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Bayesian Estimation for Two Parameters of Gamma Distribution under Generalized Weighted Loss Function

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Abstract
This paper deals with, Bayesian estimation of the parameters of Gamma distribution under Generalized Weighted loss function, based on Gamma and Exponential priors for the shape and scale parameters, respectively. Moment, Maximum likelihood estimators and Lindley's approximation have been used effectively in Bayesian estimation. Based on Monte Carlo simulation method, those estimators are compared in terms of the mean squared errors (MSE's).

Keywords: Gamma distribution; Maximum likelihood estimator; Hessian matrix; Generalized Weighted loss function; Lindley's approximation.

التقدير البيزي لمعلمتي توزيع كاما تحت دالة الخسارة الموزونة المعممة

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الخلاصة
يتناول هذا البحث، التقدير البيزي لمعلمتي توزيع كاما تحت دالة الخسارة الموزونة المعممة، على اساس دالتي أسبقية كاما والأسي لكل من معلمتي القياس والشكل على التوالي. مقدرات العزوم والإمكان الأعظم وتقريب ليندلي تم استخدامها بكفاءة في التقدير البيزي. استناداً الى طريقة مونت كارلو للمحاكاة فإن هذه المقدرات، تمت مقارنتها بالاعتماد على متوسط مربعات الخطأ (MSE's).

1. Introduction

Gamma distribution is extremely important in statistical inferential problems. It is widely used in reliability analysis and as a conjugate prior in Bayesian statistics.

It is a good alternative to the popular Weibull distribution; also, it is a flexible distribution that commonly offers a good fit to any variable such as in environmental, meteorology, climatology and other physical situation [1]. Therefore, the successful estimation of two unknown parameters of Gamma distribution will be very important. Even though the theoretical and practical sides of Gamma distribution, remain poorly studied.

The probability density function of the Gamma distribution is defined as follows

$$f(x;\alpha,\beta)=\frac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)} \quad ; \quad x>0 \ , \quad \alpha>0 \ , \quad \beta>0 \qquad (1)$$

Where, α and β are the shape and scale parameters, respectively.

The Gamma function is defined as

$\Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}e^{-x}\,dx$, for $\alpha>0$

The formula for the cumulative distribution can be written as

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$F(x; \alpha, \beta) = \int_0^x \frac{\beta^\alpha}{\Gamma(\alpha)} u^{\alpha-1} e^{-u\beta} du = \frac{\gamma(\alpha,\beta X)}{\Gamma(\alpha)}$
Where $\gamma(\alpha,\beta X)$ is the lower incomplete gamma function.
Therefore, the reliability function for $\Gamma(\alpha,\beta)$ is

$$R(x;\alpha,\beta) = 1 - F(x;\alpha,\beta) = 1 - \frac{\gamma(\alpha,\ \beta X)}{\Gamma(\alpha)}$$

If α is a positive integer (i.e., the distribution is an Erlang distribution), the cumulative distribution function has the following series expansion

$F(x; \alpha, \beta) = 1 - \sum_{j=0}^{\alpha-1} \frac{(\beta x)^j}{j!} e^{-\beta x} = \sum_{j=\alpha}^{\infty} \frac{(\beta x)^j}{j!} e^{-\beta x}$

And the reliability function will be:[2]

$R(x; \alpha, \beta) = \sum_{j=0}^{\alpha-1} \frac{(\beta x)^j}{j!} e^{-\beta x}$

The maximum likelihood estimators are obtained through Newton-Raphson method as follows [3, 4]

$$\begin{bmatrix}\alpha_{k+1}\\ \beta_{k+1}\end{bmatrix} = \begin{bmatrix}\alpha_k\\ \beta_k\end{bmatrix} - \varphi$$

Where,

$$\varphi = \frac{1}{-n\Psi'(\alpha)\frac{n\alpha}{\beta^2} - (\frac{n}{\beta})^2}\begin{bmatrix} -n\frac{n\alpha}{\beta^2} - n\Psi'(\alpha_k) + n\frac{n\alpha}{\beta^2}\ln\beta_k + \frac{n\alpha}{\beta^2}\sum_{i=1}^{n}\ln x_i - \frac{n(\frac{n}{\beta})\alpha_k}{\beta_k} + n(\frac{n}{\beta})\bar{x} \\ n(\frac{n}{\beta})\Psi'(\alpha_k) - n(\frac{n}{\beta})\ln\beta_k - (\frac{n}{\beta})\sum_{i=1}^{n} \ln x_i + \frac{n - n\Psi'(\alpha)\alpha_k}{\beta_k} - n - n\Psi'(\alpha)\bar{x}\end{bmatrix}$$

The initial value for MLE for α and β are the moment estimators which are given by[5]

$$\hat{\alpha} = \frac{n\bar{x}^2}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

$$\hat{\beta} = \frac{n\bar{x}}{\sum_{i=1}^{n} x_i^2 - n\bar{x}^2}$$

2. Bayesian Estimation

In this section, we obtain some Bayes estimators for α and β based on the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ as the initial values for α and β respectively.

2.1 Posterior Density Functions Using Gamma and Exponential Priors

To estimate α and β parameters for Gamma distribution, we assume that α has a prior $\pi_1(\cdot)$, which follows Gamma(a,b). Also, we assume that, the prior on β is $\pi_2(\cdot)$ and the density function of $\pi_2(\cdot)$ is Exponential and it is independent of $\pi_1(\cdot)$.

$$\pi_1(\alpha) = \begin{cases} \frac{(b)^a(\alpha)^{a-1}e^{-b\alpha}}{\Gamma(a)} & ; \quad a > 0,\ b > 0, \alpha > 0 \\ 0 & \quad 0.w \end{cases} \tag{2}$$

$$\pi_2(\beta) = \begin{cases} c\, e^{-\beta c} & ; \quad c > 0, \beta \geq 0 \\ 0 & ; \quad o.w \end{cases} \tag{3}$$

Where, a, b, c are known parameters. The marginal p.d.f. of $(x_1, x_2, \dots, x_n)$ is given by

$$f(x_1, x_2, \dots, x_n) = \int_0^\infty \int_0^\infty L(x_1, x_2, \dots, x_n; \alpha, \beta)\pi_1(\alpha)\, \pi_2(\beta) d\alpha\, d\beta$$

The joint posterior density functions of α and β is defined as follows

$$h(\alpha, \beta | x_1, x_2, \dots, x_n) = \frac{L(x_1, x_2, \dots, x_n; \alpha, \beta)\, \pi_1(\alpha)\, \pi_2(\beta)}{\int_0^\infty \int_0^\infty L(x_1, x_2, \dots, x_n; \alpha, \beta)\pi_1(\alpha)\, \pi_2(\beta) d\alpha d\beta}$$

$$= \frac{\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \pi_{i=1}^{n} x_i^{\alpha-1} e^{-\beta\sum_{i=1}^{n} x_i}\ \frac{(b)^a(\alpha)^{a-1}e^{-b\alpha}}{\Gamma(a)} c\, e^{-\beta c}}{\int_0^\infty \int_0^\infty \frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n} \pi_{i=1}^{n} x_i^{\alpha-1} e^{-\beta\sum_{i=1}^{n} x_i}\ \frac{(b)^a(\alpha)^{a-1}e^{-b\alpha}}{\Gamma(a)} c\, e^{-\beta c}\ d\alpha\, d\beta}$$

2.2 Bayes Estimators under Generalized weighted loss function

Rasheed (2016) suggested a new loss function which is called the Generalized weighted loss function and introduced as follows [6]

$$L(\widehat{\theta},\theta)=\frac{\left(\sum_{j=0}^{k}a_j\theta^j\right)\left(\hat{\theta}-\theta\right)^2}{\theta^c}\quad,\ \theta>0\,,\qquad a_j\ ;j=1,2,\dots,k\ \ is\ a\ constant$$

k is an positive integer number and c is a constant.
The Risk function is defined as

$$R(\hat{\theta},\theta)=E[L(\hat{\theta},\theta)]=\int_0^\infty\frac{1}{\theta^C}\left(\sum_{j=0}^{k}a_j\theta_j\right)\left(\hat{\theta}-\theta\right)^2h(\theta|\underline{x})d\theta$$

Hence, Bayes estimator using Generalized Weighted Loss function is given as follows [6]

$$\hat{\boldsymbol{\theta}}=\frac{a_0E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right)+a_1E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right)+\cdots+a_KE\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right)}{a_0E\left(\frac{1}{\theta^{C}}|\underline{x}\right)+a_1E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right)+\cdots+a_KE\left(\frac{1}{\theta^{C-K}}|\underline{x}\right)}\tag{4}$$

In general,

$$E[\mathrm{u}(\alpha,\beta)]=\int_0^\infty\int_0^\infty \mathrm{u}(\alpha,\beta)\,\mathrm{h}(\alpha,\beta|\mathrm{x}_1,\dots\mathrm{x}_\mathrm{n})d\alpha\,d\beta$$

Where u(α,β) be any function for α and β. Therefore,

$$E[\mathrm{u}(\alpha,\beta)]=\frac{\int_0^\infty\int_0^\infty \mathrm{u}(\alpha,\beta)\,L(x_1,x_2,\dots,x_n;\alpha,\beta)\pi(\alpha,\beta)\,d\alpha\,d\beta}{\int_0^\infty\int_0^\infty L(x_1,x_2,\dots,x_n;\alpha,\beta)\pi(\alpha,\beta)\,d\alpha\,d\beta}$$

i) Bayesian estimation for α under Generalized Weighted loss function

Assume that, k=1 and c = 0 then, equation (4) becomes as follows:

$$\hat{\alpha}_{10}=\frac{a_0E(\alpha|\underline{x})+a_1E(\alpha^2|\underline{x})}{a_0+a_1E(\alpha|\underline{x})}\tag{5}$$

Firstly, we need to derive $E(\alpha|\underline{x})$ and $E(\alpha^2|\underline{x})$

Assume that, u(α,β) = α

Therefore,

$$E(\alpha|\underline{x})=\frac{\int_0^\infty\int_0^\infty\alpha\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\pi_{i=1}^{n}x_i{}^{\alpha-1}e^{-\beta\sum_{i=1}^{n}x_i}\,(\alpha)^{a-1}e^{-b\alpha}\ e^{-\beta c}\ \mathrm{d}\alpha\,\mathrm{d}\beta}{\int_0^\infty\int_0^\infty\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\pi_{i=1}^{n}x_i{}^{\alpha-1}e^{-\beta\sum_{i=1}^{n}x_i}\,(\alpha)^{a-1}e^{-b\alpha}\ e^{-\beta c}\ d\alpha\,d\beta}$$

Notice that, it is difficult to find the solution of the ratio of two integrals. Therefore, the approximation form Lindley will be used to get $E(\alpha|\underline{x})$ as follows

$$E(\alpha|\underline{x})\approx\hat{\alpha}+\tfrac{1}{2}(\mathrm{u}_{11}\sigma_{11})+p_1\mathrm{u}_1\sigma_{11}+\tfrac{1}{2}(L_{30}\mathrm{u}_1\sigma_{11}^2)+\tfrac{1}{2}(L_{12}\mathrm{u}_1\sigma_{11}\sigma_{22})\tag{6}$$

Where,

$$\mathrm{L_{ij}}=\frac{\partial^{i+j}}{\partial\alpha^i\partial\beta^j}lnL(\alpha,\beta)\qquad;\quad \mathrm{i}\,,\mathrm{j}=0,1,2,3$$

$$lnL(\alpha,\beta)=\mathrm{n}\alpha\ \ln\beta-\mathrm{n}\ln\Gamma(\alpha)-\beta\textstyle\sum_{i=1}^{n}\mathrm{x_i}+(\alpha-1)\sum_{i=1}^{n}\ln\mathrm{x_i}$$

$$\mathrm{L}_{12}=\frac{\partial^3 lnL(\alpha,\beta)}{\partial\alpha\partial\beta^2}=-\frac{\mathrm{n}}{\beta^2}\qquad,\quad L_{21}=\frac{\partial^3\ln L(\alpha,\beta)}{\partial^2\partial\beta}=0$$

$$L_{03}=\frac{\partial^3\ln L(\alpha,\beta)}{\partial\beta^3}=\frac{2n\alpha}{\beta^3}\qquad,\quad L_{30}=\frac{\partial^3\ln L(\alpha,\beta)}{\partial\alpha^3}=-\mathrm{n}\,\Psi''(\alpha)$$

$$L_{20}=\frac{\partial^2\ln L(\alpha,\beta)}{\partial\alpha^2}=-\mathrm{n}\Psi'(\alpha)\quad,\quad L_{02}=\frac{\partial^2\ln L(\alpha,\beta)}{\partial\beta^2}=\frac{-n\alpha}{\beta^2}$$

$$\sigma_{11}=-\frac{1}{L_{20}}=\frac{1}{\mathrm{n}\Psi'(\alpha)}\qquad,\quad\sigma_{22}=-\frac{1}{L_{02}}=\frac{\beta^2}{n\alpha}$$

$$\mathrm{u}_1=\frac{\partial\mathrm{u}(\alpha,\beta)}{\partial\alpha}=1\qquad,\quad\mathrm{u}_{11}=\frac{\partial^2u(\alpha,\beta)}{\partial\alpha^2}=0$$

$u_2 = \frac{\partial u(\alpha,\beta)}{\partial \beta} = 0 \quad , \quad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial \beta^2} = 0$

$p = ln\pi(\alpha,\beta) = (a-1)ln\alpha + alnb - b\alpha - ln\Gamma(\alpha) + lnc - c\beta$

$p_1 = \frac{\partial p}{\partial \alpha} = \frac{a-1}{\alpha} - b \quad , \quad p_2 = \frac{\partial p}{\partial \beta} = -c$

Hence, we can apply Lindley's form (6), as follows

$$E(\alpha|\underline{x}) \approx \hat{\alpha} + \frac{1}{2}(u_{11}\sigma_{11}) + p_1 u_1 \sigma_{11} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2) + \frac{1}{2}(L_{12}u_1\sigma_{11}\sigma_{22})$$
$$\approx \hat{\alpha} + \left(\frac{a-1}{\hat{\alpha}} - b\right)\frac{1}{n\Psi'(\hat{\alpha})} + \frac{1}{2}\left(-n\,\Psi''(\hat{\alpha})\frac{1}{(n\Psi'(\hat{\alpha}))^2}\right) + \frac{1}{2}\left(-\frac{n}{\hat{\beta}^2}\frac{1}{n\Psi''(\hat{\alpha})}\ \frac{\hat{\beta}^2}{n\hat{\alpha}}\right)$$
$$= \hat{\alpha} + \frac{1}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})}{2n\Psi'(\hat{\alpha})} - \frac{1}{2\hat{\alpha}}\right) \quad (7)$$

Now, to get $E(\alpha^2|\underline{x})$, assume that, $u(\alpha,\beta) = \alpha^2$, then

$$E(\alpha^2|\underline{x}) = \frac{\int_0^\infty\int_0^\infty \alpha^2\ \frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\ \pi_{i=1}^n x_i^{\alpha-1} e^{-\beta\sum_{i=1}^n x_i}\ (\alpha)^{a-1}e^{-b\alpha}\ e^{-\beta c}\ d\alpha\, d\beta}{\int_0^\infty\int_0^\infty \frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\ \pi_{i=1}^n x_i^{\alpha-1} e^{-\beta\sum_{i=1}^n x_i}\ (\alpha)^{a-1}e^{-b\alpha}\ e^{-\beta c}\ d\alpha\, d\beta}$$

$u_1 = \frac{\partial u(\alpha,\beta)}{\partial \alpha} = 2\alpha \quad , \quad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial \alpha^2} = 2$

$u_2 = \frac{\partial u(\alpha,\beta)}{\partial \beta} = 0 \quad , \quad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial \beta^2} = 0$

$$E(\alpha^2|\underline{x}) \approx \hat{\alpha}^2 + \frac{1}{2}(u_{11}\sigma_{11}) + p_1 u_1\sigma_{11} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2) + \frac{1}{2}(L_{12}u_1\sigma_{11}\sigma_{22})$$
$$= \hat{\alpha}^2 + \frac{2\hat{\alpha}}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})\hat{\alpha}}{(n\Psi'(\hat{\alpha}))^2}\right) \quad (8)$$

By substituting (7) and (8) into (5) yields,

$$\hat{\alpha}_{10} \approx \frac{a_0\left(\hat{\alpha} + \frac{1}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})}{2n\Psi'(\hat{\alpha})} - \frac{1}{2\hat{\alpha}}\right)\right) + a_1\left(\hat{\alpha}^2 + \frac{2\hat{\alpha}}{n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})\hat{\alpha}}{(n\Psi'(\hat{\alpha}))^2}\right)}{a_0 + a_1\left(\hat{\alpha} + \frac{1}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})}{2n\Psi'(\hat{\alpha})} - \frac{1}{2\hat{\alpha}}\right)\right)}$$

Where $\hat{\alpha}_{10}$ represents the Bayesian estimation for the shape parameter α under Generalized Weighted loss function with assuming that, k=1 and c = 0

Now, another estimator under Generalized Weighted loss function will be derived by letting k=1 and c=1, so, the equation (4) becomes as follows:

$$\hat{\alpha}_{11} = \frac{a_0 + a_1 E(\alpha|\underline{x})}{a_0 E\left(\frac{1}{\alpha}|\underline{x}\right) + a_1} \quad (9)$$

It is clear that, we need $E(\frac{1}{\alpha})$ which can be derived by assuming that, $u(\alpha,\beta) = \frac{1}{\alpha}$, then,

$u_1 = \frac{\partial u(\alpha,\beta)}{\partial \alpha} = -\alpha^{-2} \quad , \quad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial \alpha^2} = 2\alpha^{-3}$

$u_2 = \frac{\partial u(\alpha,\beta)}{\partial \beta} = 0 \quad , \quad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial \beta^2} = 0$

$$E\left(\left(\frac{1}{\alpha}\right)\Big|\underline{x}\right) \approx \frac{1}{\hat{\alpha}} + \frac{1}{2}(u_{11}\sigma_{11}) + p_1 u_1\sigma_{11} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2) + \frac{1}{2}(L_{12}u_1\sigma_{11}\sigma_{22})$$
$$\approx \frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{1}{2}\left(\frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^2(n\Psi'(\hat{\alpha}))^2}\right) + \frac{1}{2}\left(\frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\right) \quad (10)$$

Now, Substituting (7) and (10) into (9) gives,

$$\hat{\alpha}_{11} \approx \frac{a_0 + a_1\left(\hat{\alpha} + \frac{1}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})}{2n\Psi'(\hat{\alpha})} - \frac{1}{2\hat{\alpha}}\right)\right)}{a_0\left(\frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{1}{2}\left(\frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^2(n\Psi'(\hat{\alpha}))^2}\right) + \frac{1}{2}\left(\frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\right)\right) + a_1}$$

When k=1 and c=2, the equation (4) becomes as follows:

$$\hat{\alpha}_{12} \approx \frac{a_0 E\left(\frac{1}{\alpha}|\underline{x}\right) + a_1}{a_0 E\left(\frac{1}{\alpha^2}|\underline{x}\right) + a_1 E\left(\frac{1}{\alpha}|\underline{x}\right)} \quad (11)$$

To derive $E\left(\frac{1}{\alpha^2}|\underline{x}\right)$, assume that

$u(\alpha,\beta) = \frac{1}{\alpha^2}$, then

$$u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = -2\alpha^{-3} \qquad , \qquad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 6\alpha^{-4}$$

$$u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = 0 \qquad , \qquad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 0$$

$$E\left(\frac{1}{\alpha^2}|\underline{x}\right) \approx \frac{1}{\hat{\alpha}^2} + \frac{1}{2}(u_{11}\sigma_{11}) + p_1 u_1\sigma_{11} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2) + \frac{1}{2}(l_{12}u_1\sigma_{11}\sigma_{22})$$

$$\approx \frac{1}{\hat{\alpha}^2} + \frac{3}{\hat{\alpha}^4 n\Psi'(\hat{\alpha})} - \frac{2}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^3\left(n\Psi'(\hat{\alpha})\right)^2} + \frac{1}{\hat{\alpha}^4 n\Psi'(\hat{\alpha})} \qquad (12)$$

Substituting (10), (12) into (11) yields

$$\hat{\alpha}_{12} \approx \frac{a_0\left(\frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{1}{2}\left(\frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^2\left(n\Psi'(\hat{\alpha})\right)^2}\right) + \frac{1}{2}\left(\frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\right)\right) + a_1}{a_0\left(\frac{1}{\hat{\alpha}^2} + \frac{3}{\hat{\alpha}^4 n\Psi'(\hat{\alpha})} - \frac{2}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^3\left(n\Psi'(\hat{\alpha})\right)^2} + \frac{1}{\hat{\alpha}^4 n\Psi'(\hat{\alpha})}\right) + a_1\left(\frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{1}{2}\left(\frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^2\left(n\Psi'(\hat{\alpha})\right)^2}\right) + \frac{1}{2}\left(\frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\right)\right)}$$

When k=2 and c=0 the equation (4) becomes as follows:

$$\hat{\alpha}_{20} = \frac{a_0 E(\alpha|\underline{x}) + a_1 E(\alpha^2|\underline{x}) + a_2 E(\alpha^3|\underline{x})}{a_0 + a_1 E(\alpha|\underline{x}) + a_2 E(\alpha^2|\underline{x})} \qquad (13)$$

Now, $E(\alpha^3|\underline{x})$ should be derived by assume that, $u(\alpha,\beta) = \alpha^3$.

Hence,

$$u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = 3\alpha^2 \qquad , \qquad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 6\alpha$$

$$u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = 0 \qquad , \qquad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 0$$

$$E(\alpha^2|\underline{x}) \approx \hat{\alpha}^3 + \frac{1}{2}(u_{11}\sigma_{11}) + p_1 u_1\sigma_{11} + \frac{1}{2}(L_{30}u_1\sigma_{11}^2) + \frac{1}{2}(L_{12}u_1\sigma_{11}\sigma_{22})$$

$$\approx \hat{\alpha}^3 + \frac{1}{2}\left(6\hat{\alpha}\frac{1}{n\Psi'(\hat{\alpha})}\right) + \left(\frac{a-1}{\hat{\alpha}} - b\right)3\hat{\alpha}^2\frac{1}{n\Psi'(\hat{\alpha})} - \frac{1}{2}\left(n\,\Psi''(\hat{\alpha})\frac{3\hat{\alpha}^2}{\left(n\Psi'(\hat{\alpha})\right)^2}\right) - \frac{1}{2}\left(\frac{n}{\hat{\beta}^2}\frac{3\hat{\alpha}^2}{n\Psi'(\hat{\alpha})}\frac{\hat{\beta}^2}{n\hat{\alpha}}\right)$$

$$\approx \hat{\alpha}^3 + \frac{3\hat{\alpha}}{n\Psi'(\hat{\alpha})} + \frac{3\hat{\alpha}^2}{n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{3\,\hat{\alpha}^2\Psi''(\hat{\alpha})}{2n\left(\Psi'(\hat{\alpha})\right)^2} - \frac{3\hat{\alpha}}{2n\Psi'(\hat{\alpha})} \qquad (14)$$

After substituting, (7), (8) and (14) into (13) yields,

$$\hat{\alpha}_{20} \approx \frac{a_0\delta_1 + a_1\delta_2 + a_2\delta_3}{a_0 + a_1\delta_1 + a_2\delta_2}$$

Where,

$$\delta_1 = \hat{\alpha} + \frac{1}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})}{2n\Psi'(\hat{\alpha})} - \frac{1}{2\hat{\alpha}}\right)$$

$$\delta_2 = \hat{\alpha}^2 + \frac{2\hat{\alpha}}{n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})\hat{\alpha}}{\left(n\Psi'(\hat{\alpha})\right)^2}$$

$$\delta_3 = \hat{\alpha}^3 + \frac{3\hat{\alpha}}{n\Psi'(\hat{\alpha})} + \frac{3\hat{\alpha}^2}{n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{3\,\hat{\alpha}^2\Psi''(\hat{\alpha})}{2n\left(\Psi'(\hat{\alpha})\right)^2} - \frac{3\hat{\alpha}}{2n\Psi'(\hat{\alpha})}$$

Letting k=2 and c=1 the equation (4) will give us

$$\hat{\alpha}_{21} = \frac{a_0 + a_1 E(\alpha|\underline{x}) + a_2 E(\alpha^2|\underline{x})}{a_0 E\left(\frac{1}{\alpha}|\underline{x}\right) + a_1 + a_2 E(\alpha|\underline{x})} \qquad (15)$$

Substituting (7), (8) and (10) into (15) yields,

$$\hat{\alpha}_{21} \approx \frac{a_0 + a_1\,\delta_1 + a_2\delta_2}{a_0\left(\frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{1}{2}\left(\frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^2\left(n\Psi'(\hat{\alpha})\right)^2}\right) + \frac{1}{2}\left(\frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\right)\right) + a_1 + a_2\delta_1}$$

When K=2 and c=2 the equation (4) becomes as follows:

$$\hat{\alpha}_{22} = \frac{a_0 E\left(\frac{1}{\alpha}|\underline{x}\right) + a_1 + a_2 E(\alpha|\underline{x})}{a_0 E\left(\frac{1}{\alpha^2}|\underline{x}\right) + a_1 E\left(\frac{1}{\alpha}|\underline{x}\right) + a_2} \qquad (16)$$

Substituting (7), (10) and (12) into (16) yields,

$$\hat{\alpha}_{22} \approx \frac{a_0 W_2 + a_1 + a_2\left(\hat{\alpha} + \frac{1}{n\Psi'(\hat{\alpha})}\left(\left(\frac{a-1}{\hat{\alpha}} - b\right) - \frac{n\,\Psi''(\hat{\alpha})}{2n\Psi'(\hat{\alpha})} - \frac{1}{2\hat{\alpha}}\right)\right)}{a_0 W_1 + a_1 W_2 + a_2}$$

Where,

$$W_1 = \frac{1}{\hat{\alpha}^2} + \frac{3}{\hat{\alpha}^4 n\Psi'(\hat{\alpha})} - \frac{2}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^3\left(n\Psi'(\hat{\alpha})\right)^2} + \frac{1}{\hat{\alpha}^4 n\Psi'(\hat{\alpha})}$$

$$W_2 = \frac{1}{\hat{\alpha}} + \frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})} - \frac{1}{\hat{\alpha}^2 n\Psi'(\hat{\alpha})}\left(\frac{a-1}{\hat{\alpha}} - b\right) + \frac{1}{2}\left(\frac{n\,\Psi''(\hat{\alpha})}{\hat{\alpha}^2\left(n\Psi'(\hat{\alpha})\right)^2}\right) + \frac{1}{2}\left(\frac{1}{\hat{\alpha}^3 n\Psi'(\hat{\alpha})}\right)$$

ii) Bayesian estimation for β under Generalized Weighted loss function

Similarly, some Bayesian estimators for β can be obtained under Generalized Weighted loss function by assuming that, k = 1, 2 and c = 0, 1, 2 as follows

When K=1 and c = 0 the equation (4) becomes as follows

$$\hat{\beta}_{10} = \frac{a_0 E(\beta|\underline{x}) + a_1 E(\beta^2|\underline{x})}{a_0 + a_1 E(\beta|\underline{x})} \tag{17}$$

To find $E(\beta|\underline{x})$ and $E(\beta^2|\underline{x})$

Let, $u(\alpha,\beta) = \beta$, then

$$u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = 0 \quad , \quad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 0$$

$$u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = 1 \quad , \quad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 0$$

$$E(\beta|\underline{x}) \approx \hat{\beta} + p_2 u_2 \sigma_{22} + \frac{1}{2}(L_{03} u_2 \sigma_{22}^2)$$

$$\approx \hat{\beta} - \frac{c\hat{\beta}^2}{n\hat{\alpha}} + \frac{\hat{\beta}}{n\hat{\alpha}} \tag{18}$$

When $u(\alpha,\beta) = \beta^2$

$$u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = 0 \quad , \quad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 0$$

$$u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = 2\beta \quad , \quad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 2$$

Thus, $E(\beta^2|\underline{x}) \approx \hat{\beta}^2 + \frac{1}{2}(u_{22}\sigma_{22}) + p_2 u_2 \sigma_{22} + \frac{1}{2}(L_{03} u_2 \sigma_{22}^2)$

$$\approx \hat{\beta}^2 + \frac{1}{2}\left(\frac{2\hat{\beta}^2}{n\hat{\alpha}}\right) + \left(-c\,\frac{2\hat{\beta}^3}{n\hat{\alpha}} + \frac{1}{2}\left(\frac{2n\hat{\alpha}}{\hat{\beta}^3}\frac{2\hat{\beta}^5}{n\hat{\alpha}}\right)\right.$$

$$\approx \hat{\beta}^2 + \frac{3\hat{\beta}^2}{n\hat{\alpha}} - \frac{2c\hat{\beta}^3}{n\hat{\alpha}} \tag{19}$$

Where $\hat{\alpha}, \hat{\beta}$ are the maximum likelihood estimators

Substituting (18), (19) into (17) yields

$$\hat{\beta}_{10} = \frac{a_0\left(\hat{\beta} - \frac{c\hat{\beta}^2}{n\hat{\alpha}} + \frac{\hat{\beta}}{n\hat{\alpha}}\right) + a_1\left(\hat{\beta}^2 + \frac{3\hat{\beta}^2}{n\hat{\alpha}} - \frac{2c\hat{\beta}^3}{n\hat{\alpha}}\right)}{a_0 + a_1\left(\hat{\beta} - \frac{c\hat{\beta}^2}{n\hat{\alpha}} + \frac{\hat{\beta}}{n\hat{\alpha}}\right)}$$

When K=1 and c=1 the equation (4) becomes as follows

$$\hat{\beta}_{11} = \frac{a_0 + a_1 E(\beta|\underline{x})}{a_0 E\left(\frac{1}{\beta}|\underline{x}\right) + a_1} \tag{20}$$

To find $E\left(\frac{1}{\beta}|\underline{x}\right)$, assume that

$$u(\alpha,\beta) = \frac{1}{\beta} \text{ then,} \quad u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = 0 \quad , \quad u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 0,$$

$$u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = -\beta^{-2} \quad , \quad u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 2\beta^{-3}$$

Thus, $E\left(\frac{1}{\beta}|\underline{x}\right) \approx \frac{1}{\hat{\beta}} + \frac{1}{2}(u_{22}\sigma_{22}) + p_2 u_2 \sigma_{22} + \frac{1}{2}(L_{03} u_2 \sigma_{22}^2) + \frac{1}{2}(L_{21} u_2 \sigma_{11}\sigma_{22})$

$$\approx \frac{1}{\hat{\beta}} + \frac{1}{2}\left(\frac{2\hat{\beta}^2}{\hat{\beta}^3 n\hat{\alpha}}\right) + \frac{c\hat{\beta}^2}{\hat{\beta}^2 n\hat{\alpha}} + \frac{1}{2}\left(\frac{-2n\hat{\alpha}}{\hat{\beta}^3}\,\frac{1}{\hat{\beta}^2}\,\frac{\hat{\beta}^4}{n^2\hat{\alpha}^2}\right)$$

$$= \frac{1}{\hat{\beta}} + \frac{c}{n\hat{\alpha}} \tag{21}$$

Where $\hat{\alpha}, \hat{\beta}$ are the maximum likelihood estimators for α and β respectively, Substituting (18) and (21) into (20) gives us,

$$\hat{\beta}_{11} \approx \frac{a_0+a_1\left(\hat{\beta}-\frac{c\hat{\beta}^2}{n\hat{\alpha}}+\frac{\hat{\beta}}{n\hat{\alpha}}\right)}{a_0\left(\frac{1}{\hat{\beta}}+\frac{c}{n\hat{\alpha}}\right)+a_1}$$

When k=1 and c=2 the equation (4) becomes as follows:

$$\hat{\beta}_{12} = \frac{a_0E\left(\frac{1}{\beta}|\underline{x}\right)+a_1}{a_0E\left(\frac{1}{\beta^2}|\underline{x}\right)+a_1E\left(\frac{1}{\beta}|\underline{x}\right)} \quad (22)$$

To obtain $\left(\frac{1}{\beta^2}|\underline{x}\right)$, assume that,

$u(\alpha,\beta) = \frac{1}{\beta^2}$, then

$u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = 0$, $u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 0$

$u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = -2\beta^{-3}$, $u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 6\beta^{-4}$

$$E\left(\frac{1}{\beta^2}|\underline{x}\right) \approx \frac{1}{\hat{\beta}^2}+\frac{1}{2}(u_{22}\sigma_{22})+p_2u_2\sigma_{22}+\frac{1}{2}(L_{03}u_2\sigma_{22}^2)+\frac{1}{2}(L_{21}u_2\sigma_{11}\sigma_{22})$$

$$\approx \frac{1}{\hat{\beta}^2}+\frac{1}{2}\left(\frac{6\hat{\beta}^2}{\hat{\beta}^4 n\hat{\alpha}}\right)+\frac{2c\hat{\beta}^2}{\hat{\beta}^3 n\hat{\alpha}}+\frac{1}{2}\left(\frac{2n\hat{\alpha}}{\hat{\beta}^3}\frac{-2}{\hat{\beta}^3}\frac{\hat{\beta}^4}{n^2\hat{\alpha}^2}\right)$$

$$\approx \frac{1}{\hat{\beta}^2}+\frac{2c}{\hat{\beta}n\hat{\alpha}}+\frac{1}{\hat{\beta}^2 n\hat{\alpha}} \quad (23)$$

After substituting (21) and (23) into (22) we get

$$\hat{\beta}_{12} \approx \frac{a_0\left(\frac{1}{\hat{\beta}}+\frac{c}{n\hat{\alpha}}\right)+a_1}{a_0\left(\frac{1}{\hat{\beta}^2}+\frac{2c}{\hat{\beta}n\hat{\alpha}}+\frac{1}{\hat{\beta}^2 n\hat{\alpha}}\right)+a_1\left(\frac{1}{\hat{\beta}}+\frac{c}{n\hat{\alpha}}\right)}$$

When k=2 and c=0 the equation (4) becomes as follows

$$\hat{\beta}_{20} = \frac{a_0E(\beta|\underline{x})+a_1E(\beta^2|\underline{x})+a_2E(\beta^3|\underline{x})}{a_0+a_1E(\beta|\underline{x})+a_2E(\beta^2|\underline{x})} \quad (24)$$

To derive $E(\beta^2|\underline{x})$, Let, $u(\alpha,\beta) = \beta^3$ then,

$u_1 = \frac{\partial u(\alpha,\beta)}{\partial\alpha} = 0$, $u_{11} = \frac{\partial^2 u(\alpha,\beta)}{\partial\alpha^2} = 0$, $u_2 = \frac{\partial u(\alpha,\beta)}{\partial\beta} = 3\beta^2$, $u_{22} = \frac{\partial^2 u(\alpha,\beta)}{\partial\beta^2} = 6\beta$

Thus, $E(\beta^3|\underline{x}) \approx \hat{\beta}^3+\frac{1}{2}(u_{22}\sigma_{22})+p_2u_2\sigma_{22}+\frac{1}{2}(L_{03}u_2\sigma_{22}^2)$

$$\approx \hat{\beta}^3+\frac{1}{2}\left(\frac{6\hat{\beta}^3}{n\hat{\alpha}}\right)+\left(-c\,\frac{3\hat{\beta}^4}{n\hat{\alpha}}\right)+\frac{1}{2}\left(\frac{2n\hat{\alpha}}{\hat{\beta}^3}\frac{3\hat{\beta}^6}{(n\hat{\alpha})^2}\right)$$

$$\approx \hat{\beta}^3+\frac{6\hat{\beta}^3}{n\hat{\alpha}}-\frac{3c\hat{\beta}^3}{n\hat{\alpha}} \quad (25)$$

Substituting, (18), (19) and (25) into (24) yields,

$$\hat{\beta}_{20} \approx \frac{a_0\left(\hat{\beta}-\frac{c\hat{\beta}^2}{n\hat{\alpha}}+\frac{\hat{\beta}}{n\hat{\alpha}}\right)+a_1\left(\hat{\beta}^2+\frac{3\hat{\beta}^2}{n\hat{\alpha}}-\frac{2c\hat{\beta}^3}{n\hat{\alpha}}\right)+a_2\left(\hat{\beta}^3+\frac{6\hat{\beta}^3}{n\hat{\alpha}}-\frac{3c\hat{\beta}^3}{n\hat{\alpha}}\right)}{a_0+a_1\left(\hat{\beta}-\frac{c\hat{\beta}^2}{n\hat{\alpha}}+\frac{\hat{\beta}}{n\hat{\alpha}}\right)+a_2\left(\hat{\beta}^2+\frac{3\hat{\beta}^2}{n\hat{\alpha}}-\frac{2c\hat{\beta}^3}{n\hat{\alpha}}\right)}$$

When K=2 and c=1 the equation (4) becomes as follows

$$\hat{\beta}_{21} = \frac{a_0+a_1E(\beta|\underline{x})+a_2E(\beta^2|\underline{x})}{a_0E\left(\frac{1}{\beta}|\underline{x}\right)+a_1+a_2E(\beta|\underline{x})} \quad (26)$$

Substituting (18), (19) and (21) into (26) gives

$$\hat{\beta}_{21} = \frac{a_0+a_1\left(\hat{\beta}-\frac{c\hat{\beta}^2}{n\hat{\alpha}}+\frac{\hat{\beta}}{n\hat{\alpha}}\right)+a_2\left(\hat{\beta}^2+\frac{3\hat{\beta}^2}{n\hat{\alpha}}-\frac{2c\hat{\beta}^3}{n\hat{\alpha}}\right)}{a_0\left(\frac{1}{\hat{\beta}}+\frac{c}{n\hat{\alpha}}\right)+a_1+a_2\left(\hat{\beta}-\frac{c\hat{\beta}^2}{n\hat{\alpha}}+\frac{\hat{\beta}}{n\hat{\alpha}}\right)}$$

When K=2 and c=2 the equation (4) becomes as follows

$$\hat{\beta}_{22} = \frac{a_0E\left(\frac{1}{\beta}|\underline{x}\right)+a_1+a_2E(\beta|\underline{x})}{a_0E\left(\frac{1}{\beta^2}|\underline{x}\right)+a_1E\left(\frac{1}{\beta}|\underline{x}\right)+a_2} \quad (27)$$

Substituting (18),(21) and (23) into (27) yields

$$\hat{\beta}_{22} \; \frac{a_0\left(\frac{1}{\hat{\beta}}+\frac{c}{n\hat{\alpha}}\right)+a_1+a_2\left(\hat{\beta}-\frac{c\hat{\beta}^2}{n\hat{\alpha}}+\frac{\hat{\beta}}{n\hat{\alpha}}\right)}{a_0\left(\frac{1}{\hat{\beta}^2}+\frac{2c}{\hat{\beta}n\hat{\alpha}}+\frac{1}{\hat{\beta}^2 n\hat{\alpha}}\right)+a_1\left(\frac{1}{\hat{\beta}}+\frac{c}{n\hat{\alpha}}\right)+a_2}$$

4. Simulation study

In this section, Monte – Carlo simulation is employed to compare the performance of different estimates, for unknown shape and scale parameters of gamma distribution based on the mean squared errors (MSE's), where,

MSE($\hat{\theta}$) = $\frac{\sum_{i=1}^{I}(\hat{\theta}_i-\theta)^2}{I}$, I is the number of replications.

We generated I = 3000 samples of size n = 20, 30, 50, and 100 to represent small, moderate and large sample sizes from Gamma distribution with α = 2, 3 and β=0.5, 1. The values of α's prior parameters are chosen as a = 3 , b=3 and for β prior parameter is c =4.

QB language has been employed for Monte-Carlo simulation study.

5. The results and discussions

The results for Monte-Carlo simulation study are summarized in Tables-(1-8), which can be summarized by the following points

1.It is clear that, the results for α (expected values and MSE) at β = 0.5 are the same as the corresponding results when β = 1,we can clarify the reason easily, according to moment method we have

$$E(X) = \frac{\sum_{i=1}^{n} x_i}{n}$$
$$= \bar{x} = \frac{\alpha}{\beta}$$

$$\alpha = \frac{\beta}{n}\sum_{i=1}^{n} x_i \qquad (28)$$

Note that, $x_1, x_2, \dots, x_n$ is a random sample from a Gamma distribution defined by (1), and each observation say x is generated by the following equation

$x = \sum_{j=1}^{\alpha} \frac{-1}{\beta} \log(u_j)$

Where, u_j is a random number followed uniform distribution with (0,1), i.e.,

$u_j \sim U(0,1)$

And, according to Monte-Carlo simulation,

e = $\frac{-1}{\beta}\log\left(u_j\right)$

is a random number from Exponential distribution with parameter β. Therefore,

$$g = \sum_{j=1}^{\alpha} \frac{-1}{\beta} \log(u_j) \qquad (29)$$

is a random variable follows the Gamma distribution defined by (1).

After substitute (28) into (29) yields, $\alpha = \frac{\beta}{n}\sum_{i=1}^{n}\sum_{j=1}^{\alpha}\frac{-1}{\beta}\log(u_j)$

Therefore, β will be canceled.

Recall that, the moment estimator for α is the initial value for the corresponding MLE and then, Bayesian estimators for α are depending on MLE for α. Therefore the results for expected values and MSE's for α are the same.

2. For estimatingα, the Bayesian estimators under Generalized Weighted loss function with applying $(k = 2, c = 0)$ $(\hat{\alpha}_{20})$ is the best, except with α=2 when n > 30, it is clear that, Bayesian estimators under Generalized Weighted loss function when $(k = 1, c = 2)$ $(\hat{\alpha}_{12})$ is the best. See Tables-(2, 4)

3.Generally, the best estimator for β is Bayesian estimation under Generalized Weighted loss function when $(k = 2, c = 0)$ $(\hat{\beta}_{20})$ for different cases and with all sample sizes.

Table 1-The expected values for different estimators for shape parameter α of Gamma distribution when α = 2

n	MO	MLE	K = 1			K = 2		
			C = 0	C = 1	C = 2	C = 0	C = 1	C = 2
20	2.486393	2.33479	2.13762	2.12004	2.10828	2.12041	2.15395	2.131833
30	2.298321	2.194657	2.07743	2.06011	2.04625	2.07635	2.09202	2.072198
50	2.183145	2.118412	2.05275	2.04011	2.02898	2.05529	2.06273	2.049095
100	2.090724	2.055311	2.02434	2.01728	2.01062	2.02652	2.02966	2.022333

Table 2- The MSE values for different estimators for shape parameter α of Gamma distribution when α = 2

n	MO	MLE	K = 1			K = 2		
			C = 0	C = 1	C = 2	C = 0	C = 1	C = 2
20	1.13161	0.80765	0.52825	0.54109	0.55376	0.47970	0.52112	0.53284
30	0.58915	0.38833	0.29378	0.29471	0.29654	0.28495	0.29499	0.29470
50	0.29714	0.18510	0.15543	0.15461	0.15426	0.15439	0.15683	0.15552
100	0.13609	0.08313	0.07634	0.07601	0.07581	0.07633	0.07680	0.07636

Table 3- The expected values for different estimators for shape parameter α of Gamma distribution when α = 3

n	MO	MLE	K = 1			K = 2		
			C = 0	C = 1	C = 2	C = 0	C = 1	C = 2
20	3.600494	3.447432	3.08889	3.08789	3.090414	3.02790	3.09277	3.08840
30	3.405721	3.299321	3.07678	3.06603	3.058438	3.05870	3.08746	3.07402
50	3.255532	3.18809	3.06179	3.05092	3.041651	3.05897	3.07131	3.05919
100	3.126059	3.089527	3.02947	3.02248	3.015975	3.03049	3.03522	3.02787

Table 4- The MSE values for different estimators for shape parameter α of Gamma distribution when α = 3

n	MO	MLE	K = 1			K = 2		
			C = 0	C = 1	C = 2	C = 0	C = 1	C = 2
20	2.06203	1.62539	1.03603	1.07908	1.11537	0.91934	1.00022	1.04861
30	1.11338	0.84883	0.62347	0.63193	0.64002	0.59647	0.61720	0.62559
50	0.61598	0.44923	0.37099	0.37154	0.37241	0.36601	0.37100	0.37115
100	0.25924	0.18543	0.16823	0.16798	0.16785	0.16780	0.16856	0.16819

Table 5-The expected values for different estimators for shape parameter β of Gamma distribution when β = 0.5

n	MO		MLE		K = 1						K = 2					
					C = 0		C = 1		C = 2		C = 0		C = 1		C = 2	
	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3
20	0.6380	0.6106	0.5987	0.5846	0.5803	0.5732	0.5691	0.5654	0.5597	0.5584	0.5913	0.5804	0.5790	0.5721	0.5680	0.5644
30	0.5846	0.5737	0.5583	0.5556	0.5478	0.5488	0.5399	0.5434	0.5329	0.5385	0.5548	0.5534	0.5465	0.5479	0.5388	0.5426
50	0.5513	0.5454	0.5351	0.5342	0.5294	0.5304	0.5246	0.5272	0.5200	0.5241	0.5335	0.5331	0.5285	0.5298	0.5237	0.5266
100	0.5247	0.5226	0.5159	0.5166	0.5133	0.5148	0.5108	0.5132	0.5084	0.5116	0.5152	0.5161	0.5127	0.5145	0.5103	0.5128

Table 6- The MSE values for different estimators for shape parameter β of Gamma distribution when β = 0.5

n	MO		MLE		K = 1						K = 2					
					C = 0		C = 1		C = 2		C = 0		C = 1		C = 2	
	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3
20	0.0813	0.0647	0.0581	0.0512	0.0586	0.0500	0.0570	0.0489	0.0558	0.0480	0.0623	0.0521	0.0599	0.0507	0.0581	0.0496
30	0.0430	0.0370	0.0303	0.0295	0.0279	0.0259	0.0272	0.0254	0.0266	0.0249	0.0294	0.0268	0.0284	0.0262	0.0277	0.0256
50	0.0221	0.0192	0.0151	0.0146	0.0139	0.0140	0.0136	0.0138	0.0134	0.0136	0.0144	0.0144	0.0141	0.0141	0.0138	0.0139
100	0.0096	0.0081	0.0064	0.0060	0.0066	0.0061	0.0065	0.0061	0.0065	0.0060	0.0067	0.0062	0.0066	0.0062	0.0066	0.0061

Table 7-The expected values for different estimators for shape parameter β of Gamma distribution when β = 1

n	MO		MLE		K = 1						K = 2					
					C = 0		C = 1		C = 2		C = 0		C = 1		C = 2	
	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3
20	1.2761	1.2212	1.1974	1.1693	1.0978	1.1063	1.0842	1.0944	1.0750	1.0851	1.1250	1.1270	1.1046	1.1118	1.0891	1.0991
30	1.1691	1.1474	1.1166	1.1112	1.0572	1.0725	1.0452	1.0634	1.0357	1.0556	1.0783	1.0872	1.0629	1.0766	1.0499	1.0670
50	1.1026	1.1474	1.0703	1.1112	1.0371	1.0466	1.0287	1.0406	1.0213	1.0352	1.0505	1.0556	1.0408	1.0491	1.0320	1.0430
100	1.0495	1.0451	1.0318	1.0332	1.0162	1.0228	1.0116	1.0196	1.0073	1.0166	1.0230	1.0274	1.0181	1.0241	1.0133	1.0209

Table 8-The MSE values for different estimators for shape parameter β of Gamma distribution when β = 1

n	MO		MLE		K = 1						K = 2					
					C = 0		C = 1		C = 2		C = 0		C = 1		C = 2	
	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3	α=2	α=3
20	1.3001	1.0351	0.9289	0.8188	0.1905	0.1753	0.1925	0.1744	0.1950	**0.1740**	0.1922	0.1802	0.1917	0.1776	0.1927	0.1760
30	0.6878	0.5922	0.4841	0.4713	0.0972	0.0943	0.0967	0.0933	0.0966	**0.0926**	0.1010	0.0976	0.0991	0.0958	0.0980	0.0946
50	0.3539	0.3074	0.2412	0.2329	0.0510	0.0529	0.0505	0.0524	**0.0502**	0.0520	0.0528	0.0543	0.0519	0.0536	0.0512	0.0530
100	0.1530	0.1293	0.1026	0.0966	0.0253	0.0238	0.0251	0.0236	0.0250	**0.0235**	0.0258	0.0242	0.0255	0.0240	0.0253	0.0238

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