Further properties of the fuzzy complete a-fuzzy normed algebra

Rasha Khudhur Abbas, Jehad R. Kider*

Branch of Mathematics, Department of Applied Sciences, University of Technology-Iraq

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Abstract

In this paper further properties of the fuzzy complete a-fuzzy normed algebra have been introduced. Then we found the relation between the maximal ideals of fuzzy complete a-fuzzy normed algebra and the associated multiplicative linear function space. In this direction we proved that if $\ell$ is character on $Z$ then $\ker\ell$ is a maximal ideal in $Z$. After this we introduce the notion structure of the a-fuzzy normed algebra then we prove that the structure, $st(Z)$ is $\omega^*$-fuzzy closed subset of $fb(Z, \mathbb{C})$ when $(Z, n_Z, \circ, \bullet)$ is a commutative fuzzy complete a-fuzzy normed algebra with identity $e$.

Keywords: Character of a-fuzzy normed algebra, Structure of a-fuzzy normed algebra, $\omega^*$-fuzzy topology on $afb(Z, \mathbb{C})$, Fuzzy Tychonoff Theorem, fuzzy Gelfand transform.

1. Introduction

In this paper we continue the previous study of fuzzy complete a-fuzzy normed algebra. We prove here the other important properties of fuzzy complete a-fuzzy normed algebra. The organization of this paper is as follows: we divided this research into four sections; the introduction will be in section one, after that in section two important properties of fuzzy length space and a-fuzzy normed space are recalled. Furthermore, basic important properties of a-fuzzy normed algebra that will be needed later can be found in section three. Moreover, further properties of fuzzy complete a-fuzzy normed algebra have been proved as a main results in the same section. Finally, in section four we highlight the conclusion for this research.

*Email: jehadkider@gmail.com
2. Preliminaries about a-fuzzy normed algebra

**Definition 2.1** [1]:
Let \( \odot : I \times I \to I \) be a binary operation function, then it is said to be continuous t-conorm (or simply t-conorm) if it satisfies the following conditions:
(i) \( p \odot q = q \odot p \);
(ii) \( p \odot [q \odot w] = [p \odot q] \odot w \);
(iii) \( \odot \) is continuous function;
(iv) \( p \odot z \geq (q \odot w) \) whenever \( p \geq q \) and \( z \geq w \).
For all \( p, q, z, w \in I = [0, 1] \).

**Definition 2.2:**
If \( L_C : \mathbb{C} \to I \) is a fuzzy set, and \( \odot \) is a t-conorm, then \( L_C \) is **a-fuzzy length on** \( \mathbb{C} \) if
(i) \( 0 < L_C(\sigma) \leq 1 \);
(ii) \( L_C(\sigma) = 0 \) if and only if \( \sigma = 0 \);
(iii) \( L_C(\sigma \tau) \leq L_C(\sigma) L_C(\tau) \);
(iv) \( L_C(\sigma + \tau) \leq L_C(\sigma) \odot L_C(\tau) \).
For all \( \sigma, \tau \in \mathbb{C} \). The \( (\mathbb{C}, L_C, \odot) \) is **a-fuzzy length space**.

**Remark 2.3:**
We will take \( \odot \) to be \( \mu \odot \sigma = \mu + \sigma - \mu \sigma \), for all \( \mu, \sigma \in I = [0, 1] \).

**Example 2.4:**
If \( L_{|.|} : \mathbb{C} \to I \) for all \( \alpha \in \mathbb{C} \), where \( |.| \) is length value on \( \mathbb{C} \) Then \( (\mathbb{C}, L_{|.|}, \odot) \) is a-fuzzy length space.

**Definition 2.5** [3]:
If \( (\mathbb{C}, L_C, \odot) \) is a-fuzzy length space, \( Z \) is a vector space over \( \mathbb{C} \), and \( \odot \) is a t-conorm and \( n_Z : Z \to I \) is a fuzzy set. Then \( n_Z \) is **a-fuzzy norm on** \( Z \) if
(i) \( 0 < n_Z(z) \leq 1 \);
(ii) \( n_Z(z) = 0 \iff z = 0 \);
(iii) \( n_Z(\mu z) \leq L_C(\mu) n(z) \) for all \( \mu \neq 0 \in \mathbb{C} \);
(iv) \( n_Z(z + y) \leq n_Z(z) \odot n_Z(y) \).
For all \( z, y \in Z \). Then \( (Z, n_Z, \odot) \) is **a-fuzzy normed space** (or simply a-FNS).

**Definition 2.7** [3]:
If \( (z_k) \) is a sequence in \( Z \), then \( (z_k) \) is fuzzy converges to the limit \( z \) as \( k \to \infty \), if for all \( \mu \in (0, 1) \), we can find \( N \in \mathbb{N} \) when \( n_Z(z_k - z) < \mu \), for all \( k \geq N \), if \( (Z, n_Z, \odot) \) is a-FNS.
If \( (z_k) \) is a fuzzy converges to \( z \) we write \( \lim_{k \to \infty} z_k = z \), or \( z_k \to z \), or \( \lim_{k \to \infty} n_Z(z_k - z) = 0 \).

**Definition 2.8** [3]:
If \( (z_k) \) is a sequence in \( Z \), then \( (z_k) \) is fuzzy Cauchy sequence in \( Z \) if for all \( \mu \in (0, 1) \), we can find \( N \in \mathbb{N} \) when \( n_Z(z_k - z_m) < s \), for all \( k, m \geq N \), if \( (Z, n_Z, \odot) \) is a-FNS.
Definition 2.9 [3]:
If for any \((z_k)\) fuzzy Cauchy in \(Z\), there is \(z \in Z\) such that \(z_k \to z\), then the a-FNS \((Z, n_Z, \ominus)\) is a fuzzy complete.

Corollary 2.10 [3]:
The a-fuzzy length space \((\mathbb{C}, L_C, \ominus)\) is a fuzzy complete.

Theorem 2.11 [4]:
When \((Z, n_Z, \ominus)\) and \((W, n_W, \ominus)\) are two a-FNS. Then the operator \(H : Z \to W\) is a fuzzy continuous at \(z \in Z\) if and only if whenever \((z_k)\) is a fuzzy converges to \(z \in Z\) then \((H(z_k))\) is a fuzzy converges to \(H(z) \in W\).

Definition 2.12 [4]:
When \((Z, n_Z, \ominus)\) and \((Y, n_Y, \ominus)\) are two a-FNS then the operator \(S : Z \to Y\) is a fuzzy bounded if \(\mu \in (0, 1)\) with \(n_Y[S(z)] < \mu n_Z(z)\), for all \(z \in Z\).

Notation [4]:
If \((Z, n_Z, \ominus)\) and \((Y, n_Y, \ominus)\) are two a-FNS then \(\text{afb}(Z, Y) = \{S : Z \to Y, S\) is a fuzzy bounded operator\}.

Theorem 2.13 [4]:
Define \(n_{\text{afb}(Z,Y)}(S) = \sup_{z \in D(S)} n_Y(Sz)\), for all \(S \in \text{afb}(Z, Y)\). Then \((\text{afb}(Z, Y), n_{\text{afb}(Z,Y)}, \ominus)\) is a-FNS. If \((Z, n_Z, \ominus)\) and \((Y, n_Y, \ominus)\) are two a-FNS.

Theorem 2.14 [4]:
The space \(\text{afb}(Z, Y)\) is a fuzzy complete if \(Y\) is a fuzzy complete when \((Z, n_Z, \ominus)\) and \((Y, n_Y, \ominus)\) are two a-FNS.

Definition 2.15 [4]:
A linear functional \(h\) from a-FNS \((Z, n_Z, \ominus)\) into the a-fuzzy length space \((\mathbb{C}, L_C, \ominus)\) is fuzzy bounded if there is \(\delta \in (0, 1)\) with \(L_C[h(u)] < \delta \cdot n_Z(u)\) for any \(u \in D(h)\). Furthermore, the a-fuzzy norm of \(h\) is \(n_{\text{afb}(Z,\mathbb{C})}(h) = \sup_{u \in D(h)} L_C(hu)\), for all \(h \in \text{afb}(Z, \mathbb{R})\) and \(\sup_{u \in D(h)} L_C(hu) < n_{\text{afb}(Z,\mathbb{R})}(h) \cdot n_C(u)\) for any \(u \in D(h)\).

Definition 2.16 [4]:
Let \((Z, n_Z, \ominus)\) be a-FNS. Then \(\text{afb}(Z, \mathbb{C}) = \{h : Z \to \mathbb{C}, h\) is fuzzy bounded and linear\}\) forms a-fuzzy normed space with a-fuzzy norm defined by \(n_{\text{afb}(Z,\mathbb{C})}(h) = \sup_{u \in D(h)} L_C(hu)\) which is called the fuzzy dual space of \(Z\).

Theorem 2.17 [4]:
If \((Z, n_Z, \ominus)\) is a-FNS then fuzzy dual space \(\text{afb}(Z, \mathbb{C})\) is a fuzzy complete.

Definition 2.18 [4]:
\(Z_D = \{z + D : z \in Z\}\) is a \(\mathbb{K}\)-space with the operations \((v + D) + (z + D) = (v + z) + D\) and \(\alpha(z + D) = (\alpha z) + D\). If \(Z\) is a vector space over the field \(\mathbb{K}\) and \(D\) is a closed subspace of \(Z\).

Definition 2.19 [5]:
Define a-fuzzy norm for the quotient space \(Z_D\) by \(q[u + D] = \inf_{d \in D} n_U[z + d]\) for all \(z + D \in Z_D\).
When \((Z, n_Z, \odot)\) be a-FNS and \(D \subseteq Z\) is a fuzzy closed in \(Z\).

**Theorem 2.20 [5]:**

The quotient space \(\left(\frac{Z}{D}, q, \odot\right)\) is a-FNS if \((Z, n_Z, \odot)\) is a-FNS and \(D \subseteq Z\) is a fuzzy closed in \(Z\).

**Remark 2.21 [5]:**

If \((Z, n_Z, \odot)\) is a-FNS and \(D \subseteq Z\) is a fuzzy closed in \(Z\). Then

(1) \(\pi: Z \rightarrow \frac{Z}{D}\) is a natural operator defined by \(\pi[z] = z + D\);

(2) \(q(z + D) \leq n_Z(z)\).

**Theorem 2.22 [5]:**

If \((\frac{Z}{D}, q, \odot)\) is a fuzzy complete then \((Z, n_Z, \odot)\) is a fuzzy complete when \((Z, n_Z, \odot)\) be a-FNS and \(D \subseteq Z\) is a fuzzy closed in \(Z\).

**Theorem 2.23 [5]:**

If \((Z, n_Z, \odot)\) is a fuzzy complete then \((\frac{Z}{D}, q, \odot)\) is a fuzzy complete when \((Z, n_Z, \odot)\) is a-FNS and \(D \subseteq Z\) is a fuzzy closed in \(Z\).

**Definition 2.24 [1]:**

Let \(\odot: I \times I \rightarrow I\) be a binary operation function then \(\odot\) is said to be continuous t-norm (or simply t-norm) if it satisfies the following conditions:

(i) \(p \odot q = q \odot p\);

(ii) \(p \odot [q \odot w] = [p \odot q] \odot w\);

(iii) \(\odot\) is continuous function;

(iv) \(p \odot 1 = p\);

(v) \(p \odot z \leq (q \odot w)\) whenever \(p \leq q\) and \(z \geq w\).

For all \(p, q, z, w \in I = [0, 1]\).

**Definition 2.25 [6]:**

The space \((Z, n_Z, \odot, \otimes)\) is called a-fuzzy normed algebra space (or simply a-FNAS) if

(1) \((Z, +, .)\) is an algebra space over the field \(\mathbb{K}\), where \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{K} = \mathbb{C}\);

(2) \((Z, n_Z, \odot)\) is a-FNS, with \(\odot\) is a t-conorm;

(3) \(\otimes\) is a t-norm;

(4) \(n_Z(p.q) \leq n_Z(p) \otimes n_Z(q)\), for all \(p, q \in Z\).

**Remark 2.26:**

In this paper we take:

(1) \(\sigma \odot \tau = \sigma \cdot \tau\), for all \(\sigma, \tau \in [0, 1]\).

(2) \(\gamma \odot \delta = \gamma + \delta - \gamma \delta\), for all \(\gamma, \delta \in [0, 1]\).

**Definition 2.29 [6]:**

The space \((Z, n_Z, \odot, \otimes)\) is a fuzzy complete a-FNA if \((Z, n_Z, \odot)\) is a fuzzy complete a-FNS. Then \((Z, n_Z, \odot, \otimes)\) is a commutative fuzzy complete a-FNA.

**Lemma 2.31 [6]:**

If \((Z, n_Z, \odot, \otimes)\) is a-FNA, then multiplication is a fuzzy continuous function.
Theorem 2.32 [6]:
An a-FNA \((Z, n_z, \odot, \oslash)\) without identity can be embedded into a-FNA \(Z_e\) having the identity \(e\), also \(Z\) is considered as an ideal in \(Z_e\).

Proposition 2.33 [6]:
The space \((Z_e, n_{Z_e}, \odot, \oslash)\) is a fuzzy complete \(\iff (Z, n_Z, \odot, \oslash)\) is a fuzzy complete.

Theorem 2.34 [6]:
Every a-FNA can be embedded as a closed subalgebra of \(afb(Z, Z)\).

Proposition 2.35 [6]:
If \((Z, n_Z, \odot, \oslash)\) is a fuzzy complete a-FNA and \(z \in Z\), then \((e-z)\) is invertible, the series \(\sum_{k=0}^\infty z^k\) is fuzzy converges, and \(\sum_{k=0}^\infty z^k = (e - z)^{-1}\).

Theorem 2.36 [6]:
The space \((\frac{Z}{D}, q, \odot, \oslash)\) is a fuzzy complete a-FNA if \((Z, n_Z, \odot, \oslash)\) is a fuzzy complete and \(D\) is a fuzzy closed ideal in \(Z\). Also \(\frac{Z}{D}\) has an identity if \(Z\) has an identity. As well as the identity of \(\frac{Z}{D}\) has a fuzzy norm equal to 1.

Remark 2.37 [6]:
If \((Z, n_Z, \odot, \oslash)\) is a fuzzy complete, then for any \(a \neq 0\), \(a^{-1}\) exists and \(a^{-1} \in Z\).

Proposition 2.38 [6]:
If \((Z, n_Z, \odot, \oslash)\) is a fuzzy complete a-FNA, then \(T(z) = z - 1\) is fuzzy continuous mapping.

Lemma 2.39 [6]:
Let \((Z, n_Z, \odot, \oslash)\) be a fuzzy complete having an identity \(e\). If \(z^{-1}\) and \(u^{-1}\) exist in \(Z\) then \((zu)^{-1}\) and \((uz)^{-1}\) are exist in \(Z\).

Proposition 2.40 [6]:
Let \((Z, n_Z, \odot, \oslash)\) be a fuzzy complete a-FNA having an identity \(e\). If \(z, u \in Z\) where \((e - zu)^{-1}\) exists. If \(d = (e - zu)^{-1}\) then \((e - uz)^{-1} = e + udz\).

Definition 2.41 [6]:
Let \(\mathcal{A} = \{A_j : j \in J\}\) be a family of subsets of a space \(Z\). The family \(\mathcal{A}\) is centered if for any finite number of sets \(A_1, A_2, \ldots, A_k \in \mathcal{A}\) we have \(\bigcap_{j=1}^k A_j \neq \emptyset\).

Definition 2.42 [6]:
Let \(Z\) be a non-empty set. A collection \(T\) of a subset of \(Z\) is said to be a fuzzy topology on \(Z\) if
(i) \(Z \in T\) and \(\emptyset \in T\);
(ii) If \(A_1, A_2, \ldots, A_n \in T\) then \(\bigcap_{i=1}^n A_i \in T\);
(iii) If \(\{A_j : j \in J\} \in T\) then \(\bigcup_{j \in J} A_j \in T\).
Then \((Z, T)\) is called a fuzzy topological space.

Theorem 2.43:
If \(Z\) is a fuzzy topological space then the following statement are equivalent:
(1) \(Z\) is a fuzzy compact;
(2) For any centred family \( \mathcal{A} \) of a fuzzy closed subset of \( Z \) we have \( \bigcap_{A \in \mathcal{A}} A \neq \emptyset \).

**Proof:** (2)\(\Rightarrow\)(1)

Let \( \mathcal{A} = \{A_j : j \in J\} \) be a fuzzy open cover of \( Z \). We need to show that \( \mathcal{A} \) has a finite subcover. For \( j \in J \), define \( G_j = \overline{A_j} \) this gives a family \( \mathcal{G} = \{G_j : j \in J\} \) of fuzzy closed sets in \( Z \). We have \( \bigcap_{j \in J} G_j = \bigcap_{j \in J} \overline{A_j} \) is not a centered family, and there exists \( G_1, G_2, \ldots, G_k \in \mathcal{G} \) such that \( \bigcap_{j=1}^k G_j \neq \emptyset \). This implies that \( \mathcal{G} \) is not a centered family, so there exists \( G_1, G_2, \ldots, G_k \in \mathcal{G} \) such that \( \bigcap_{j=1}^k G_j = \emptyset \). This gives, \( \emptyset = \bigcap_{j=1}^k G_j = \bigcap_{j \in J} \overline{A_j} \), so for all \( j \), \( \overline{A_j} \neq \emptyset \). Therefore, \( Z = \bigcup_{j=1}^k A_j \) and so \( Z \) is a fuzzy compact since \( \{A_j : j=1, 2, \ldots, k\} \) is a finite subcover of \( \mathcal{A} \).

(1)\(\Rightarrow\)(2) Follows from a similar argument.

**Fuzzy Tychonoff Theorem 2.44:**

If \( \{Z_j : j \in J\} \) is a family of fuzzy topological spaces and \( Z_j \) is a fuzzy compact \( \forall j \in J \), then the product space \( \prod_{j \in J} Z_j \) is a fuzzy compact.

**Proof:**

Let \( Z = \prod_{j \in J} Z_j \) where \( Z_j \) is a fuzzy compact \( \forall j \in J \). Let \( \mathcal{A} \) be a centred family of fuzzy closed subset of \( Z \). We will show that there exists \( z = (z_j)_{j \in J} \) such that \( z \in \bigcap_{A \in \mathcal{A}} A \). Let \( D \) denote the set consisting of all centred families \( \mathcal{F} \) of \( \{\overline{A_j} : j \in J\} \) and \( \mathcal{F} \) is an upper bound of \( \{\mathcal{F}_j : j \in J\} \). Now by Zorn’s Lemma we obtain that the set \( D \) contains a maximal element \( \mathcal{M} \).

We will show that every chain in \( D \) has an upper bound. Indeed, if \( \{\mathcal{F}_j : j \in J\} \) is a chain in \( D \) then take \( \mathcal{F} = \bigcup_{j \in J} \mathcal{F}_j \). Since \( \mathcal{F} \) is a centred family and \( \mathcal{F} \subseteq \mathcal{F} \) for all \( j \) thus \( \mathcal{F} \) is the upper bound of \( \{\mathcal{F}_j : j \in J\} \). Now by Zorn’s Lemma we obtain that the set \( D \) contains a maximal element \( \mathcal{M} \).

We will show that there exists \( z \in Z \) such that \( z \in \bigcap_{M \in \mathcal{M}} M \). Since \( \mathcal{A} \subseteq \mathcal{M} \) and \( \mathcal{A} \) contains of fuzzy closed sets we have, \( \bigcap_{M \in \mathcal{M}} M = \bigcap_{A \in \mathcal{A}} A \). Therefore, it will follow that \( z \in \bigcap_{A \in \mathcal{A}} A \) and \( \bigcap_{A \in \mathcal{A}} A \neq \emptyset \).

Construction of the element \( z \) proceed as follows. For \( j \in J \) let \( p_j : Z \to Z_j \) be the projection onto the \( j \)-th coordinate. Now for each \( j \in J \) the family \( \{p_j(M) : M \in \mathcal{M}\} \) is centred family of fuzzy closed subsets of \( Z_j \), so by the fuzzy compactness of \( Z_j \) there exists \( z_j \in Z_j \) such that \( z_j \in \bigcap_{M \in \mathcal{M}} M \). We set \( z = (z_j)_{j \in J} \).

In order to see that \( z \in \bigcap_{M \in \mathcal{M}} M \) notice that \( \mathcal{M} \) the following property:

If \( B \subseteq Z \) and \( B \cap M \neq \emptyset \) for all \( M \in \mathcal{M} \) then \( B \subseteq \mathcal{M} \) \((\ast)\)

Indeed if \( \mathcal{M}' = \mathcal{M} \cup \{B\} \) then \( \mathcal{M}' \in \mathcal{D} \), so by maximality of \( \mathcal{M} \) we must have \( \mathcal{M} \subseteq \mathcal{M}' \). For \( j \in J \) let \( U_j \subseteq Z_j \) be a fuzzy open neighborhood of \( z_j \). Since \( z_j \in \overline{p_j(M)} \) for all \( M \in \mathcal{M} \), thus \( U_j \cap p_j(M) \neq \emptyset \) for all \( M \in \mathcal{M} \).

Equivalently \( p_j^{-1}(U_j) \cap M \neq \emptyset \) for all \( M \in \mathcal{M} \). By property \( \ast \) we obtain that \( p^{-1}(U_j) \in \mathcal{M} \) for all \( j \in J \). Since \( \mathcal{M} \) is a centred family we obtain

\( p^{-1}(U_j) \cap p^{-1}(U_2) \cap \ldots \cap p^{-1}(U_k) \cap M \neq \emptyset \) for all \( M \in \mathcal{M} \) \((\ast\ast)\)

Now the sets of the form \( p^{-1}(U_1) \cap p^{-1}(U_2) \cap \ldots \cap p^{-1}(U_k) \) are precisely the fuzzy open neighbourhood of \( z \) that belong to the basis of the product fuzzy topology on \( Z \), and thus any fuzzy open neighbourhood of \( Z \) contains a neighbourhood of this type. Therefore, using \((\ast\ast)\) we obtain that if \( M \in \mathcal{M} \) then for any fuzzy open neighborhood \( U \) of \( z \) we have \( M \cap U \neq \emptyset \).
This means that for any $M \in \mathcal{M}$ we have $z \in \bar{M}$, and hence $z \in \bigcap_{M \in \mathcal{M}} \bar{M}$.

3. Further properties of fuzzy complete a-fuzzy normed algebra

Definition 3.1:
An ideal $J$ in an algebra $(Z, +, \cdot)$ is maximal if $J \subseteq Z$ (that is $J \neq Z$), and if there is an ideal $\mathcal{T}$ with $J \subseteq \mathcal{T}$ then $\mathcal{T} = Z$.

Proposition 3.2:
Every maximal ideal $J$ in $Z$ where $(Z, n_Z, \oplus, \ominus)$ is fuzzy complete a-fuzzy normed algebra with an identity $e$, is fuzzy closed.

Proof:
If $J$ be a maximal ideal in $Z$, then $J$ must does not contains any invertible element, otherwise $J = Z$. This implies that $J \subseteq Z \setminus \mathcal{G}(Z)$. But $\mathcal{G}(Z)$ is fuzzy open so $Z \setminus \mathcal{G}(Z)$ is fuzzy closed, hence $J \subseteq Z \setminus \mathcal{G}(Z)$. As special case, $J \neq Z$. Since $J \subseteq Z$ so $J = J$ but $J$ is maximal ideal. Hence $J$ is a fuzzy closed.

Proposition 3.3:
If $(Z, n_Z, \oplus, \ominus)$ is a fuzzy complete a-FNA, then every homomorphism $\theta : Z \rightarrow \mathbb{C}$, is fuzzy continuous.

Proof:
The case when $\theta = 0$ then it is fuzzy continuous. Let $\theta \neq 0$ and $Z$ has an identity $e$. Now $\forall u \in Z$, $\theta(u) = \theta(u, e) = \theta(u, \cdot, e)$, and so $\theta(e) = 1$. If $u \in Z$ with $\theta(u) \neq 0$, then $u = u - \theta(u, e) \in \ker \theta$ and so $b$ is not invertible [or $1 = \theta(bb^{-1}) = \theta(b, b^{-1})$ which is not correct]. Therefore, $\theta(u) \in \sigma_Z(u)$ and this implies that $L_C[\theta(u)] \leq n_Z(u)$. This inequality still true when $\theta(u) = 0$ and hence $\varphi$ is fuzzy continuous on $Z$. [if $(z_k)$ be a sequence in $Z$ converge to $z \in Z$, that is $\lim_{k \to \infty} n_Z(z_k - z) = 0$, then $\lim_{k \to \infty} L_C[\theta(z_k) - \theta(z)] = 0$ that is $\theta(z_k) \rightarrow \theta(z)$].

If $Z$ does not have an identity, we consider $Z_e$ instead. Define $\theta' : Z_e \rightarrow \mathbb{C}$ by: $\theta'[(u, \alpha)] = \theta(u) + \alpha$ for all $(u, \alpha) \in Z_e$. Then $\varphi'$ is a homomorphism is clear and therefore by the first part of the prove, $\varphi'$ is fuzzy continuous on $Z_e$. specially, its restriction to $Z$ in $Z_e$ is fuzzy continuous i.e., $\theta$ is a fuzzy continuous.

Definition 3.4:
A homomorphism $\kappa : Z \rightarrow \mathbb{C}$ where $(Z, n_Z, \oplus, \ominus)$ is a fuzzy complete a-FNA is called a character. Character is fuzzy continuous by Proposition 3.3.

Theorem 3.5:
If $\kappa$ is a character on $Z$, then $\ker \kappa$ is a maximal ideal in $Z$, and every maximal ideal has this form for some unique character, when $(Z, n_Z, \oplus, \ominus)$ is a commutative fuzzy complete a-FNA with identity $e$.

Proof:
If $\kappa : Z \rightarrow \mathbb{C}$ is a character and $J = \ker \ell$, it is clear that $J \neq Z$ because $\kappa = 0$. If $z \notin J$ then for any $u \in Z$ is represented by $u = z - \kappa(u) \frac{z}{\kappa(z)} [u - \kappa(u) \frac{z}{\kappa(z)}] \in \ker \kappa = J$, we see that $Z = Cz + J$ and therefore $J$ is a maximal ideal. This implies that $J$ is fuzzy closed and hence $\frac{Z}{J}$ is fuzzy.
complete a-FNA. Now we will show that the maximality of \( J \) implies that every non-zero element of \( \frac{Z}{J} \) is invertible. To prove this, let \((z + J)\neq 0\) and \((z + J)^{-1} \) does not exists.

Thus \( J \subseteq (J + z) \subseteq Z \) \( \forall e \in (J + z) \) because \((z + J)^{-1} \notin \frac{Z}{J} \). But this is not true since \( J \) is maximal by our assumption. This implies that every element of \( \frac{Z}{J} \) is \( \lambda(e + j) \) for \( \lambda \in \mathbb{C} \).

If \( \theta: \frac{Z}{J} \rightarrow \mathbb{C} \) represent this isomorphism, and if \( \pi: Z \rightarrow \frac{Z}{J} \) is the canonical projection.

Then \( \theta \circ \pi: Z \rightarrow \mathbb{C} \) is a homomorphism with \( \ker \theta = J \);
\[
\theta \circ \pi(p,q) = \theta[(p + q) + J] = \theta[(p + J)(q + J)] = [\theta(p + J)][\theta(q + J)] = \theta(p) \cdot \theta(q).
\]

Also, \( \theta \circ \pi(p) = 0 \iff \pi(p) = 0 \iff p \in J \).

Hence there is a correspondence between maximal ideals \( J \) and the characters \( \mathcal{J} \) with \( \ker \mathcal{J} = J \).

This correspondence is one-to-one because \( \mathcal{J} \) is uniquely determined by its Kernel. If \( \mathcal{J} \) and \( \mathcal{J}' \) are two character with \( \ker \mathcal{J} = \ker \mathcal{J}' \) then for any \( w \in Z \), \((w - \mathcal{J}(w)e) \in \ker \mathcal{J} \) and thus \( \mathcal{J}(w) = \mathcal{J}'(w) \) because \( \mathcal{J}(e) = 1 \).

**Theorem 3.6:**

Every commutative fuzzy complete a-FNA \((Z, n_Z, \mathcal{Q}, \bigodot)\) with an identity \( e \) has at least one character.

**Proof:**

If \( u^{-1} \) exists for all \( u \in Z \) then \( Z \cong \mathbb{C} \) and the isomorphism \( \mu: Z \rightarrow \mathbb{C} \) is a character. On the other hand, if \( \exists x \in Z \) such that \( u^{-1} \) does not exists then \( xZ \subset J \) where \( J \) is a maximal proper ideal, by Zorn’s lemma the set \( \mathcal{T} = \{J \subset L: L \text{ is ideal}\} \) is partially ordered by \( \subseteq \), thus \( \cup_{L \in \mathcal{T}} L \) is an ideal and \( J \subseteq \cup_{L \in \mathcal{T}} L \). Since \( e \notin \cup_{L \in \mathcal{T}} L \). By Zorn’s Lemma states that there exists a maximal \( \mathcal{K} \) with \( \subseteq \mathcal{K} \). But \( J = \ker \mathcal{K} \) where \( \mathcal{K} \) is a character on \( Z \).

If \( Z \) is not commutative, then we may does not find a character at all on the a-FNA.

**Example 3.7**

If \( Z = M_k(\mathbb{C}) \) where \( k > 1 \), then assume that \( E_{ij} = (e_{ij}) \in M_k(\mathbb{C}) \) where \( e_{ij} = 0 \), except for the i-j position is equal to 1. Now let \( \mathcal{K} \) is a character on \( Z \), then for \( i \neq j \), \( E_{ij}^2 = 0 \) which imply that \( \mathcal{K}(E_{ij}) = 0 \). But \( E_{ij} = E_{ji}. E_{ji} \), when i\( \neq j \), and this imply that \( \mathcal{K}(E_{ii}) = 0 \) for \( j = 1, 2, ..., k \).

Hence, \( \mathcal{K}(1) = \mathcal{K}(E_{11}) + \mathcal{K}(E_{22}) + ... + \mathcal{K}(E_{kk}) = 0 \). But this is not true.

Thus \( Z = M_k(\mathbb{C}) \) with \( k > 1 \), does not has a characters.

**Definition 3.8:**

If \((Z, n_Z, \mathcal{Q}, \bigodot)\) is a fuzzy complete a-FNA has an identity \( e \). Then \{ \( \mathcal{K}: \mathcal{K} \) is a characters of \( Z \) \} is called structure of \( Z \) and is denoted by \( \text{st}(Z) \).

**Definition 3.9:**

(1) The \( \omega^* - \)fuzzy topology on \( \text{af}(Z, \mathbb{C}) \) is generated by \( N(q, A, \varepsilon) = \{g \in \text{af}(Z, \mathbb{C}): L_g|g(a) - q(a)| \leq \varepsilon, \forall a \in A \} \), where \( q \in \text{af}(Z, \mathbb{C}) \), and \( A \subset Z \) is finite.

(2) A set \( E \) in \( \text{af}(Z, \mathbb{C}) \) is fuzzy open in \( \omega^* - \)fuzzy topology \( \iff \forall \theta \in E, \exists N(\theta, A, \varepsilon) \subseteq E \).
Proposition 3.10:
If \((Z, n_Z, \Theta, \bigodot)\) is a fuzzy complete a-FNA having an identity \(e\) then \(\omega^*\) –fuzzy topology on \(afb(Z, \mathbb{C})\) is a Hausdorff space.

Proof:
If \(h_1, h_2 \in afb(Z, \mathbb{C})\) with \(h_1 \neq h_2\) then there exist \(a \in Z\) such that \(h_1(a) \neq h_2(a)\). Let \(L_z[h_1(a) - h_2(a)] = r\), for some \(0 < r < 1\), then \(\forall r < r_0 < 1, \exists r_1\) satisfying \(r_1 \odot r_1 < r_0\).

Now consider \(N(h_1, \{a\}, r_1)\) and \(N(h_2, \{a\}, r_1)\). Clearly, \(N(h_1, \{a\}, r_1) \cap N(h_2, \{a\}, r_1) = \emptyset\) if there exists \(y \not\in N(h_1, \{a\}, r_1) \cap N(h_2, \{a\}, r_1)\) then \(r = L_z[h_1(a) - h_2(a)]\).

\[\leq L_z[h_1(a) - y(a)] \odot L_z[h_2(a) - y(a)]\]

\[\leq r_1 \odot r_1 < r.\]
But this is not true. Hence the proof is complete.

Proposition 3.11:
If \((Z, n_Z, \Theta, \bigodot)\) is a fuzzy complete a-FNA having identity \(e\) then \(st(Z)\) is \(\omega^*\)-fuzzy closed subset of \(afb(Z, \mathbb{C})\).

Proof:
If \((h_k)\) is a sequence in \(st(Z)\) converging to \(h \in \bigcap_{f \in \mathbb{C}} \{h \in afb(Z, \mathbb{C})\}\) for each \(z \in Z\). Now for any \(x, y \in Z\) we have

\[h(xy) = \lim_{k \to \infty} h_k(xy) = \lim_{k \to \infty} h_k(x). \lim_{k \to \infty} h_k(y) = h(x)h(y)\]

It follows that \(h \in st(Z)\). Hence \(st(Z)\) is a fuzzy closed.

In the next result we prove fuzzy Banach-Alaoglu’s Theorem:

Theorem 3.12
If \((Z, n_Z, \Theta, \bigodot)\) is a fuzzy complete a-FNS, then the fuzzy closed unit ball \(B_{\omega^*}(Z, \mathbb{C}) = \{h \in \omega^* afb(Z, \mathbb{C})\} : \exists_{n_{\omega^* afb(Z, \mathbb{C})}}(h) \leq 1\}\) of \(afb(Z, \mathbb{C})\) is a \(\omega^*\) –fuzzy compact.

Proof:
Let \(z \in Z\), define \(D_z = \{a \in \mathbb{C} : L_z(a) \leq n_z(z)\} \subseteq \mathbb{C}\). Then \(D_z\) is a fuzzy compact which implies \(D = \bigcap_{z \in Z} D_z\) is a fuzzy compact in the product fuzzy topology by fuzzy Tychonoff Theorem 2.44.

Let \(B^d\) denotes the fuzzy closed unit ball \(B_{\omega^* afb(Z, \mathbb{C})}\).

Define \(\theta: B^d \to D\) by \(\theta(\eta) = (\eta(u))_{u \in Z} \forall \eta \in B^d\). We will prove that \(\theta\) is one-to-one and fuzzy continuous. It is clear that \(\theta\) is Linear. If \(\theta(\eta) = 0\) then \(\eta(u) = 0\) \(\forall u \in Z\). Hence, \(\eta = 0\) and from this we obtain \(\eta\) is one-to-one.

To prove \(\theta\) is a fuzzy continuous, now if \((\sigma_z) \in B^d\) satisfying \(\sigma_z \to \omega^* \eta\). Then \(\eta_k(u) \to \eta(\eta) \forall u \in Z\). Consequently, \(\theta(\eta_k) = (\eta_k(u))_{u \in Z} \to (\eta(u))_{u \in Z} = \theta(\eta)\). Hence, \(\theta\) is a fuzzy continuous. If \(\theta(B^d)\) is a fuzzy closed subset of \(D\) and \(D\) being fuzzy compact, then \(\theta(B^d)\) is fuzzy compact.

Thus our next step is to prove \(\theta(B^d)\) is a fuzzy closed. If \(\sigma = (\sigma_z) \in D\) and \(\sigma \in \theta(B^d)\) then define \(\eta \in Z \to \mathbb{C}\) by \(\eta(u) = \sigma_u \forall u \in Z\). The map \(\eta\) is linear, if \(x, y \in Z\) and \(a, b \in \mathbb{C}\) then \(\forall k \in \mathbb{N}\), choose \(\eta_k \in B^d\) thus

\[\eta(ax + by) = \lim_{k \to \infty} \eta_k(ax + by) = ax \lim_{k \to \infty} \eta_k(x) + bx \lim_{k \to \infty} \eta_k(y) = ax \sigma_f(x) + bx \sigma_f(y) \quad \text{[since \(\eta_k\) is linear]}\]

Thus \(\eta\) is linear. Since \(L_z(\sigma_z) \leq n_z\), so \(\eta \in B^d\). Now by the definition of \(\eta\), we see that \(\sigma = \theta(\eta) \in B^d\). Hence, \(\theta(B^d)\) is a fuzzy closed. But \(B^d\) and \(\theta(B^d)\) are homeomorphic, so \(B^d\) must be fuzzy compact.
Proposition 3.13:
If \((Z, n_Z, \mathcal{O}, \mathcal{O})\) is a fuzzy complete a-FNA having an identity \(e\) then \(st(Z)\) is a \(\omega^*\)-fuzzy closed of \(B_{afb}(Z, \mathcal{C})=\{h \in afb(Z, \mathcal{C}): n_{afb}(Z, \mathcal{C})(h) \leq 1\}\) and hence is a fuzzy compact.

Proof:
Suppose that \((\ell_k)\) be a sequence in \(st(Z)\) which is fuzzy converge to \(\theta \in afb(Z, \mathcal{C})\). Then \(\ell_k(z) \to \theta(z) \forall z \in Z\).

Since \(\forall x, y \in Z, \theta(xy)=\lim_{k \to \infty} \ell_k(xy)=\lim_{k \to \infty} \ell_k(x) \cdot \lim_{k \to \infty} \ell_k(y) = \theta(x) \cdot \theta(y)\), so we conclude that \(\theta \in st(Z)\). Here \(\theta \neq 0\) since \(\theta(1)=1\). Thus \(st(Z)\) is a \(\omega^*\)-fuzzy closed. Therefore, \(st(Z)\) is a fuzzy compact because it is a fuzzy closed subset of a fuzzy compact set.

Theorem 3.14:
If \((Z, n_Z, \mathcal{O}, \mathcal{O})\) is a commutative fuzzy complete a-FNA having an identity \(e\), then for each \(z \in Z\) and \(h \in st(Z)\) we define \(\Psi_z: st(Z) \to \mathcal{C}\) by \(\Psi_z(h)=h(z)\). Then the range of the function \(\Psi_z\) on \(st(Z)\) satisfies \(R(\Psi_z)=\sigma_Z(z)\). Furthermore, the map \(\Psi: Z \to C(st(Z))\) and \(n_{afb}(st(Z), \mathcal{C})(\Psi_z) \leq n_Z(z)\) for all \(z \in Z\). The map \(\Psi\) is called Gelfand transform.

Proof:
If \(u \in Z\) and \(h \in st(Z)\) then \(h(x) \in \sigma_Z(u)\) that is \(\Psi_z(u) \in \sigma_Z(u)\) and so the range of \(\Psi_z\) satisfies the inclusion \(R(\Psi_z) \subseteq \sigma_Z(u)\).

Let \(\alpha \in \sigma_Z(u)\), so \((u-\alpha)e\) does not exists and \((u-\alpha)e \notin J\) where \(J\) is some maximal ideal, say \([\text{Since } (u-\alpha)e \notin Z(x-\alpha) \subset J]\).

If \(h \in st(Z)\) with \(\ker h=J\) then \((x-\alpha)e \in J\) which implies that \(h(u)=\alpha\). Thus \(\Psi_z(h)=h(u)=\alpha\) and so \(R(\Psi_z)=\sigma_Z(u)\). But \(\Psi\) is a homomorphism;
\[\Psi_{zy}(h)=h(zy)=h(z)h(y)=\Psi_z(h)\Psi_y(h) \forall z, y \in Z, h \in st(Z)\] and so \(\Psi_{zy} \in \Psi_z, \Psi_y\).

Similarly, we can show that \(\Psi_{az+\beta y}=a\Psi_z + b\Psi_y(h)\), thus \(\Psi_z\) is linear.

To prove that \(\Psi_z \in C(st(Z))\), if \(\Omega\) is a fuzzy open set in \(\mathcal{C}\) we will prove that \(\Psi_z^{-1}(\Omega)\) is a fuzzy open in \(st(Z)\). When \(\Psi_z^{-1}(\Omega)=\emptyset\), the proof is end. If \(\Psi_z^{-1}(\Omega)\neq \emptyset\). Assume that \(g \in \Psi_z^{-1}(\Omega)\). So \(\exists \delta \in \Omega\) such that \(\Psi_z(g) \in \delta\). Since \(\Omega\) is a fuzzy open set in \(\mathcal{C}\), \(\exists 0<\varepsilon<1\) with \(N_\varepsilon(\delta) = \{\alpha \in \mathcal{C}: L_C(\alpha-\delta) < \varepsilon\}\).

If \(\psi=\theta(g, \{u\}, \varepsilon)\), then \(\theta(\psi) \in \psi \in \Psi_z^{-1}(\Omega)\). Hence, \(\Psi_z^{-1}(\Omega)\) is a fuzzy open set in \(st(Z)\) and therefore \(\Psi_z: st(Z) \to \mathcal{C}\) is a fuzzy continuous function thus \(\Psi_z(u) \in C(st(Z))\).

On the other hand, we can introduce another prove for the fuzzy continuity of \(\Psi_z\) by using sequences. If \(g_k \to g\) in \(st(Z)\) then \(\Psi_z(g_k) \to \Psi_z(g) \Rightarrow g(u)\), hence \(\Psi_z\) is a fuzzy continuous.

Now, \(R(\Psi_z)=\sigma_Z(u)\subseteq \{\alpha \in \mathcal{C}: L_C(\alpha) \leq n_Z(u)\}\) and thus \(L_C(\Psi_z(g)) \leq n_Z(u)\) for all \(g \in st(Z)\). Hence, \(n_{afb}(st(Z), \mathcal{C})(\Psi_z) \leq n_Z(u)\) for all \(u \in Z\).

Theorem 2.3.15:
If \((Z, n_Z, \mathcal{O}, \mathcal{O})\) is a commutative fuzzy complete a-FNA having an identity \(e\) and \(Z=uZ\), that is, the set of polynomials in \(z\) is fuzzy dense in \(Z\). Then the map \(\Psi_u: st(Z)\to \sigma_Z(u) \subseteq \mathcal{C}\) is a homeomorphism.

Proof:
Since \(\Psi_u\) is fuzzy continuous function on \(st(Z)\) satisfying \(R(\Psi_u) = \sigma_Z(u)\) i.e., \(\Psi_u: st(Z)\to \sigma_Z(u)\) is a fuzzy continuous and onto. But \(st(Z)\) and \(\sigma_Z(u)\) are fuzzy compact Hausdorff spaces, thus it remains only to prove that \(\Psi_u\) is injective. Now if \(\Psi_u(\ell_1) = \Psi_u(\ell_2)\), so that \(\ell_1(u) = \ell_2(u)\), by using the multiplicativelty of \(\ell_1\) and \(\ell_2\) we see that for given \(k \in \mathbb{N}\) and \(c_0, c_1, \ldots, c_k\) in \(\mathcal{C}\),
\[\ell_1(\sum_{j=0}^{k} c_j u^k) = \ell_2(\sum_{j=0}^{k} c_j u^k)\]. Since \(\ell_1\) and \(\ell_2\) are fuzzy continuous and \(u\) generates \(Z\), it follows that \(\ell_1 = \ell_2\).
Example 2.3.16:
Let $Z=\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}: \alpha, \beta \in \mathbb{C}\}$ be a subalgebra of $M_2(\mathbb{C})$. Then $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}=\alpha I + \beta q$, where $q=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We note that $q^2=0$.
Evidently, $Z$ is two-dimensional commutative fuzzy complete $a$-fuzzy normed algebra with identity $I$. We shall compute the spectrum $\sigma_Z(x)$ for $x=\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$. Indeed, for $\lambda \in \mathbb{C}$, $x-\lambda I=\begin{pmatrix} \alpha - \lambda & \beta \\ 0 & \alpha - \lambda \end{pmatrix}$ is invertible in $M_2(\mathbb{C})$ if $\alpha \neq \lambda$. If $\alpha \neq \lambda$, then in fact

$$(x-\lambda I)^{-1}=\begin{pmatrix} (\alpha-\lambda)^{-1} & -\beta (\alpha-\lambda)^{-1} \\ 0 & (\alpha-\lambda)^{-1} \end{pmatrix},$$
which belongs to $Z$. Hence, $\sigma_Z(x)=\sigma_Z(\alpha I + \beta q)=\{\alpha\}$. In particular $\sigma_Z(q)=\sigma_Z(\beta q)=\{0\}$, but $q\neq 0$. If $\hat{h}$ is a character of $Z$ then $\hat{h}(uv)\hat{h}(v)\hat{h}(u)$ implies $\hat{h}(q^2)=\hat{h}(q)\hat{h}(q)$. But $q^2=0$ and so $\hat{h}(q)=0$. Since $\hat{h}(I)=1$, we find that $\hat{h}(\alpha I + \beta q)=\alpha$ for any $\alpha, \beta \in \mathbb{C}$. Thus, there is just one character on $Z$ so $st(Z) = \{\hat{h}\}$, where $\hat{h}$ is given uniquely by the action $\hat{h}(I)=1$ and $\hat{h}(q)=0$.
The fuzzy Gelfand transform is the map $z \mapsto \Psi_z$, $(\alpha I + \beta q)\mapsto \alpha \Psi_I + \beta \Psi_q$. But $\Psi_I=1$ and $\Psi_q(\hat{h})=\hat{h}(q)=0$ so that $\Psi_q=0$ and we have $\Psi_{(\alpha I + \beta q)}=\alpha$ for any $\alpha, \beta \in \mathbb{C}$.

The transform $\Psi_q$ has kernel $\{\beta q: \beta \in \mathbb{C}\}$, so $\Psi_q$ is not an isomorphism. The algebra $Z$ has exactly one maximal ideal, namely, the Kernel of $\hat{h}$. As $Z$ is an algebra with identity generated by $q$ and so $st(Z) \equiv \sigma_Z(q)$, throw $\Psi_{\alpha I + \beta q} \mapsto \sigma_Z(q)$, $\hat{h} \mapsto \Psi_q(\hat{h})=0$.
Thus, the two sets $st(Z)$ and $\sigma_Z(q)$ are singleton sets.

On the other hand, we can calculate the spectrum of $x \in Z$ using $R(\Psi_x)=\sigma_Z(x)$. For $x=\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ we have, $\sigma_Z(x) = \{\Psi_x(\hat{h})\} = \{\hat{h}(x)\} = \{\hat{h}(\alpha I + \beta q)\} = \{\alpha \hat{h}(I) + \beta \hat{h}(q)\} = \{\alpha\}$
Since $\hat{h}(q)=0$.

4. Conclusions
In [6] we proved some properties of fuzzy complete $a$-fuzzy normed algebra. In this paper we recall the definition of $a$-fuzzy normed algebra in order to prove other properties of fuzzy complete $a$-fuzzy normed algebra.

References