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Inspection Times and Optimal Censoring Scheme for Generalized Inverted Exponential Distribution with Progressive type I Interval Censored Data

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Abstract

In this article, two issues related to the design of the experiment are investigated; namely the inspection times and optimal censoring. In the first issue, we study four different approaches to determine the inspection times, namely; pre-specified, equally spaced, optimally spaced, and equal probability using two optimality criteria. In the second issue, we identify the optimal censoring scheme by finding the expected numbers of removals or proportions that attain a specific optimality criterion using two optimality criteria considered. Numerical comparisons using the Monte Carlo simulations are provided of the inspection times, optimum control schemes issues for different parameters and different sample sizes. Statistical measures such as bias and root mean square error for both pre-specified and equal space methods are calculated and the inspection times for different values of the censoring percentage h are obtained.

Keywords: The generalized inverted exponential distribution, progressive type I interval censored, optimal censoring, inspection times.

أوقات الفحص وخطة الرقابة المثلى للتوزيع الأسي المعكوس المعمم مع بيانات الفاصل الزمني التدريجي من النوع الأول الخاضعة للرقابة

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الخلاصة

في هذه البحث ، تمت دراسة مسالتين تتعلقان بتصميم التجربة الاحصائيه وهما أوقات الفحص والرقابة المثلى. في القسم الأول ، قمنا بدراسة أربعة طرائق مختلفة لتحديد أوقات التفتيش ، وهي ؛ امحددة مسبقًا ، ومتباعدة بشكل متساوٍ ، ومتباعدة بشكل مثالي ، ومتساوية وباستخدام معيارين للأمثلية. في القسم الثاني ، تم تحديد نظام الرقابة الأمثل من خلال إيجاد الأعداد المتوقعة من عمليات الإزالة (أو النسب) التي تحقق معيارًا أمثلًا محددًا باستخدام معيارين للأمثلية. كما تم انشاء مقارنات عددية باستخدام محاكاة مونت كارلو للمسألتين المذكورتين أعلاه لمعلمات مختلفة وأحجام عينات مختلفة. قيم التحيز وجذر مربع معدل الخطأ لطريقتي الاوقات المحدده والمسافات المتساويه تم احتسابها واوقات الفحص لقيم مختلفه لنسب الرقابه h تم الحصول عليها.

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1 Introduction

A random variable x^X of the generalized inverted exponential distribution (GIED) with shape parameter α and scale parameter λ has the following expressions of c.d.f. and p.d.f.

$$F(x) = 1 - (1 - e^{-\lambda/x})^{\alpha}, x > 0, \alpha > 0, \lambda > 0, \quad (1)$$

$$f(x) = \frac{\alpha\lambda}{x^2} e^{-\lambda/x}(1 - e^{-\lambda/x})^{\alpha}, x > 0, \alpha > 0, \lambda > 0 \; . \quad (2)$$

The hazard function of GIED distribution is given by the following [1]:

$$\frac{f(x)}{1-F(x)} = \frac{\alpha\lambda}{x^2(e^{-\lambda/x}-1)}.$$

Clearly, the hazard function of the GIED can be increasing or decreasing which depends on the shape parameter α. The GIED was used in many applications, for instance; in horse racing, supermarkets queue, sea currents, and wind speeds. It is also suitable for modeling for the applications of agriculture, botany, economics, medicine, psychology, zoology, life testing and reliability of mechanical or electrical components lying in the life testing experiment. It is also observed that this distribution may provide a better fit than gamma, Weibull, and generalized exponential distributions in many situations [2]. For more properties of the GIED in applications, one can refer to [3] , [4] , [5] , and [6]. The most common censoring schemes in life-testing and reliability studies are type I and type II censoring. However, type I and type II schemes do not have the ability to allow the removal of units at points other than the terminal point of the experiment. The progressive type I interval censored scheme which was proposed by [2] can be described as follows: Assume n units are put on test at time $t_0 = 0$ and each unit is followed until it fails or is censored. Units are observed at pre-specified times $t_1 < t_2 < \cdots < t_m$, to the end. Let d_j be the number of units which are failed in $[t_{j-1}, t_j)$ and r_j denote the number of units which is removed from the experiment at time t_i , $i = 1,2,\dots,m$. Clearly $n = \sum_{i=1}^{m}(r_i + d_i)$.

Hence, our observations consist of $D = \{(t_i, d_i, r_i); i = 1, \dots, m\}$. The numbers of removal items $r_1, \dots, r_m$ are expressed as nonnegative integers. Alternatively, the removal numbers may determine by pre-specified percentages of the remaining surviving units as follows. Let $p = (p_1, p_2, \dots. p_m)$ be pre-specified percentages with $p_m = 1$. At time t_i, $\lceil p_i \times (number\ of\ surviving\ unit\ time\ t_i)\rceil$ from the remaining surviving units that are removed from the experiment, where $\lceil w \rceil$ denotes the largest integer, which is smaller than or equal to w.

In this article, two issues related to the design of the experiment are investigated; inspection times and optimal censoring. First, Section 1 is devoted to the maximum likelihood estimation of the parameters of the GIED under progressive interval-censored data. In Section 2, we study four different approaches to determine the inspection times, namely; pre-specified, equally spaced, optimally spaced, and equal probability using two optimality criteria. In Section 3, we identify the optimal censoring scheme by finding the expected numbers of removals (or proportions) that attain a specific optimality criterion using two optimality criteria considered. In Section 4, numerical comparisons using Monte Carlo simulations are provided of the two aforementioned issues for different parameters and different sample sizes.

2 Maximum likelihood estimation

Based on the observed progressive the type I interval censored sample $D = \{(t_i, d_i, \dots. r_i); i = 1, \dots, m\}$, the log-likelihood function of α and λ can be written as follows:

$$l(\alpha,\lambda|D) = \sum_{i=}^{m} d_i \log\left(F(t_i) - F(t_{i-1})\right) + \sum_{i=}^{m} r_i \log\left(1 - F(t_i)\right)$$

$$= \sum_{i=}^{m} d_i \log\left((1 - e^{-\lambda/t_{i-1}})^{\alpha} - (1 - e^{-\lambda/t_i})^{\alpha}\right) + \alpha \sum_{i=}^{m} r_i \log\left(1 - e^{-\lambda/t_i}\right). \quad (3)$$

Where $t_0 = 0$. To compute the MLEs of the unknown parameters, $\hat{\alpha}$ and $\hat{\lambda}$, we need to find α and λ that maximize the log-likelihood function i.e.

$$(\hat{\alpha},\hat{\lambda}) = arg_{max_{(\alpha,\lambda)}} l(\alpha,\lambda|D)$$

In order to obtain the MLEs, we propose to use the optim() function in R language to solve the optimization problem.

3 Inspection times

We usually, in progressive type I interval censored, identified inspection times by fixed quantities before the start of the experiment. However, it is important to investigate the effect of different inspection times on the efficiency of obtained estimators. This problem under progressive interval censored observations has not received much attention in the literature. The authors determined [7] the optimally spaced inspection times for the two-parameter lognormal distribution under progressive type I interval censored plan. Recently, in [8] , [9] and [10], the authors obtained various inspection times by using the expected Fisher information matrix for Burr XII, inverse Weibull and truncated normal distributions, respectively.

In the following, we study four different approaches to determine the inspection times, namely; the pre-specified, equally spaced, optimally spaced and equal probability. In the pre-specified approach, time points are commonly pre-determined on the basis of the available knowledge about the experiment. The equally spaced inspection times are identified by constructing inspection intervals of equal length in which time points to be included are considered. Specifically, if t_m is the termination time of the experiment, time points can be obtained by $t_i = \frac{i}{m} t_m, i = 1, \dots, m.$ The authors mentioned [9] that when units on the test have a decreasing failure rate, the equally spaced inspection times may provide efficient estimates.

In the optimally spaced approach, time points are obtained in order to achieve some optimality criteria. To study the problem of selecting the inspection times, we consider the following optimality criteria:

Criterion I: Minimizing the trace of the expected variance covariance matrix of the MLEs.
Criterion II: Maximizing the determinant of the expected Fisher information matrix of the MLEs.
It is known that the expected variance covariance matrix of the MLEs can be obtained by inverting the expected Fisher information matrix. Let $p = (p_1, \dots, p_m)$ be a censoring scheme. Observe that the probability that a unit fails in the interval $(0, t_1]$ is
$P(0 < T \le t_1 | T > 0) = \frac{F(t_1) - F(0)}{1 - F(0)} = F(t_1).$

Then $D_1 \sim Binomial(n, F(t_1))$ and $R_1|D_1 \sim Binomial(n - D_1, p_1)$ Consequently, the expected number of failures in the interval $(0, t_1]$ is $\zeta_1 = E(D_1) = nF(t_1)$ and the expected number of removed units is $\tau_1 = E(R_1|D_1)|_{\zeta_1} = (n - \zeta_1)p_1$. Subsequently, the probability that a unit fails in the interval $(t_{i-1}, t_i]$

$P(t_{i-1} < T \le t_i | T > t_{i-1}) = \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})}, i = 1,2,\dots,m.$

Then the conditional distributions of D_i and R_i are given by

$$D_i|(D_{i-1}, R_{i-1}, \dots, D_1, R_1) \sim Binomial(n - \sum_{j=1}^{i-1}(D_j + R_j), \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})} \quad , \tag{4}$$

$$R_i| D_i, D_{i-1}, R_{i-1}, \dots, D_1, R_1 = R_i \sim Binomial\left(n - \sum_{j=1}^{i} D_j - \sum_{j=1}^{i-1} R_j, p_i\right), \tag{5}$$

(5)

and the expected number of failures and the expected number of removed items are respectively computed by

$$\zeta_i = E(D_i | D_{i-1} R_{i-1}, \dots, D_1, R_1)|_{(\zeta_{i-1}, r_{i-1}, \dots, \zeta_1, r_1)}$$
$$= (n - \sum_{j=1}^{i-1}(\zeta_j + \tau_j)) \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})} \quad . \tag{6}$$

$$\tau_i = E(R_i | D_i, D_{i-1}, R_{i-1}, \dots, D_1, R_1)|_{(\zeta_{i-1}, r_{i-1}, \dots, \zeta_1, r_1)}$$
$$= (n - \sum_{j=1}^{i-1}(\zeta_j + \tau_j) - \zeta_i) p_i. \quad . \tag{7}$$

Therefore, the expected Fisher information matrix can be obtained as follows

$$I(\alpha, \lambda) = \begin{bmatrix} -l_{\alpha\alpha} & -l_{\alpha\lambda} \\ -l_{\alpha\lambda} & -l_{\lambda\lambda} \end{bmatrix}$$

where

$$l_{\alpha\alpha} := \frac{\partial^2 l(\alpha, \lambda|D)}{\partial \alpha^2} = \sum_{i=1}^{m} \xi_i \frac{A_i A_{i,\alpha\alpha} - A_{i,\alpha}^2}{A_i^2}, \tag{8}$$

$$l_{\alpha\lambda} := \frac{\partial^2 l(\alpha, \lambda|D)}{\partial \alpha \partial \lambda} = \sum_{i=1}^{m} \xi_i \frac{A_i A_{i,\alpha\lambda} - A_{i,\alpha} A_{i,\lambda}}{A_i^2} + \sum_{i=1}^{m} \tau_i \frac{B_{i,\lambda}}{B_i} \tag{9}$$

$$l_{\lambda\lambda} := \frac{\partial^2 l(\alpha, \lambda|D)}{\partial \lambda^2} = \sum_{i=1}^{m} \xi_i \frac{A_i A_{i,\lambda\lambda} - A_{i,\lambda}^2}{A_i^2} + \alpha \sum_{i=1}^{m} \tau_i \frac{B_i B_{i,\lambda\lambda} - B_{i,\lambda}^2}{B_i^2}, \tag{10}$$

where

$$A_i = (1 - e^{-\lambda/t_{i-1}})^\alpha - (1 - e^{-\lambda/t_i})^\alpha,$$

$$B_i = 1 - e^{-\lambda/t_i},$$

$$A_{i,\alpha} := \frac{\partial A_i}{\partial \alpha} = (1 - e^{-\lambda/t_{i-1}})^\alpha log\left(1 - e^{-\lambda/t_{i-1}}\right) - (1 - e^{-\lambda/t_i})^\alpha log(1 - e^{-\lambda/t_i}),$$

$$A_{i,\lambda} := \frac{\partial A_i}{\partial \lambda} = \frac{\alpha}{t_{i-1}} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}})^{\alpha-1} - \frac{\alpha}{t_i} e^{-\lambda/t_i} (1 - e^{-\lambda/t_i})^{\alpha-1},$$

$$B_{i,\lambda} := \frac{\partial B_i}{\partial \lambda} = \frac{1}{t_i} e^{-\lambda/t_i},$$

$$A_{i,\alpha\alpha} := \frac{\partial^2 A_i}{\partial \alpha^2} = (log(1 - e^{-\lambda/t_{i-1}}))^2 (1 - e^{-\lambda/t_{i-1}})^\alpha - (log(1 - e^{-\lambda/t_i}))^2 (1 - e^{-\lambda/t_i})^\alpha$$

$$A_{i,\alpha\lambda} := \frac{\partial^2 A_i}{\partial \alpha \partial \lambda} = \frac{1}{t_{i-1}} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}})^{\alpha-1} \left[1 + \alpha log\left(1 - e^{-\lambda/t_{i-1}}\right)\right]$$

$$-\frac{1}{t_i} e^{-\lambda/t_i}(1-e^{-\lambda/t_i})^{\alpha-1}\left[1+\alpha log\left(1-e^{-\lambda/t_i}\right)\right],$$

$$A_{i,\lambda\lambda} := \frac{\partial^2 A_i}{\partial \lambda^2} = \frac{\alpha}{t_{i-1}}\left(\frac{\alpha-1}{t_{i-1}}(e^{-\lambda/t_{i-1}})^2(1-e^{-\lambda/t_{i-1}})^{\alpha-2} - \frac{1}{t_{i-1}} e^{-\lambda/t_{i-1}}(1-e^{-\lambda/t_{i-1}})^{\alpha-1}\right)$$

$$-\frac{\alpha}{t_i}\left(\frac{\alpha-1}{t_i}(e^{-\lambda/t_i})^2(1-e^{-\lambda/t_i})^{\alpha-2} - \frac{1}{t_i} e^{-\lambda/t_i}(1-e^{-\lambda/t_i})^{\alpha-1}\right),$$

$$B_{i,\lambda\lambda} := \frac{\partial^2 B_i}{\partial \lambda^2} = -\frac{1}{t_i^2} e^{-\lambda/t_i}.$$

It is easy to observe that the computing of the optimally spaced inspection times is a constraint optimization problem due to the condition $t_i > t_{i-1}, i = 1,2,\dots,m$. Hence, in order to remove the monotonicity constraints, we consider the transformation of t_i 's as $t_i = \sum_{k=1}^{i} e^{s_k}$. With the use of new variables s_i 's, a genetic algorithm is used for the determination of the optimally spaced inspection times via GA() package.

In the last approach, the equal probability, we study inspection times for a pre-specified percentage of the censoring observations quantity h satisfying the expression $\sum_{i=1}^{m} \tau_i = nh$. Note that $\sum_{i=1}^{m} \zeta_i = n(1-h)$ since $\sum_{i=1}^{m} \zeta_i + \sum_{i=1}^{m} \tau_i = n$. Furthermore, we consider the probability of the expected number of failures in each inspection interval is considered to be the same. As consequence, the problem of finding the equal probability inspection times reduces to compute t_i's such that $\zeta_1 = \zeta_2 = \dots = \zeta_m$ and $\sum_{i=1}^{m} \zeta_i = n(1-h)$. Observe that, by solving (6) for t_i, we obtain

$$t_i = F^{-1}\left[\frac{\zeta_i[1-F(t_{i-1})]}{n-\sum_{j=1}^{i-1}(\zeta_j+\tau_j)} + F(t_{i-1})\right], i = 1,2,\dots,m.$$

Hence, we propose the following algorithm to obtain equal probability inspection times (see for example [9]).

Input: Choose $n \in Z^+, m \in Z^+, m \le n, h \in [0,1]$ and $p = (p_1, p_2, \dots, p_m)$, where Z^+ is the set of positive integers and $p_i \in [0,1]$.
Initialize: Set $\zeta_i = \frac{n(1-h)}{m}, i = 1,\dots,m$. Compute $t_1 = F^{-1}(\frac{\zeta_1}{n})$ and $\tau_1 = (n-\zeta_1)p_1$.
Repeat step 1 to step 3 for $i = 2,\dots,m$.
Step 1: Obtain

$$t_i = F^{-1}\left(\frac{\zeta_i\left(1-F(t_{i-1})\right)}{n-\sum_{j=1}^{i-1}(\zeta_j+\tau_j)} + F(t_{i-1})\right)$$

Step 2: Compute

$$\tau_i = \left\lceil n - \frac{i * n(1-h)}{m} - \sum_{j=1}^{i-1} \tau_j \right\rceil p_j.$$

Step 3: If $\sum_{j=1}^{i} \tau_j > nh$ then set $\tau_i = nh - \sum_{j=1}^{i} \tau_j, \tau_k = 0$ for $k = i+1,\dots,m$ and stop.
Here $\lceil x \rceil$denotes the greatest integer less than or equal to x.

4 Optimal censoring

It is common in the analysis of real life experiments to consider the censoring scheme as fixed and pre-specified. However, in the estimation problem, we may choose the censoring scheme among a set of possible schemes in order to improve the estimations of parameters. It is known that under progressive type I interval censored, the number of the removed units R_i at each inspection time t_i, can be a constant number or a pre-specified proportion p_i of surviving units. Optimal censoring can be described as finding the expected numbers $R = (R_1, R_2, \dots, R_m)$ (or proportions $p = (p_1, p_2, \dots, p_m)$) which attains to a specific optimality criterion. The issue of identifying the optimal censoring scheme for different distributions under progressive type I interval censored has received little attention in the statistical literature. See [8] for Burr XII and [9] for inverse Weibull distribution.

The problem of selecting the optimal censoring method under progressive type I interval censored observation can be described as follows. For given n and h, the optimal censoring scheme is the one among all possible censoring schemes which satisfy the conditions $\sum_i^m R_i = \lceil nh \rceil$ and $\sum_{i=1}^m (\zeta_i + \tau_i) = n$, where ζ_i and τ_i are defined in (6) and (7). Recall that the number of all possible censoring schemes satisfying the relation $\sum_i^m R_i = \lceil nh \rceil$ is $\frac{(\lceil nh \rceil + m - 1)!}{(m-1)! \lceil nh \rceil !}$.

First, we consider the optimal censoring with the pre-specified inspection times, i.e. t includes the pre-specified quantities. Assume that $\psi(\zeta, \tau, t)$ is the objective function that needs to be minimized (or maximized). Following [9] , we use the following algorithm to get the optimal censoring scheme based on the pre-specified inspection times.

Step 1. Set the values of n, m, h and $t = (t_1, t_2, \dots, t_m)$.
Step 2. Calculate $W = \frac{(\lceil nh \rceil + m - 1)!}{(m-1)! \lceil nh \rceil !}$ and set $c = 0$ and $k = 1$.
Step 3. Generate $\sum_{i=1}^m R_i = \lceil nh \rceil$ and consider $\tau_i = R_i$.
Step 4. Compute the $\zeta_i, i = 1,2, \dots, m$ using (6).
Step5. If $\sum_{i=1}^m \zeta_i - n + \lceil nh \rceil \le \varepsilon$, set $c = c + 1$ and compute $\psi_k(\zeta, \tau, t)$ else set $k = k + 1$ and go to Step 3.
Step 6. If $\psi_k(\zeta, \tau, t) > (or <) \psi_{k-1}(\zeta, \tau, t)$, then update the optimal censoring scheme $(R_1, R_2, \dots, R_m)$ and go to Step 3 with $k = k + 1$ until $k = w$.

Here, ε is a pre-specified quantity and $\psi_k(.)$ is the value of $\psi(.)$ at the $k - th$ iteration. Next, we utilize the following algorithm to obtain the optimal censoring scheme based on the equal probability inspection times (see, [9]).

Step 1. Select the values of n, m and h.

Step 2. Set $\zeta_i = \frac{n - \lceil nh \rceil}{m}$, $i = 1,2, \dots, m$.

Step 3. Calculate $W = \frac{(\lceil nh \rceil + m - 1)!}{(m-1)! \lceil nh \rceil !}$ and set $k = 1$.

Step 4. Generate $(R_1, R_2, \dots, R_m)$ such that $\sum_{i=1}^m R_i = \lceil nh \rceil$ and consider $\tau_i = R_i, i = 1,2, \dots, m$.

Step 5. Compute

$$t_i = F^{-1}\left(\frac{\zeta_i (1 - F(t_i - 1))}{n - \sum_{j=1}^{i-1} (\zeta_i + \tau_i)} + F(t_i - 1)\right), i = 2,3, \dots, m$$

Where $t_0 = 0$.
Step 6. Given the values of τ_i, ζ_i and $t_i, i = 1,2, \dots, m$, compute $\psi_k(\zeta, \tau, t)$.

Step 7. If $\psi_k(\zeta,\tau,t) > (or <)\psi_{k-1}(\zeta,\tau,t)$ then update the optimal censoring scheme $(R_1,R_2,\ldots,R_m)$ and equal probability inspection times $(t_1,t_2,\ldots,t_m)$. Further, set $k=k+1$ and go to Step 4 until $k=w$.

Based on the above algorithms, we suggest to consider the following two criteria.
Criterion(I): Minimizing the objective function $\psi(.)$ which is the trace of the expected variance covariance matrix of the MLEs.
Criterion(II): Maximizing the objective function $\psi(.)$ which is the determinant of the expected Fisher information matrix of the MLEs.

It is clear that for a large value of m, the total number of sampling schemes can be quite large. For example when $n=25, m=10$ and $h=0.3$ the possible number of censoring schemes is $\binom{\lceil nh\rceil+m-1}{m-1}=\binom{29}{9}=10015005$.

Following [11], we propose to use a sub-optimal censoring problem in which the optimal censoring scheme belongs to the convex hull generated by the points $(\lceil nh\rceil,0,\ldots,0),(0,\lceil nh\rceil,0,\ldots,0),\ldots,(0,\ldots,0,\lceil nh\rceil)$. Therefore, the sub-optimal censoring scheme can be obtained by choosing the optimal censoring scheme among these extreme points on the convex hull. In addition, for generating censoring schemes $(R_1,R_2,\ldots,R_m)$ satisfies the condition $\sum_{i=1}^{m} R_i=\lceil nh\rceil$, we may utilize the function compositions() from partition package in R language.

5 Simulation

In this section, our objective is to compare the performance of the different methods of the inspection times and optimal censoring schemes of the GIED under progressive type I interval censored through the Monte-Carlo simulation study. At first, data is simulated by employing an algorithm proposed by [11] to generate the number of failures $d_1,d_2,\ldots,d_m$ in each interval $(t_{i-1},t_i]$, for $i=1,\ldots,m$ from the sample of size n. The data generation algorithm is described as follows. Given n,m and $p=(p_1,\ldots,p_m)$ where $0\le p_i\le 1$ and $p_m=1$.

Step (i): Generate $t_1^*,\ldots,t_m^*$ from GIED (α,λ) using $t_i^*=-\frac{\lambda}{\log\left(1-U_i^{1/\alpha}\right)}$, where $U_i: U(0,1)$.

Step(ii) : Arrange $t_1^*,\ldots,t_m^*$ as $t_1<t_2<\cdots<t_m$.

Step(iii) : Compute $F_i=F(t_i), i=1,\ldots,m$ using (1).

Step(iv) : Set $d_0=r_0=F_0=0$ and $i=1$.

Step(v) : Generate

$$d_i\,|\,(d_0,\cdots,d_{i-1},r_0,\cdots,r_{i-1}):\ \text{binomial}(n-\sum_{j=0}^{i-1}(d_j+r_j),q_i),$$

where $q_i=\frac{F_i-F_{i-1}}{1-F_{i-1}}$.

Step(vi) :Compute

$$r_i=[\,p_i(n-\sum_{j=0}^{i}d_j-\sum_{j=0}^{i-1}r_j)],$$

where $[x]$ denotes the largest integer not greater than x.

Step(vii) :If $i<m$, then replace i by $i+1$ and go to Step(v), otherwise stop.

We consider different parameter values and sample sizes such as $(\alpha,\lambda)=(0.5,0.5),(1.5,1)$ and $n=25,50,100$. The number of inspection times m is considered to be 5 or 10. Four

different censoring schemes are adopted here for each value of m. When $m = 5$, we consider the following censoring schemes
Scheme 1: $p^1 = (0.25, 0.25, 0.5, 0.5, 1)$.
Scheme 2: $p^2 = (0.5, 0.5, 0.25, 0.25, 1)$.
Scheme 3: $p^3 = (0, 0, 0, 0, 1)$.
Scheme 4: $p^4 = (0.25, 0, 0, 0, 1)$.
and when $m = 10$, we consider the following censoring schemes.
Scheme 1: $p^1 = (0.25, 0.25, 0.25, 0, 0, 0, 0, 0, 0, 1)$.
Scheme 2: $p^2 = (0, 0, 0, 0.25, 0.25, 0.25, 0, 0, 0, 1)$.
Scheme 3: $p^3 = (0, 0, 0, 0, 0, 0, 0.25, 0.25, 0.25, 1)$.
Scheme 4: $p^4 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$.

The above schemes are chosen to specify the percentage of surviving units to be withdrawn at the m censoring and monitoring points. Further, we consider $h = 0.3, 0.5, 0.8$. First, we consider the numerical results concerned with different inspection times reported in Tables 1-7. In Tables 1-2, we compare the performance of the MLEs based on the pre-specified with the MLEs based on the equally spaced inspection times in terms of Bias and RMSE for $n = 25, 50, 100.$ The results of the two tables show the Bias and RMSE values for $m = 10$ are closer to each other for both the pre-specified and equally spaced methods than $m = 5$.
In Tables 3-4, we obtain the equal probability inspection times for different values of the percentage of censoring observations quantity h. Some items of these tables are presented as "-" which represents the situations that the experiment can be terminated only after the failure of all remaining units. It can be seen that some equal probability inspection times are not available for censoring schemes p_1 and p_2 (or p_5 and p_6 for $m = 10$) and $h \leq 0.5$. Moreover, the first equal probability inspection times t_1 is the same for all the censoring schemes and the scheme p_4 has the largest values of the equal probability inspection times among the other scheme for a fixed value of h. Clearly, the values of inspection times are decreasing with the values of the percentage of censoring, h. Tables 5-7 include the optimally spaced inspection times based on criteria I and II for $m = 5, 10$ and $n = 25, 50, 100$ The main observation from these tables is that the first inspection times for criterion I is less than that of criterion II for all the cases.

Next, we consider the numerical results of optimal censoring schemes reported in Tables 8-9. In Table 8, we have reported the optimal censoring schemes for $m = 5$ and in Table 9, we have reported the sub-optimal censoring schemes for $m = 10$. It can be seen that, for both tables, by changing the sample size, the censoring scheme patterns, in general, do not affect. However, from Table 8, the reported censoring schemes for almost all the cases are the same or very close to each other under criteria I and II. Moreover, most of the unites are removed in the first and the last stages. From Table 9, the censoring scheme patterns for both criteria are showed that the units are removed in the $i-th$ stage, $i = 1, 2, 3,$ except for a few cases for $h = 0.3$. Furthermore, to investigate the optimal proportion of the removed units instead of the optimal number, one may consider the expression (7).

Table 1: Bias and RMSE of α and λ using the pre-specified and equally spaced for $m = 5$

n		$\alpha = 0.5$				$\lambda = 0.5$			
		The pre-specified		The equally spaced		The pre-specified		The equally spaced	
	Sch.	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
25	p^1	0.032	0.403	0.010	0.402	0.028	0.393	0.017	0.410
	p^2	0.002	0.377	0.019	0.382	0.019	0.354	0.021	0.387
	p^3	0.005	0.308	0.014	0.326	0.018	0.344	0.005	0.373
	p^4	0.008	0.341	0.003	0.362	0.020	0.355	0.011	0.384
50	p^1	0.013	0.294	0.022	0.286	0.014	0.295	0.019	0.298
	p^2	0.024	0.352	0.004	0.351	0.003	0.308	0.007	0.343
	p^3	0.027	0.236	0.034	0.245	0.014	0.259	0.023	0.276
	p^4	0.024	0.256	0.029	0.264	0.016	0.269	0.024	0.288
100	p^1	0.024	0.221	0.036	0.227	0.026	0.226	0.034	0.241
	p^2	0.002	0.245	0.016	0.251	0.017	0.228	0.020	0.255
	p^3	0.034	0.195	0.038	0.201	0.027	0.210	0.034	0.221
	p^4	0.030	0.203	0.036	0.211	0.026	0.214	0.034	0.229
		$\alpha = 1.5$				$\lambda = 1$			
25	p^1	0.277	1.151	0.318	1.294	0.057	0.440	0.054	0.506
	p^2	0.207	0.997	0.171	1.046	0.011	0.427	0.006	0.506
	p^3	0.206	0.726	0.218	0.912	0.059	0.356	0.061	0.428
	p^4	0.262	0.958	0.512	8.706	0.088	0.385	0.067	0.455
50	p^1	0.122	0.721	0.109	0.649	0.020	0.299	0.017	0.323
	p^2	0.296	1.089	0.193	0.994	0.055	0.373	0.018	0.401
	p^3	0.073	0.456	0.087	0.479	0.026	0.238	0.021	0.261
	p^4	0.010	0.522	0.110	0.567	0.032	0.243	0.022	0.285
100	p^1	0.070	0.414	0.107	0.469	0.019	0.188	0.0277	0.224
	p^2	0.111	0.571	0.097	0.568	0.021	0.233	0.018	0.270
	p^3	0.051	0.300	0.042	0.330	0.018	0.166	0.009	0.192
	p^4	0.049	0.353	0.073	0.387	0.008	0.178	0.022	0.207

Table 2: Bias and RMSE of α and λ using the pre-specified and equally spaced for $m = 100$

n		$\alpha = 0.5$				$\lambda = 0.5$			
		The pre-specified		The equally spaced		The pre-specified		The equally spaced	
	Sch.	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**	**Bias**	**RMSE**
25	p^5	0.078	0.540	0.036	0.418	0.026	0.425	0.013	0.388
	p^6	0.066	0.412	0.008	0.395	0.094	0.387	0.025	0.339
	p^7	0.008	0.316	0.009	0.285	0.022	0.363	0.010	0.326
	p^8	0.003	0.338	0.003	0.316	0.027	0.369	0.009	0.351
50	p^5	0.001	0.338	0.001	0.290	0.020	0.310	0.010	0.272
	p^6	0.015	0.380	0.028	0.354	0.053	0.332	0.010	0.297
	p^7	0.028	0.237	0.032	0.226	0.009	0.266	0.024	0.249
	p^8	0.017	0.250	0.023	0.239	0.005	0.272	0.019	0.253
100	p^5	0.010	0.256	0.020	0.230	0.021	0.242	0.030	0.222
	p^6	0.032	0.343	0.006	0.294	0.010	0.268	0.034	0.240
	p^7	0.036	0.198	0.042	0.199	0.026	0.215	0.038	0.211
	p^8	0.029	0.202	0.035	0.198	0.022	0.215	0.033	0.207
		$\alpha = 1.5$				$\lambda = 0.5$			
25	p^5	0.265	1.152	0.372	1.351	0.027	0.447	0.067	0.452
	p^6	0.036	1.136	0.117	1.090	0.149	0.563	0.055	0.487
	p^7	0.197	0.742	0.245	0.766	0.059	0.350	0.099	0.337
	p^8	0.251	0.864	0.255	0.861	0.070	0.388	0.085	0.355
50	p^5	0.130	0.769	0.127	0.662	0.004	0.315	0.024	0.289
	p^6	0.447	1.756	0.369	1.246	0.016	0.483	0.059	0.380
	p^7	0.091	0.449	0.088	0.425	0.038	0.242	0.031	0.227
	p^8	0.102	0.518	0.107	0.492	0.033	0.248	0.033	0.239
100	p^5	0.082	0.533	0.089	0.455	0.016	0.240	0.024	0.194
	p^6	0.162	0.893	0.166	0.698	0.014	0.337	0.028	0.251
	p^7	0.040	0.310	0.025	0.310	0.017	0.177	0.003	0.181
	p^8	0.049	0.346	0.038	0.345	0.019	0.182	0.008	0.187

Table 3: The equal probability inspection times for $m = 5$

		$(\alpha, \lambda) = (0.5, 0.5)$				
h		t_1	t_2	t_3	t_4	t_5
0.3	p_1	0.372	0.828	-	-	-
	p_2	0.372	1.219	-	-	-
	p_3	0.372	0.684	1.219	2.324	5.302
	p_4	0.372	0.828	1.85	5.302	38.677
0.5	p_1	0.301	0.564	1.174	-	-
	p_2	0.301	0.743	-	-	-
	p_3	0.301	0.489	0.743	1.12	1.738
	p_4	0.301	0.564	0.975	1.738	3.463
0.8	p_1	0.196	0.290	0.417	0.763	2.733
	p_2	0.196	0.336	0.684	1.685	9.873
	p_3	0.196	0.267	0.336	0.409	0.489
	p_4	0.196	0.290	0.384	0.489	0.613
		$(\alpha, \lambda) = (1.5, 1)$				
0.3	p_1	0.426	0.684	-	-	-
	p_2	0.426	0.841	-	-	-
	p_3	0.426	0.615	0.841	1.158	1.682
	p_4	0.426	0.684	1.037	1.682	3.748
0.5	p_1	0.372	0.55	0.825	1.448	8.166
	p_2	0.372	0.644	-	-	-
	p_3	0.372	0.505	0.644	0.805	1.006
	p_4	0.372	0.550	0.748	1.006	1.393
0.8	p_1	0.276	0.362	0.458	0.586	0.779
	p_2	0.276	0.399	0.615	1.277	-
	p_3	0.276	0.343	0.399	0.453	0.505
	p_4	0.276	0.362	0.435	0.505	0.577

Table 4: The equal probability inspection times for $m = 10$

		$(\alpha,\lambda) = (0.5,0.5)$									
h		t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
0.3	p^5	0.250	0.415	-	-	-	-	-	-	-	-
	p^6	0.250	0.511	-	-	-	-	-	-	-	-
	p^7	0.250	0.372	0.511	0.684	0.911	1.219	1.66	2.324	3.396	5.302
	p^8	0.250	0.415	0.622	0.911	1.348	2.069	3.396	6.279	14.625	61.478
0.5	p^5	0.215	0.330	0.501	-	-	-	-	-	-	-
	p^6	0.215	0.390	-	-	-	-	-	-	-	-
	p^7	0.215	0.301	0.390	0.489	0.605	0.743	0.911	1.12	1.388	1.738
	p^8	0.215	0.330	0.455	0.605	0.795	1.045	1.388	1.879	2.622	3.826
0.8	p^5	0.155	0.209	0.271	0.353	0.474	0.675	1.340	-	-	-
	p^6	0.155	0.232	0.372	-	-	-	-	-	-	-
	p^7	0.155	0.196	0.232	0.267	0.301	0.336	0.372	0.409	0.448	0.489
	p^8	0.155	0.209	0.255	0.301	0.348	0.396	0.448	0.504	0.564	0.630
		$(\alpha,\lambda) = (1.5,1)$									
0.3	p^5	0.328	0.457	-	-	-	-	-	-	-	-
	p^6	0.328	0.519	-	-	-	-	-	-	-	-
	p^7	0.328	0.426	.519	0.615	0.721	0.841	0.983	1.158	1.38	1.682
	p^8	0.328	0.457	0.582	0.721	0.886	1.095	1.38	1.809	2.566	4.461
0.5	p^5	0.295	0.395	0.513	-	-	-	-	-	-	-
	p^6	0.295	0.439	-	-	-	-	-	-	-	-
	p^7	0.295	0.372	0.439	0.505	0.573	0.644	0.721	0.805	0.899	1.006
	p^8	0.295	0.395	.483	0.573	0.669	0.776	0.899	1.045	1.225	1.457
0.8	p^5	0.232	0.289	0.346	0.412	0.496	0.61	0.883	-	-	-
	p^6	0.232	0.312	0.426	-	-	-	-	-	-	-
	p^7	0.232	0.276	0.312	0.343	0.372	0.399	0.426	0.453	0.479	0.505
	p^8	0.232	0.289	0.333	0.372	0.408	0.444	0.479	0.514	0.55	0.587

Table 5: The optimally spaced inspection times for $m = 5$

(α, λ)	n		Crit.I					Crit.II				
(0.5, 0.5)	25	p_1	1.7	3.8	7.0	10.4	14.1	1.8	4.7	8.7	12.4	15.2
		p_2	0.4	2.9	4.9	5.6	7.9	2.1	5.2	8.7	12.4	16.0
		p_3	1.1	1.8	3.3	5.9	9.4	1.5	3.2	6.0	9.9	13.6
		p_4	1.5	2.7	4.7	7.5	11.3	1.7	3.4	5.8	9.2	13.2
	50	p_1	1.7	3.8	7.1	10.8	14.0	1.9	4.8	8.7	12.5	16.1
		p_2	0.3	1.3	3.4	5.0	6.4	2.0	5.1	8.8	12.5	16.3
		p_3	1.0	1.7	3.0	5.3	9.0	1.5	3.3	6.0	9.9	13.7
		p_4	0.2	4.0	6.5	9.0	10.0	1.8	3.4	5.9	9.5	13.5
	100	p_1	1.7	3.9	7.2	11.0	14.8	1.8	4.5	8.5	11.7	15.5
		p_2	0.4	2.9	4.8	6.7	8.6	2.1	5.1	8.9	12.8	16.8
		p_3	1.0	1.7	3.1	5.4	9.2	1.4	3.0	5.6	9.4	13.3
		p_4	0.2	3.2	6.0	9.7	12.4	1.7	3.5	6.2	9.8	13.7
(1.5,1)	25	p_1	0.9	3.3	4.4	5.7	6.1	1.5	2.4	3.8	5.5	7.8
		p_2	1.2	3.6	5.7	8.2	8.8	1.7	2.8	3.9	5.4	7.4
		p_3	0.6	2.8	4.6	5.9	7.7	1.2	1.7	2.3	3.5	5.6
		p_4	0.9	2.6	4.3	6.1	6.4	1.5	1.7	2.6	4.0	5.8
	50	p_1	0.9	1.2	4.1	4.7	5.6	1.5	2.5	3.8	5.6	7.8
		p_2	1.2	3.1	5.9	8.4	9.9	1.7	2.8	3.9	5.4	7.6
		p_3	0.6	3.3	4.2	5.5	7.3	1.2	1.6	2.3	3.6	5.6
		p_4	0.9	2.7	4.5	7.6	9.9	1.5	1.8	2.7	4.1	6.3
	100	p_1	0.9	2.3	4.6	6.9	8.0	1.5	2.4	3.8	5.5	7.5
		p_2	1.2	3.4	4.4	5.7	7.4	1.7	2.7	3.9	5.2	7.2
		p_3	0.6	3.3	4.3	5.6	7.3	1.2	1.8	2.4	3.8	5.8
		p_4	0.9	2.9	4.3	6.4	8.7	1.5	2.0	2.7	4.3	6.4

Table 6: The optimally spaced inspection times for $(\alpha,\lambda)=(0.5,0.5)$ and $m=10$

n		Crit	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
25	p^5	I	1.6	3.7	6.6	10.0	13.8	17.7	21.0	23.0	25.1	26.8
		II	1.8	4.6	8.2	12.2	15.9	19.5	22.4	25.0	26.6	28.9
	p^6	I	0.4	3.0	3.9	5.8	7.2	8.8	10.2	11.5	12.6	13.9
		II	2.0	5.1	9.0	12.6	16.2	19.8	21.7	24.2	25.9	27.9
	p^7	I	1.0	1.5	2.2	3.7	5.6	7.3	9.7	12.9	16.6	19.9
		II	1.5	2.7	4.2	6.3	8.7	12.0	15.7	19.2	23.1	26.3
	p^8	I	0.2	3.9	5.9	7.7	9.9	12.4	15.6	18.5	21.7	25.3
		II	1.8	3.1	4.8	6.6	9.3	12.7	16.2	20.1	23.8	27.4
50	p^5	I	1.6	3.7	6.6	10.0	13.8	17.7	21.0	23.0	25.1	26.8
		II	1.8	4.6	8.2	12.2	15.9	19.5	22.4	25.0	26.6	28.9
	p^6	I	0.4	3.0	3.9	5.8	7.2	8.8	10.2	11.5	12.6	13.9
		II	2.0	5.1	9.0	12.6	16.1	19.8	21.7	24.2	25.9	27.9
	p^7	I	1.0	1.5	2.2	3.7	5.6	7.3	9.7	12.9	16.6	19.9
		II	1.5	2.7	4.2	6.3	8.7	12.0	15.7	19.2	23.1	26.3
	p^8	I	0.2	3.9	5.9	7.7	9.9	12.4	15.7	18.5	21.7	25.3
		II	1.8	3.1	4.8	6.6	9.3	12.7	16.2	20.1	23.8	27.4
100	p^5	I	1.6	3.6	6.3	9.6	13.3	17.0	19.9	22.2	24.3	25.9
		II	1.8	4.4	7.9	11.6	15.5	19.0	22.3	23.9	26.9	29.2
	p^6	I	0.4	3.1	4.8	6.7	9.2	10.3	12.3	12.6	13.6	14.3
		II	2.0	5.1	8.8	12.3	16.0	19.4	21.0	22.4	25.3	27.4
	p^7	I	1.0	1.5	2.2	3.5	5.4	7.7	10.3	13.5	16.9	20.7
		II	1.4	2.4	3.8	5.8	8.1	11.1	14.7	18.4	21.6	25.5
	p^8	I	1.5	2.2	3.2	4.7	6.5	8.7	11.6	15.1	18.8	22.4
		II	1.8	2.8	4.2	6.0	8.5	11.2	14.4	18.0	22.0	25.8

Table 7: The optimally spaced inspection times for $(\alpha, \lambda) = (1.5,1)$ and $m = 10$

n		**Crit**	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
25	p_5	I	0.9	3.9	4.7	6.4	7.5	8.3	8.8	10.9	13.2	15.8
		II	1.5	2.3	3.4	4.7	6.3	8.4	10.7	13.4	15.7	17.4
	p_6	I	1.2	3.3	5.2	7.5	10.0	10.1	11.3	12.8	14.0	14.6
		II	1.7	2.8	4.3	5.9	8.0	9.1	10.9	13.1	14.8	16.8
	p_7	I	0.5	2.3	5.5	7.4	8.7	10.9	12.4	14.2	15.1	18.4
		II	1.2	1.5	2.1	2.9	4.2	5.9	7.9	9.5	11.6	13.9
	p_8	I	0.9	2.7	5.2	8.9	11.8	14.7	18.2	19.8	21.6	23.5
		II	1.5	1.9	2.1	2.8	3.8	5.0	6.4	7.5	9.0	11.5
50	p_5	I	0.9	2.8	4.8	7.0	9.0	10.5	11.9	13.1	15.1	18.1
		II	1.5	2.4	3.6	5.1	6.7	8.6	10.2	12.9	14.8	16.6
	p_6	I	1.2	4.3	4.6	5.0	5.7	7.1	8.2	9.6	11.6	13.9
		II	1.7	2.8	4.1	5.7	7.7	9.3	11.6	12.9	14.4	16.0
	p_7	I	0.6	3.6	5.2	7.3	8.9	11.3	12.8	14.6	15.8	17.6
		II	1.2	1.4	2.0	2.7	3.8	5.4	7.3	9.3	11.5	13.9
	p_8	I	0.9	3.7	5.9	7.7	9.9	12.0	14.8	15.6	16.3	18.4
		II	1.5	1.8	2.6	3.5	4.6	6.3	7.9	9.9	11.8	14.5
100	p_5	I	0.9	2.8	3.6	6.4	7.7	10.2	11.0	11.5	12.3	12.9
		II	1.5	2.3	3.5	4.9	6.8	8.9	10.8	13.1	15.3	17.0
	p_6	I	1.2	2.6	4.2	6.6	8.4	9.9	11.4	12.1	13.5	15.0
		II	1.7	2.9	4.3	6.0	7.7	9.3	11.7	14.2	15.9	17.6
	p_7	I	0.6	3.0	5.6	6.7	9.1	11.2	13.5	14.3	17.0	17.2
		II	1.2	1.8	2.3	2.9	3.7	5.2	6.6	8.3	10.6	11.9
	p_8	I	0.9	3.0	4.2	6.5	9.7	11.9	13.5	15.0	16.4	18.6
		II	1.5	1.7	2.3	3.2	4.6	6.2	8.2	10.3	12.6	14.4

Table 8: Optimal censoring schemes under the pre-specified and equal probability inspection times for $m = 5$

(α, λ)		n	h	$CritI = (R_1, R_2, \dots, R_5)$	$CritII = (R_1, R_2, \dots, R_5)$
(0.5, 0.5)	PS	25	0.3	(0 ,0, 0, 0, 7)	(0 ,0, 0, 0,7)
			0.5	(7, 0, 0, 0, 5)	(6 , 0, 0, 2,4)
			0.8	(18,0,0 ,0 ,2)	(17,1 ,0 ,0,2)
		50	0.3	(1 ,0,0, 0,14)	(0 ,0 ,1,1,13)
			0.5	(15,0,0,0,10)	(14 ,1,0,0,10)
			0.8	(36 ,0 ,0 ,0,4)	(35 ,1,0, 0,4)
		100	0.3	(2 ,0,0 ,0,28)	(2,0 ,0 ,0,28)
			0.5	(30,0 ,0,0,20)	(30, 0,0,0,20)
			0.8	(30 ,0,0,0,20)	(30, 0,0,0,20)
	EP	25	0.3	(1, 0,0,0,14)	(0 ,0,1,1,13)
			0.5	(15 , 0, 0 , 0,10)	(14 , 1 , 0, 0 ,10)
			0.8	(36 , 0 , 0 , 0 ,4)	(35 , 1 , 0 , 0 ,4)
		50	0.3	(1 , 0, 0 , 0, 14)	(0 , 0 , 1 , 1, 13)
			0.5	(15 , 0, 0 , 0, 10)	(14 , 1 , 0, 0 ,10)
			0.8	(36 , 0 , 0 , 0 , 4)	(35 , 1 , 0 , 0 ,4)
		100	0.3	(1 , 0, 0 , 0 , 14)	(0 , 0 , 1 , 1, 13)
			0.5	(15 , 0, 0 , 0, 10)	(14 , 1 , 0, 0 ,10)
			0.8	(36 , 0 , 0 , 0 , 4)	(35 , 1 , 0 , 0 ,4)
(1.5,1)	PS	25	0.3	(6 , 0 , 0 , 0 , 1)	(6, 0 , 0, 0 , 1)
			0.5	(11 , 0 , 0 , 1, 0)	(11, 0, 0, 1, 0)
			0.8	(15 , 4 , 1 , 0, 0)	(19, 1, 0, 0, 0)
		50	0.3	(13 , 0 , 0 , 0, 2)	(13, 0, 0, 0, 2)
			0.5	(1, 21, 3 , 0 , 0)	(23, 0, 0, 1, 1)
			0.8	(30, 9, 1 , 0 , 0)	(39, 0, 0, 1, 0)
		100	0.3	(25, 0, 1 , 0 , 4)	(23, 3, 0, 0, 4)
			0.5	(47, 0 , 0 , 0, 3)	(47, 0, 0, 0, 3)
			0.8	(60, 18, 2, 0, 0)	(79, 0, 0 , 0, 1)
	EP	25	0.3	(6, 0 , 0 , 0 , 1)	(6, 0 ,0, 0 ,1)
			0.5	(11, 0 , 0, 0, 1)	(11, 0, 0, 0, 1)
			0.8	(19, 0, 0, 0, 1)	(19, 0, 0, 0, 1)
		50	0.3	(13, 0, 0, 0, 2)	(12, 0, 0, 0, 3)
			0.5	(23, 0, 0, 0, 2)	(23, 0, 0, 0, 2)
			0.8	(39, 0, 0, 0, 1)	(39, 0, 0, 0, 1)
		100	0.3	(26, 0, 0, 0, 4)	(25, 0, 0, 0, 5)
			0.5	(47, 0, 0, 0, 3)	(46, 0, 0, 0, 4)
			0.8	(57, 0, 0, 0, 3)	(57, 0, 0, 0, 3)

Table 9: Optimal censoring schemes under the pre-specified and equal probability inspection

times for $m = 10$

(α,λ)		n	h	$CritI = (R_1, R_2, \ldots, R_5)$	$CritII = (R_1, R_2, \ldots, R_5)$
(0.5, 0.5)	The pre-specified	*25*	0.3	(0, 0 ,0 ,0, 0, 0, 0, 0, 7, 0)	(0, 0, 0, 0, 0, 0, 0, 0, 7, 0)
			0.5	(0, 0, 0, 12, 0, 0, 0, 0, 0, 0)	(0, 0, 0, 12, 0, 0, 0, 0, 0, 0)
			0.8	(0, 20, 0, 0, 0, 0, 0, 0, 0, 0)	(20, 0, 0, 0, 0, 0, 0, 0, 0, 0)
		50	0.3	(0, 0, 0, 0, 0, 0, 15, 0, 0, 0)	(0, 0, 0, 0, 0, 0, 15, 0, 0, 0)
			0.5	(0, 25, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 25, 0, 0, 0, 0, 0, 0, 0)
			0.8	(0, 40, 0, 0, 0, 0, 0, 0, 0, 0)	(40, 0, 0, 0, 0, 0, 0, 0, 0, 0)
		100	0.3	(0, 30, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 30, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(0, 0, 50, 0, 0, 0, 0, 0, 0, 0)	(0, 50, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(0, 80, 0, 0, 0, 0, 0, 0, 0, 0)	(80, 0, 0, 0, 0, 0, 0, 0 , 0 , 0)
	The equal probability	*25*	0.3	(7, 0, 0, 0, 0, 0 ,0 ,0 ,0 ,0)	(0, 7, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(12, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 12, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(20, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 20, 0, 0, 0, 0, 0, 0, 0)
		50	0.3	(15, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 15, 0, 0, 0 , 0 , 0, 0, 0, 0)
			0.5	(25, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 25, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(40, 0, 0, 0, 0, 0, 0,0, 0, 0,)	(0, 0, 40, 0, 0, 0, 0, 0, 0, 0)
		100	0.3	(30, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 30, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(50, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 50 , 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(80, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 80, 0, 0, 0, 0, 0, 0, 0)
		25	0.3	(7, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(7, 0, 0 ,0 ,0 ,0 ,0 ,0 ,0 ,0)
			0.5	(0, 0, 12, 0, 0,0, 0, 0, 0, 0)	(12, 0, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(0, 20, 0, 0, 0, 0, 0, 0, 0, 0)	(20, 0, 0, 0, 0, 0, 0, 0, 0, 0)
(1.5,1)	The pre-specified	*50*	0.3	(15, 0, 0,0, 0, 0, 0, 0, 0, 0)	(15, 0, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(0 , 0, 25, 0, 0, 0, 0,0, 0, 0)	(25, 0, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(0, 40, 0, 0, 0, 0, 0, 0, 0, 0)	(40, 0, 0, 0, 0, 0, 0, 0 , 0 , 0)
		100	0.3	(0, 30, 0, 0, 0, 0, 0,0, 0, 0)	(0, 30, 0, 0, 0, 0 , 0 , 0, 0, 0)
			0.5	(0, 0, 50, 0,0, 0, 0, 0, 0, 0)	(0, 50, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(0, 80, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 80, 0, 0, 0, 0, 0, 0, 0, 0)
	The equal probability	*25*	0.3	(7, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 7, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(12, 0, 0, 0, 0, 0,0, 0, 0, 0)	(0, 12, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(20, 0, 0 , 0, 0 , 0,0 ,0, 0,0)	(0, 0, 20, 0, 0, 0 , 0, 0, 0, 0)
		50	0.3	(15, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 15, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(25, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(25, 0, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(40, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 0, 40, 0, 0, 0, 0, 0, 0, 0)
		100	0.3	(30, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0, 30, 0, 0, 0, 0, 0, 0, 0, 0)
			0.5	(50, 0, 0, 0, 0, 0, 0, 0, 0, 0)	(0 ,50, 0, 0, 0, 0, 0, 0, 0, 0)
			0.8	(80, 0, 0, 0, 0, 0, 0, 0, 0,0)	(0, 0, 80, 0, 0, 0, 0, 0, 0, 0)

6 Concluding remarks

Selecting the inspection times is an important practical issue to improve the efficiency of the obtained estimators. By considering such an issue, we investigate the pre-specified, equally spaced, optimally spaced and equal probability methods to determine the inspection times. The first equal probability inspection times t_1 is the same for all the censoring schemes and the scheme p_4 has the largest values of the equal probability inspection times among the other scheme for a fixed value of h. We notice the optimally spaced inspection times based on criteria I and II for $m = 5,10$ and $n = 25,50,100$. The main observation from these tables is that the first inspection times for criterion I are less than that of criterion II for all the cases. Clearly, the values of inspection times are decreasing with the values of the percentage of censoring h. In regard to optimal censoring, the censoring schemes with most of the removal units are appeared in the first stages (at most the first three stages) is the most preferred ones among the other schemes based on all criteria. However, the considered censoring schemes are almost the same under criteria I and II for almost all cases.

We hope that the methodologies proposed in this work will be useful to applied statisticians. It will be interesting to study the same methodology under hybrid censored data. The work is in the progress and it will be reported later.

References

[1] A. N. S. Taha Anwar Taha, "Comparison Different Estimation Method for Reliability Function of Rayleigh Distribution Based On Fuzzy Lifetime Data," *Iraqi Journal of Science,* pp. 1707-1719, 2022..

[2] A. A. A. M. Abouammoh, "“Reliability estimation of generalized inverted exponential distribution," *Journal of Statistical Computation and Simulation,* p. 1301–1315, 2009..

[3] H. K. K. Krishna, "Reliability estimation in generalized inverted exponential distribution with progressively type ii censored sample," *Journal of Statistical Computation and Simulation,* p. 1007–1019, ,2013.

[4] S. ,. T. Dey, "Generalized inverted exponential distribution: Different methods of estimation," *American Journal of Mathematical and Management Sciences,* p. 194–215, 2014a ..

[5] S. T. Y. M. J. C.-H. Singh, "Sampling plans based on truncated life test for a generalized inverted exponential distribution," *Industrial Engineering and Management Systems,* p. 183–195, 2015.

[6] M. K. H. G. R. Dube, "Generalized inverted exponential distribution under progressive first-failure censoring," *Journal of Statistical Computation and Simulation,* p. 1095–1114, 2016.

[7] C.-T. W. S. J. B. Lin, "Planning life tests with progressively type-i interval censored data from the lognormal distribution," *Journal of Statistical Planning and Inference,* p. 139:54–61, 2009.

[8] R. N. A. M. S. S. Arabi Belaghi, "On estimating the parameters of the burr xii model under progressive type-i interval censoring," *Journal of Statistical Computation and Simulation,* p. 3132–3151, 2017.

[9] S. T. Y. M. Singh, "Estimating the parameters of an inverse weibull distribution under progressive type-i interval censoring," *Statistical Papers,* p. 21–56, 2018.

[10] C. ,. T. Y. M. Lodhi, "“Inference on a progressive type i interval-censored truncated normal distribution,," *Journal of Applied Statistics,* p. 1402–1422, 2020.

[11] R. ,. K. T. Aggarwal, "Estimates using a simultaneous equations model," *Journal of Banking & Finance,* p. 1139–1160, 2001.

[12] R. Aggarwala, "Progressive interval censoring: some mathematical results with applications to inference," *Communications in Statistics-Theory and Methods,,* p. 1921–1935 , 2001 .

[13] R. N. A. M. S. S. Arabi Belaghi, "On estimating the parameters of the burr xii model under progressive type-i interval censoring," *Journal of Statistical Computation and Simulation,* p. 3132–3151.