $\cap_{j=1}^{k}$ [Z− A_j] = Z−[$\cup_{j=1}^{k}$ A_j]. Therefore, Z=$\cup_{j=1}^{k}$ A_j and so Z is a fuzzy compact since {A_j: j=1,2, …, k} is a finite subcover of $\mathcal{A}$.

(1)⟹(2) Follows from a similar argument.

Fuzzy Tychonoff Theorem 2.44:

If {Z_j: j∈J} is a family of fuzzy topological spaces and Z_j is a fuzzy compact ∀ j∈J, then the product space $\Pi_{j\in J}Z_j$ is a fuzzy compact.

Proof:

Let Z=$\Pi_{j\in J}Z_j$ where Z_j is a fuzzy compact ∀ j∈J. Let $\mathcal{A}$ be a centred family of fuzzy closed subset of Z. We will show that there exists z = (z_j), j∈J ∈Z such that z ∈ $\cap_{A\in\mathcal{A}}$ A. Let D denote the set consisting of all centred families $\mathcal{F}$[not necessarily fuzzy closed] of subset of Z such that $\mathcal{A} \subseteq \mathcal{F}$. The set D is partially ordered set by ⊆ .

We will show that every chain in D has an upper bound. Indeed, if {$\mathcal{F}_j$: j∈J} is A chain in D then take $\mathcal{F}$= $\cup_{j\in J}$ $\mathcal{F}_j$. Since $\mathcal{F}$ is centred family and $\mathcal{F}_j \subseteq \mathcal{F}$ for all j∈J thus $\mathcal{F}$ is an upper bound of {$\mathcal{F}_j$: j∈J}. Now by Zorn's Lemma we obtain that the set D contains a maximal element $\mathcal{M}$. We will show that there exists z ∈Z such that z∈ $\cap_{M\in\mathcal{M}}$ $\overline{M}$. Since $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{A}$ contains of fuzzy closed sets we have, $\cap_{M\in\mathcal{M}}$ $\overline{M}$ ⊆ $\cap_{A\in\mathcal{A}}$ A. Therefore, it will follow that z∈ $\cap_{A\in\mathcal{A}}$ A and $\cap_{A\in\mathcal{A}}$ A ≠ ∅.

Construction of the element z proceed as follows. For j∈J let p_j:Z→ $Z_j$be the projection onto the jth coordinate. Now for each j∈J the family {$\overline{p_j(M)}$: $M \in \mathcal{M}$} is centred family of fuzzy closed subsets of Z_j, so by the fuzzy compactness of Z_j there exists $z_j \in Z_j$such that $z_j \in \cap_{M\in\mathcal{M}}$ $\overline{p_j(M)}$. We set z = (z_j), j∈J.

In order to see that z∈ $\cap_{M\in\mathcal{M}}$ $\overline{M}$ notice that $\mathcal{M}$ the following property:

If B⊆Z and B∩M≠ ∅ for all M ∈ $\mathcal{M}$then B ∈ $\mathcal{M}$ ……..(∗)

Indeed if $\mathcal{M}'$=$\mathcal{M}$ ∪ {B} then $\mathcal{M}'$ ∈D, so by maximality of $\mathcal{M}$we must have $\mathcal{M}$=$\mathcal{M}'$. For j∈J let $U_j \subseteq Z_j$be a fuzzy open neighborhood of z_j. Since $z_j \in \overline{p_j(M)}$ for all $M \in \mathcal{M}$, thus U_j ∩ $p_j(M)$ ≠ ∅ for all $M \in \mathcal{M}$.

Equivalently $p_j^{-1}(U_j)$∩ M≠ ∅ for all $M \in \mathcal{M}$. By property ∗ we obtain that $p^{-1}(U_j) \in \mathcal{M}$ for all j∈J. Since $\mathcal{M}$is a centred family we obtain

$p^{-1}(U_1) \cap p^{-1}(U_2) \cap\ldots\cap p^{-1}(U_k)$ ∩ M ≠ ∅, for all $M \in \mathcal{M}$ …….(∗∗)

Now the sets of the form, $p^{-1}(U_1) \cap p^{-1}(U_2) \cap\ldots\cap p^{-1}(U_k)$ are precisely the fuzzy open neighbourhood of z that belong to the basis of the product fuzzy topology on Z, and thus any fuzzy open neighbourhood of Z contains a neighbourhood of this type. Therefore, using (∗∗) we obtain that if $M \in \mathcal{M}$ then for any fuzzy open neighbourhood U of z we have M∩U ≠ ∅.

This means that for any $M \in \mathcal{M}$ we have z$\in \overline{M}$, and hence z $\in \cap_{M\in\mathcal{M}} \overline{M}$.

3.Further properties of fuzzy complete a-fuzzy normed algebra

Definition 3.1:

An ideal $\mathcal{J}$ in an algebra (Z, +, .) is **maximal** if $\mathcal{J} \subset$Z (that is $\mathcal{J} \neq$Z), and if there is an ideal $\mathcal{T}$ with $\mathcal{J} \subset \mathcal{T}$ then $\mathcal{T}$ =Z.

Proposition 3.2:

Every maximal ideal $\mathcal{J}$ in Z where (Z, n_Z ,$\circledast$, $\odot$) is fuzzy complete a-fuzzy normed algebra with an identity e, is fuzzy closed.

Proof:

If $\mathcal{J}$ be a maximal ideal in Z, then $\mathcal{J}$ must does not contains any invertible element, otherwise $\mathcal{J}$=Z. This implies that $\mathcal{J} \subseteq$Z$-$ $\mathcal{G}$(Z). But $\mathcal{G}$(Z) is fuzzy open so Z$-$ $\mathcal{G}$(Z) is fuzzy closed, hence $\mathcal{J} \subseteq \bar{\mathcal{J}} \subseteq$ Z$-$ $\mathcal{G}$(Z). As special case, $\mathcal{J} \neq$ Z. Since $\mathcal{J} \subseteq \bar{\mathcal{J}}$ so $\bar{\mathcal{J}}$=$\mathcal{J}$ but $\mathcal{J}$ is maximal ideal. Hence $\mathcal{J}$ is a fuzzy closed.

Proposition 3.3:

If (Z, n_Z ,$\circledast$, $\odot$) is a fuzzy complete a-FNA, then every homomorphism θ:Z$\rightarrow$ $\mathbb{C}$, is fuzzy continuous

Proof:

The case when θ=0 then it is fuzzy continuous. Let $\theta \neq$0 and Z has an identity e. Now $\forall$ u$\in$Z, θ(u)= θ(u.e) = θ(u). $\theta(e)$, and so $\theta(e)$=1. If u$\in$Z with θ(u)$\neq$0, then b=u$-$ θ(u).e $\in$ ker θ and so b is not invertible [or 1= θ(bb^{-1})=θ(b). $\theta(b^{-1})$ which is not correct]. Therefore, θ(u) $\in$ σ_Z(u) and this implies that $L_{\mathbb{C}}[\theta(\text{u})] \leq n_Z$(u). This inequality stall true when θ(u)=0 and hence φ is fuzzy continuous on Z. [if (z_k) be a sequence in Z converge to z$\in$Z that is $\lim_{k\to\infty} n_Z\,(z_k - z)$=0, then $\lim_{k\to\infty} L_{\mathbb{C}}\,[\theta(z_k) - \theta(z)]$ = 0 that is $\theta(z_k) \rightarrow \theta(z)$].

If Z does not have an identity, we consider Z_e instead. Define θ': $Z_e \rightarrow \mathbb{C}$ by: θ'[(u, α)] = θ(u) + α for all (u, α) $\in Z_e$. Then φ' is a homomorphism is clear and therefore by the first part of the prove, θ' is fuzzy continuous on Z_e. specially, its restriction to Z in Z_e is fuzzy continuous i.e., θ is a fuzzy continuous.

Definition 3.4:

A homomorphism $\mathscr{k}$:Z$\rightarrow$ $\mathbb{C}$ where (Z, n_Z ,$\circledast$, $\odot$) is a fuzzy complete a-FNA is called a **character**. Character is fuzzy continuous by Proposition 3.3.

Theorem 3.5:

If $\hbar$ is a character on Z, then ker $\hbar$ is a maximal ideal in Z, and every maximal ideal has this form for some unique character, when (Z, n_Z ,$\circledast$, $\odot$) is a commutative fuzzy complete a-FNA with identity e.

Proof:

If $\hbar$:Z$\rightarrow$ $\mathbb{C}$ is a character and $\mathcal{J}$=ker ℓ, it is clear that $\mathcal{J} \neq$Z because $\mathscr{k}$ =0. If z $\notin \mathcal{J}$ then for any u $\in$ Z is represented by u=z$\frac{\hbar(u)}{\mathscr{k}(z)}$ + [u$-$ z$\frac{\mathscr{k}(u)}{\mathscr{k}(z)}$] since [u$-$ z$\frac{\mathscr{k}(u)}{\mathscr{k}(z)}$] $\in$ ker $\mathscr{k}$=$\mathcal{J}$, we see that Z= $\mathbb{C}$z + $\mathcal{J}$ and therefore $\mathcal{J}$ is a maximal ideal. This implies that $\mathcal{J}$ is fuzzy closed and hence $\frac{Z}{J}$ is fuzzy

complete a-FNA. Now we will show that the maximality of $\mathcal{J}$ implies that every non-zero element of $\frac{Z}{J}$ is invertible. To prove this, let (z + $\mathcal{J}$)≠0 and $(z + \mathcal{J})^{-1}$ does not exists.
Thus $\mathcal{J} \subset (\mathcal{J}$ +zZ) ⊂ Z [e∉ ($\mathcal{J}$ + z) because $(z + \mathcal{J})^{-1} \notin \frac{Z}{J}$]. But this is not true since $\mathcal{J}$ is maximal by our assumption. This implies that every element of $\frac{Z}{J}$ is $\lambda(e$ + j) for $\lambda \in \mathbb{C}$.
If $\theta: \frac{Z}{J} \rightarrow \mathbb{C}$ represent this isomorphism, and if π:Z→ $\frac{Z}{J}$ is the canonical projection.
Then $\theta \circ \pi$: Z → ℂ is a homomorphism with ker $\theta=\mathcal{J}$;
$\theta \circ \pi$(p.q) = $\theta[\pi$(p.q)] = θ[(p.q) + $\mathcal{J}$] = θ[(p + $\mathcal{J}$)(q + $\mathcal{J}$)] = [θ(p + $\mathcal{J}$)].[θ(q + $\mathcal{J}$)]
= $\theta \circ \pi$(p). $\theta \circ \pi$(q).
Also, $\theta \circ \pi$(p) = 0 ⟺ π(p) = 0 ⟺ p∈ $\mathcal{J}$.

Hence there is a correspondence between maximal ideals $\mathcal{J}$ and the characters $\hbar$ with ker $\hbar=\mathcal{J}$.

This correspondence is one-to-one because $\hbar$ is uniquely determined by its Kernel. If $\hbar$ and ℓ are two character with ker$\hbar$= kerℓ then for any w∈Z, (w − $\hbar$(w)e) ∈ ker $\hbar$= kerℓ and thus $\hbar$(w)= ℓ(w) because $\ell(e)$=1.

Theorem 3.6:
Every commutative fuzzy complete a-FNA (Z, n_Z , ⊛, ⊙) with an identity e has at least one character.

Proof:
If u^{-1} exists for all u∈ Z then Z≅ ℂ and the isomorphism μ: Z→ ℂ is a character. On the other hand, if ∃ x∈Z such that u^{-1} does not exists then xZ ⊂ $\mathcal{J}$ where $\mathcal{J}$ is a maximal proper ideal, by Zorn's lemma the set $\mathcal{T}$={$\mathcal{J} \subset \mathcal{L}$: $\mathcal{L}$ is ideal } is partially ordered by ⊆, thus $\cup_{\mathcal{L}\in\mathcal{T}} \mathcal{L}$ is an ideal and $\mathcal{J} \subset \cup_{\mathcal{L}\in\mathcal{T}} \mathcal{L}$ Since e ∉ $\cup_{\mathcal{L}\in\mathcal{T}} \mathcal{L}$. By Zorn's Lemma states that there exists a maximal $\mathcal{K}$ with ⊂ $\mathcal{K}$. But $\mathcal{J}$=ker $\hbar$ where $\hbar$ is a character on Z.
If Z is not commutative, then we may does not find a character at all on the a-FNA..

Example 3.7
If Z=$M_k(\mathbb{C})$ where k>1, then assume that E_{ij} = (e_{ij}) ∈ $M_k(\mathbb{C})$ where e_{ij} = 0, except for the ij-position is equal to 1. Now let $\hbar$ is a character on Z, then for i≠j, E_{ij}^2=0 which imply that $\hbar(E_{ij})$=0. But E_{ii}= E_{ij}. E_{ji}, when i≠j, and this imply that $\hbar(E_{ii})$=0 for j=1, 2, …, k.
Hence, $\hbar$(I)= $\hbar(E_{11})$ + $\hbar(E_{22})$ + … + $\hbar(E_{kk})$=0. But this is not true.
Thus Z=$M_k(\mathbb{C})$ with k>1, does not has a characters.

Definition 3.8:
If (Z, n_Z , ⊛, ⊙) is a fuzzy complete a-FNA has an identity e. Then {$\hbar$: $\hbar$ is a characters of Z} is called **structure** of Z and is denoted by st(Z).

Definition 3.9:
(1)The ω^* −fuzzy topology on afb(Z, ℂ) is generated by N(q, A, ε)={g∈ afb(Z, ℂ): $L_{\mathbb{C}}$[g(a)-q(a)]≤ ε, for all a∈A}, where q ∈ afb(Z, ℂ), and A ⊂Z is finite.
(2)The a set E in afb(Z, ℂ) is fuzzy open in ω^* −fuzzy topology ⟺ ∀ ϑ ∈ E, ∃ N(ϑ, A, ε) ⊆E.

Proposition 3.10:
If (Z, n_Z , ⊛, ⨀) is a fuzzy complete a-FNA having an identity e then ω^* −fuzzy topology on afb(Z, ℂ) is a Hausdorff space.

Proof:
If $h_1, h_2 \in$ afb(Z, ℂ) with $h_1 \neq h_2$ then there exist a∈Z such that h_1(a)≠ h_2(a). Let $L_{\mathbb{C}}[h_1(a) - h_2(a)]$ = r, for some $0 < r < 1$, then ∀ $r < r_o < 1$, ∃ r_1 satisfying $r_1 \circledast r_1 < r_o$.
Now consider N[h_1 , {a}, r_1) and N(h_2, {a}, r_1). Clearly, N[h_1 , {a}, r_1) ∩ N(h_2, {a}, r_1)= ∅ if there exists y∈ N[h_1 , {a}, r_1) ∩ N(h_2, {a}, r_1) then
r=$L_{\mathbb{C}}[h_1(a) - h_2(a)]$
$\leq L_{\mathbb{C}}[h_1(a) - y(a)] \circledast L_{\mathbb{C}}[h_2(a) - y(a)]$
$\leq r_1 \circledast r_1$ <r.
But this is not true. Hence the proof is complete.

Proposition 3.11:
If (Z, n_Z , ⊛, ⨀) is a fuzzy complete a-FNA having identity e then st(Z) is ω^*-fuzzy closed subset of afb(Z, ℂ).

Proof:
If (h_k) is a sequence in st(Z) converging to h∈ fb(Z, ℂ), then h_k(z)→h(z) for each z∈ Z.
Now for any x, y∈Z we have
h(xy)=$\lim_{k\to\infty} h_k(xy)= \lim_{k\to\infty} h_k(x). \lim_{k\to\infty} h_k(y)$= h(x).h(y)
It follows that h∈ st(Z). Hence st(Z) is a fuzzy closed.
In the next result we prove fuzzy Banach-Alaouglu's Theorem:

Theorem 3.12
If (Z, n_Z , ⊛) is a fuzzy complete a-FNS, then the fuzzy closed unit ball $B_{afb(Z,\mathbb{C})}$={h∈afb(Z, ℂ): $n_{\text{afb}(Z,\mathbb{C})}$(h)≤1} of afb(Z, ℂ) is a ω^* −fuzzy compact.

Proof:
Let z∈Z, define D_z={$\alpha \in \mathbb{C}$: $L_{\mathbb{C}}(\alpha) \leq n_Z$(z)}⊂ ℂ. Then D_z is a fuzzy compact which implies D=$\Pi_{z\in Z} D_z$ is a fuzzy compact in the product fuzzy topology by fuzzy Tychonoff Theorem 2.44.
Let B^d denotes the fuzzy closed unit ball $B_{afb(Z,\mathbb{C})}$.
Define θ: B^d →D by $\theta(\eta)=(\eta(u))_{u\in Z}$ ∀ $\eta \in B^d$.We will prove that θ is one-to-one and fuzzy continuous. It is clear that θ is Linear. If $\theta(\eta)$=0 then $(\eta(u))_{u\in Z}$= (0) which implies that η(u)=0 ∀ u∈Z. Hence, η=0 and from this we obtain η is one-to-one.
To prove θ is a fuzzy continuous, now if $(\eta_k) \subset B^d$ satisfying $\eta_k \to^{\omega^*} \eta$. Then η_k(u)→ η(u) ∀ u∈Z. Consequently, $\theta(\eta_k)$= $(\eta_k(u))_{u\in Z} \to (\eta(u))_{u\in Z}$= $\theta(\eta)$. Hence, θ is a fuzzy continuous.
If $\theta(B^d)$ is a fuzzy closed subset of D and D being fuzzy compact, then $\theta(B^d)$ is fuzzy compact. Thus our next step is to prove $\theta(B^d)$ is a fuzzy closed. If σ = (σ_z)∈D and $\sigma \in \overline{\theta(B^d)}$ then define η:Z→ ℂ by η(u)= σ_u ∀ u∈Z. The map η is linear, if x, y∈Z and $\alpha, \beta \in \mathbb{C}$ then ∀ k∈ ℕ, choose $\eta_k \in B^d$ thus
$\eta(\alpha x+\beta y) = \lim_{k\to\infty} \eta_k(\alpha x+\beta y) = \alpha \lim_{k\to\infty} \eta_k(x) + \beta \lim_{k\to\infty} \eta_k(y)$
$= \alpha f(x)+\beta f(y)$ [since η_k is linear].
Thus η is linear. Since $L_{\mathbb{C}}(\sigma_u) \leq n_Z$(u), so $\eta \in B^d$. Now by the definition of η, we see that σ= $\theta(\eta) \in B^d$. Hence, $\theta(B^d)$ is a fuzzy closed. But B^d and $\theta(B^d)$ are homeomorphic, so B^d must be fuzzy compact.

Proposition 3.13:
If (Z, n_Z , ⊛, ⊙) is a fuzzy complete a-FNA having an identity e then st(Z) is a ω^* −fuzzy closed of $B_{afb(Z,\mathbb{C})}$={h∈afb(Z, ℂ): $n_{\text{afb(Z,}\mathbb{C})}$(h)≤1} and hence is a fuzzy compact.
Proof:
Suppose that (ℓ_k) be a sequence in st(Z) which is fuzzy converge to θ ∈afb(Z, ℂ). Then ℓ_k(z)→ θ(z) ∀ z∈Z. Since ∀ x, y ∈Z, θ(xy)=$\lim_{k\to\infty} \ell_k(xy)$= $\lim_{k\to\infty} \ell_k(x)$. $\lim_{k\to\infty} \ell_k(y)$= θ(x). θ(y) so, we conclude that θ ∈st(Z). Here θ ≠0 since θ(I)=1. Thus st(Z) is a ω^* −fuzzy closed. Therefore, st(Z) is a fuzzy compact because it is a fuzzy closed subset of a fuzzy compact set.

Theorem 3.14:
If (Z, n_Z , ⊛, ⊙) is a commutative fuzzy complete a-FNA having an identity e, then for each z∈Z and h∈ st(Z) we define Ψ_z:st(Z)→ ℂ by Ψ_z(h)=h(z). Then the range of the function Ψ_z on st(Z) satisfies R(Ψ_z)=σ_Z(z). Furthermore, the map Ψ is homomorphisum, Ψ:Z→C(st(Z)) and $n_{afb(st(Z),\mathbb{C})}$ (Ψ_z) ≤ n_Z(z) for all z∈Z. The map Ψ is called Gelfand transform.

Proof:
If u∈Z and h∈st(Z) then h(x)∈ σ_Z(x) that is Ψ_u(h) ∈ σ_Z(u) and so the range of Ψ_u satisfies the inclusion R(Ψ_u) ⊆ σ_Z(u).
Let $\alpha \in \sigma_Z$(u), so (u − αe) $^{-1}$ does not exists and (u−αe) ∉ $\mathcal{J}$ where $\mathcal{J}$ is some maximal ideal, say [Since (u−αe) ∈ Z(x−αe) ⊂ $\mathcal{J}$].
If $\hbar$ ∈ st(Z) with ker$\hbar$=$\mathcal{J}$ then (x−αe) ∈ $\mathcal{J}$ which implies that $\hbar$(u)= α. Thus Ψ_u($\hbar$)= $\hbar$(u)= α and so R(Ψ_u)= σ_Z(u). But Ψ is a homomorphism;
Ψ_{zy}(h) = h(zy) = h(z).h(y) = Ψ_z(h).Ψ_y(h) ∀ z, y ∈Z, h∈st(Z) and so Ψ_{zy}= Ψ_z.Ψ_y.
Similarly, we can show that $\Psi_{\alpha z+\beta y}$= $\alpha\Psi_z$ + $\beta\Psi_y$(h), thus Ψ_z is linear.
To prove that Ψ_u ∈ C(st(Z)), if Ω is a fuzzy open set in ℂ we will prove that $\Psi_u{}^{-1}$(Ω) is a fuzzy open in st(Z). When $\Psi_x{}^{-1}$(Ω)=∅, the proof is end. If $\Psi_u{}^{-1}$(Ω)≠ ∅. Assume that g ∈ $\Psi_u{}^{-1}$(Ω). So ∃ δ ∈ Ω such that Ψ_u(g)= δ. Since Ω is a fuzzy open in ℂ, ∃ 0< ε <1 with $N_\varepsilon(\delta)$={$\alpha \in$ ℂ: $L_{\mathbb{C}}(\alpha-\delta) < \varepsilon$}⊂ Ω. If V=N($g$:{u}, ε)={ω ∈st(Z): $L_{\mathbb{C}}$(ω(u)− g(u))< ε}.
Then ω(u)= $\Psi_u(\omega)$ ∈ Ω, ∀ ω ∈V, so g ∈V⊆ $\Psi_u{}^{-1}$(Ω). Hence, $\Psi_u{}^{-1}$(Ω) is a fuzzy open in st(Z) and therefore Ψ_u:st(Z)→ ℂ is a fuzzy continuous thus Ψ_u(.)∈ C(st(Z)).

On other hand, we can introduce another prove for the fuzzy continuity of Ψ_x by using sequences. If $g_k \to g$ in st(Z) then $\Psi_u(g_k) \to$ Ψ_u(g)= g(u), hence Ψ_x is a fuzzy continuous.
Now, R(Ψ_u)=σ_Z(u) ⊆ { α ∈ ℂ: $L_{\mathbb{C}}(\alpha)$≤ n_Z(u)} and thus $L_{\mathbb{C}}$(Ψ_u(g))≤ n_Z(u), ∀ g ∈st(Z). Hence, $n_{afb(st(Z),\mathbb{C})}$($\Psi_u$) ≤ n_Z(u), ∀ u∈Z.

Theorem 2.3.15:
If (Z, n_Z , ⊛, ⊙) is a commutative fuzzy complete a-FNA having an identity e and Z=uZ, that is, the set of polynomials in z is fuzzy dense in Z. Then the map Ψ_u:st(Z)→ σ_Z(u) ⊂ ℂ is a homeomorphism.

Proof:
Since Ψ_u is fuzzy continuous function on st(Z) satisfying R(Ψ_u) = σ_Z(u) i.e., Ψ_u:st(Z)→ σ_Z(u) is a fuzzy continuous and onto. But st(Z) and σ_Z(u) are fuzzy compact Hausdorff spaces, thus it remains only to prove that Ψ_u is injective. Now if $\Psi_u(\ell_1)$ = $\Psi_z(\ell_2)$, so that ℓ_1(u)= ℓ_2(u), by using the multiplicatively of ℓ_1 and ℓ_2 we see that for given k∈ ℕ and c_0, c_1, …, c_k in ℂ, $\ell_1(\sum_{j=0}^{k} c_j u^k)$ = $\ell_2(\sum_{j=0}^{k} c_j u^k)$. Since ℓ_1 and ℓ_2 are fuzzy continuous and u generates Z, it follows that ℓ_1 = ℓ_2.

Example 2.3.16:

Let Z= $\{\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}$: α, $\beta \in \mathbb{C}\}$ be a subalgebra of $M_2(\mathbb{C})$. Then $\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}$=$\alpha$I + βq, where q=$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$. We note that q^2=0.

Evidently, Z is two-dimensional commutative fuzzy complete a-fuzzy normed algebra with identity I. We shall compute the spectrum σ_Z(x) for x= $\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}$. Indeed, for $\lambda \in \mathbb{C}$, x$-\lambda$I=$\begin{pmatrix}\alpha-\lambda & \beta\\ 0 & \alpha-\lambda\end{pmatrix}$ is invertible in $M_2(\mathbb{C}) \Leftrightarrow$ if $\alpha \neq \lambda$. If $\alpha \neq \lambda$, then, in fact (x$-\lambda$I$)^{-1}$ =$\begin{pmatrix}(\alpha-\lambda)^{-1} & -\beta(\alpha-\lambda)^{-1}\\ 0 & (\alpha-\lambda)^{-1}\end{pmatrix}$, which belongs to Z. Hence, σ_Z(x)= $\sigma_Z(\alpha$I + βq)=$\{\alpha\}$.In particularσ_Z(q)= $\sigma_Z(\beta$q) =$\{0\}$, but q≠0. If $\hbar$ is a character of Z then $\hbar$(uv)= $\hbar$(u) $\hbar$(v) implies $\hbar(q^2)$= $\hbar$(q) $\hbar$(q). But q^2=0 and so $\hbar$(q)=0. Since $\hbar$(I)=1, we find that $\hbar(\alpha$I + βq) = α for any α, $\beta \in \mathbb{C}$. Thus, there is just one character on Z so st(Z) = { $\hbar$}, where $\hbar$ is given uniquely by the action $\hbar$(I)=1 and $\hbar$(q)=0.
The fuzzy Gelfand transform is the map z $\longmapsto \Psi_z$, (αI + βq) $\longmapsto \alpha\Psi_I+\beta\ \Psi_q$. But Ψ_I=1 and $\Psi_q(\hbar)$= $\hbar$(q)=0 so that Ψ_q=0 and we have $\Psi_{(\alpha I+\beta q)}$= α for any α, $\beta \in\ \mathbb{C}$.
The transform Ψ_q has kernel { βq: $\beta \in \mathbb{C}$}, so Ψ_q is not an isomorphism. The algebra Z has exactly one maximal ideal, namely, the Kernel of $\hbar$. As Z is an algebra with identity generated by q and so st(Z)≅ σ_Z(q), throw Ψ_z:st(Z) → σ_Z(q), $\hbar \longmapsto \Psi_q(\hbar)$=0.
Thus, the two sets st(Z) and σ_Z(q) are singleton sets.

On the other hand, we can calculate the spectrum of x∈Z using R(Ψ_x)= σ_Z(x). For x= $\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}$ we have, σ_Z(x)={ $\Psi_x(\hbar)$} = { $\hbar$(x)}={ $\hbar(\alpha$I + βq)}={α $\hbar$(I) + β $\hbar$(q)}={α}
Since $\hbar$(q)=0.

4. Conclusions

In [6] we proved some properties of fuzzy complete a-fuzzy normed algebra. In this paper we recall the definition of a-fuzzy normed algebra in order to prove other properties of fuzzy complete a-fuzzy normed algebra.

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