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Numerical Approximations of a One-Dimensional Time-Fractional Semilinear Parabolic Equation

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Abstract

The time fractional order differential equations are fundamental tools that are used for modeling neuronal dynamics. These equations are obtained by substituting the time derivative of order α, where $0 < \alpha < 1$, in the standard equation with the Caputo fractional formula. In this paper, two implicit difference schemes: the linearly Euler implicit and the Crank-Nicolson (CN) finite difference schemes, are employed in solving a one-dimensional time-fractional semilinear equation with Dirichlet boundary conditions. Moreover, the consistency, stability and convergence of the proposed schemes are investigated. We prove that the IEM is unconditionally stable, while CNM is conditionally stable. Furthermore, a comparative study between these two schemes will be conducted via numerical experiments. The efficiency of the proposed schemes in terms of absolute errors, order of accuracy and computing time will be reported and discussed.

Keywords: Fractional order equation**,** Caputo fractional formula**,** Finite difference shames, Semilinear parabolic equation, Implicit Euler scheme, Crank-Nicolson method.

التقريبات العددية لمعادلة قطع مكافئ شبه خطية أحادية البعد كسرية بالنسبه للزمن

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الخلاصة

تعتبر المعادلات التفاضلية الكسرية بالنسبة للزمن أدوات أساسية تستخدم لنمذجة ديناميات الخلايا العصبية. يتم الحصول على هذه المعادلات عن طريق استبدال المشتقة الزمنية للرتبة α ، حيث $0 < \alpha < 1,$ ، في المعادلة القياسية مع صيغة Caputo الكسرية. في هذا البحث ، تم استخدام طريقتين ضمنتين للفرقات المنتهية: وهما طريقة أويلر الخطية الضمنية وطريقة Crank–Nicolson ، لحل معادلة شبه خطية جزئية أحادية البعد كسرية بالنسبة للزمن مع شروط Dirichlet الحدودية. علاوة على ذلك ، تم التحقق من اتساق واستقرار وتقارب الطرق المقترحة. نقوم باثبات أن طريقة اويلر الضمنية مستقرة دون قيد أو شرط ، في حين أن طريقة Crank–Nicolsonمستقرة بشكل مشروط. علاوة على ذلك ، سيتم إجراء دراسة مقارنة بين هذين الطريقتين من خلال

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التجارب العددية. تم التحري ومناقشة كفاءة الطرق المقترحة من حيث الأخطاء المطلقة ودرجة الدقه والوقت المطلوب للحسابات العددية.

1. Introduction

Fractional calculus is a field of mathematics that is concerned with the properties of both derivatives and integrals of non-integer orders. In fact, this field focuses on solving time-dependent fractional differential equations (PDEs involving fractional derivatives). The study of fractional calculus was established when classical calculus started.

Since the last century, fractional calculus was built on important foundations by many researchers, such as Heaviside, Lagrange Riemann, Liouville, Gr¨unwald, Euler, Fourier, Abel etc. see for instance [1]

Nowadays, fractional calculus has become very popular and it has many applications, due to the fact that the so called differintegral which is an operator that includes both integer-order derivatives and integrals as special cases. The fractional integral may be used for describing the cumulation of some quantity, when the order of integration is unknown, it can be determined as a parameter of a regression model as Podlubn´y presents in [2,3]. Moreover, the fractional derivative is sometimes used for describing damping. In addition, other applications can be occurred in: the control theory of dynamical systems, optics and signal processing ,fluid flow, diffusive transport akin to diffusion, probability and statistics, viscoelasticity, electrical networks, dynamical processes in self-similar and porous structures, electrochemistry of corrosion, rheology, etc.

Since the last decades, there are various analytical and numerical techniques have been employed in solving fractional differential equations. Analytical methods include Fourier and Laplace transformations, and the Green function method [4-6]. However, most fractional differential equations cannot be solved analytically. Therefore, it is essential to develop numerical schemes for solving these equations. Many effective methods, such as the finite difference method, spectral method and finite element method, have been used to solve time (space) fractional differential equations, see for instance [7-9]. In fact, many authors have focused on numerical solutions of linear types of time (space)-fractional differential equations, see for instance [9-16], whereas other semilinear or nonlinear types have been considered by only a few authors, see for instance [17-21]. However, they are still in the early stage of research.

This paper is concerned with the numerical solutions of a one-dimensional time fractional semilinear parabolic equation, using finite difference schemes. Namely, two numerical approximations are proposed: linearly implicit Euler scheme, and Crank-Nicolson scheme.

The aim of this paper is to show that the proposed schemes are consistent, stable and convergent. Moreover, they can efficiently be used to find the numerical solutions of the governed equation.

The rest of this paper is organized as follows: Section two presents the mathematical formulation of the governed problem. In section three the derivation, consistency, stability and convergent of the two proposed finite difference schemes are considered. Two numerical experiments are studied in section four. Finally, some conclusions are given in the last section.

2. Mathematical Formulation

We consider a one-dimensional time-fractional semilinear diffusion equation with Dirichlet boundary conditions:

$$u_t^{\alpha} = u_{xx} + f(x,t,u), 0 < x < 1, \quad t \in (0,T) \tag{1}$$

$$u(0,t) = a(t), \quad u(1,t) = b(t), 0 < t < T \tag{2}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1 \tag{3}$$

where $a, b \in C(R), 0 \le u_0$, $f \in C^1([0,1] \times (0,T) \times [0,\infty))$ and satisfies Lipshtiz condition with respect to u:

$$|f(x,t,u_1) - f(x,t,u_2)| \le L\,|u_1 - u_2|, \ \forall\, u_1, u_2 \ge 0, L > 0 \tag{4}$$

And $\alpha \in (0,1)$, is the order of the time fractional derivative in Caputo sense [11, 22], which takes the form:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \frac{\partial \xi}{(t-\xi)^{\alpha}} \tag{5}$$

In fact, equation (1) has many applications, such as it is used to describe transport processes with long memory, where the rate of diffusion is inconsistent with the classical Brownian motion model [9].

The existence, uniqueness and stability of problem (1)-(3) can be guaranteed by some references, see for instance [23, 24]. Throughout this paper, we assume that the solution to problem (1)-(3) is positive for nonzero time values.

3. Numerical finite difference schemes

In this section, two numerical approximations of problem (1)-(3) are proposed. Namely, the linearly implicit Euler scheme, and the Crank-Nicolson scheme.

In this segment, the grid dimensions in relation to space and time for the positive integers I and N are respectively represented by $h = \frac{1}{I}$ and $k = \frac{T}{N}$ .The grid point in the space interval [0,1] is denoted $x_i = ih$, $i = 0,1,2,\ldots,I$ and the grid points for time are designated $t_n = nk$, $n = 0,1,2 \ldots N$. In addition, we denote $u_i^n = u(x_i, t_n)$.

3.1 Linearly Implicit Euler Scheme

In this method, equation (1) is approximated at the mesh point (x_i, t_{n+1}) as follows.

- The backward approximated formula for the time fractional derivative in equation (1) can be obtained by approximating the time-derivate in Caputo equation (5), using the backward finite difference formula [10,25]:

$$u_t^{\alpha}(x_i, t_{n+1}) = \frac{1}{k^{\alpha}\Gamma(2-\alpha)} \sum_{s=0}^{n} b_s (u_i^{n+1-s} - u_i^{n-s}) + O(k) \tag{6}$$

$b_s = (s+1)^{(1-\alpha)} - s^{(1-\alpha)}$, $s = 0,1,2,\ldots I$.

- In the right hand side of equation (1) the second order space-derivative is approximated using the common central difference formula, as follows:

$$u_{xx}(x_i, t_{n+1}) = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + O(h^2) \tag{7}$$

- Finally, the nonlinear reaction term is approximated as follows:

$$f(x_i, t_{n+1}, u_i^{n+1}) = f(x_i, t_{n+1}, u_i^n) + O(k) \tag{8}$$

By substituting equations (6), (7) and (8) in (1) it follows that

$$\frac{1}{k^{\alpha}\Gamma(2-\alpha)} \sum_{s=0}^{n} b_s (u_i^{n+1-s} - u_i^{n-s}) = \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{h^2} + f(x_i, t_{n+1}, u_i^n) \tag{9}$$

$$\sum_{s=1}^{n} b_s (u_i^{n+1-s} - u_i^{n-s}) = r[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] + \delta\, f(x_i, t_{n+1}, u_i^n).$$

where $r = \frac{\delta}{h^2}$, $\delta = k^{\alpha} \Gamma(2-\alpha)$.
$b_s = (s+1)^{(1-\alpha)} - s^{(1-\alpha)}$, $s = 0,1,2,\ldots\ldots I$.
$x_i = x_0 + ih$, , $t_{n+1} = t_n + k = (n+1)k$,
$i = 0,1,2,\ldots.,I, n = 0,1,\ldots,N,\ \ x_0 = 0$, $t_0 = 0$, $x_I = 1$, $t_N = T$
So, equation (8) becomes

$$[u_i^{n+1} - u_i^n] + \sum_{s=1}^{n} b_s(u_i^{n+1-s} - u_i^{n-s})$$
$$= r[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] + \delta f(x_i, t_{n+1}, u_i^n).$$

So, $[u_i^{n+1} - u_i^n] - \sum_{s=1}^{n} b_s(u_i^{n-s} - u_i^{n+1-s})$
$$= r[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] + \delta f(x_i, t_{n+1}, u_i^n)$$
$$[u_i^{n+1} - u_i^n] = \sum_{s=1}^{n} b_s(u_i^{n-s} - u_i^{n+1-s})$$
$$+ r[u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}] + \delta f(x_i, t_{n+1}, u_i^n) .$$

Thus, $(1+2r)u_i^{n+1} - r[u_{i+1}^{n+1} + u_{i-1}^{n+1}]$
$$= \sum_{s=1}^{n} b_s(u_i^{n-s} - u_i^{n+1-s}) + u_i^n + \delta f(x_i, t_{n+1}, u_i^n) .$$
$I = 1,2,\ldots\ldots, I-1$, $n = 0,1,\ldots\ldots N-1$
For $n = 0$
$$(1+2r)u_i^1 - r\,[u_{i+1}^1 + u_{i-1}^1] = u_{i,j}^0 + \delta\, f(x_i, t_1, u_i^0) \tag{10}$$
For $n > 0$
$$(1+2r)u_i^{n+1} - r\,[u_{i+1}^{n+1} + u_{i-1}^{n+1}]$$
$$= u_i^n + \sum_{s=1}^{n} b_s(u_i^{n-s} - u_i^{n+1-s}) + \delta f(x_i,, t_{n+1}, u_i^n)$$
So, $(1+2r)u_i^{n+1} - r\,[u_{i+1}^{n+1} + u_{i-1}^{n+1}] = (2 - 2^{1-\alpha})u_i^n$
$$+ \sum_{s=1}^{n-1}(b_s - b_{s+1})u_i^{n-s} + b_n\, u_i^0 + \delta f(x_i, t_{n+1}, u_i^n) \tag{11}$$
$i = 0,1,2,\ldots.,I-1$, $n = 0,1,2,\ldots.,N$,

Let $U^n = \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ u_{I-1}^n \end{bmatrix}$ $F^n = \begin{bmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_{I-1}^n \end{bmatrix}$ $V^n = \begin{bmatrix} u_0^n \\ \vdots \\ \vdots \\ u_I^n \end{bmatrix}$, $f_i^n = f(x_i, t_{n+1}, u_i^n)$
$i = 0,1,2,\ldots.,I-1$, $n = 0,1,2,\ldots.,N$.
The above equation can be written in a matrix form as follows:
$$\left\{\begin{matrix} A\,U^1 = U^0 + \delta F^0 + rV^1 \\ AU^{n+1} = \sum_{s=0}^{n-1}(b_s - b_{s+1})U^{n-s} + b_n\, U_i^0 + \delta F^n + rV^{n+1} \end{matrix}\right\}. \tag{12}$$
Where:
$$A = \begin{bmatrix} (1+2r) & -r & 0 & \cdots & \cdots & 0 \\ -r & (1+2r) & -r & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & -r & (1+2r) \end{bmatrix}_{(I-1)\times(I-1)}$$

Lemma 3.1 [26]
For $s = 0,1,2,\ldots n$
- $b_s > b_{s+1}$ *i.e.* $(b_s^{-1} \le b_n^{-1})$
- $b_0 = 1$.
- $b_s > 0$.
- $\sum_{s=1}^{n-1}(b_s - b_{s+1}) + b_n = b_1$
- $\sum_{s=0}^{n-1}(b_s - b_{s+1}) + b_n = 1$

Theorem 3.1 At each advanced time-level $(n+1)$, the linear system (12) is uniquely solvable.

Proof : Since A is diagonally dominant with positive real diagonal entries, then A is positive definite and nonsingular [27].
Hence, the linear system (12) is uniquely solvable.

3.1.1 Stability Analysis of Linearly implicit Euler scheme

We suppose that $\tilde{u}_i^n$ is the approximate solution of problem (1)-(3).
Set $e_i^n = \tilde{u}_i^n - u_i^n$, $i = 0,1,2,\ldots,I-1$, $n = 0,1,2,\ldots,N$.
Clearly, e_i satisfies equations (10) and (11):
$(1+2r)e_i^1 - r\,[e_{i+1}^1 + e_{i-1}^1] = e_i^0 + \delta\, f(x_i,t_1,\tilde{u}_i^0) - \delta\, f(x_i,t_1,u_i^0)$
$(1+2r)e_i^{n+1} - r\,[e_{i+1}^{n+1} + e_{i-1}^{n+1}] = (2-2^{1-\alpha})e_i^n + \sum_{s=1}^{n-1}(b_{s-1} - b_s)e_i^{n-s} + b_n\, e_i^0 + \delta\left[f(x_i,t_{n+1},\tilde{u}_i^n) - \delta f(x_i,y_j,t_{n+1},u_i^n)\right]$.
Based on (12), the above equation can be written in a matrix form as follows:

- $AE^1 = E^0 + \delta[f(x_i,t_1,\tilde{u}_i^0) - f(x_i,t_1,u_i^0)]$.
- $AE^{n+1} = (b_0 - b_1)E^n + (b_1 - b_2)E^{n-1} + \cdots\cdots +$
$(b_{n-1} - b_n)E^1 + b_nE^0 + \delta[f(x_i,t_{n+1},\tilde{u}_i^n) - f(x_i,t_{n+1},u_i^n)]$ (13)
- E^0 is given, where $E^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{I-1}^n \end{bmatrix}$

Definition 3.1 [17] For any arbitrary initial rounding error E^0, the difference approximation $AU^{n+1} = BU^n + b^n$ is stable, if there exists > 0 , independent on h , k such that

$$\|E^n\| \le C\|E^0\|, \quad \text{or} \quad \|(A^{-1}B)^n\| \le C,\ n = 0,1,2,\ldots,N .$$

Theorem 3.2. The fully implicit finite difference formula (10)-(11) is unconditionally stable.

Proof: In order to prove this theorem, we apply the maximum error stability technique [25]. By mathematical induction, we can show that

$$\|E^{n+1}\| \le (1+\delta L)^{n+1}\|E^0\|_\infty$$

For $n = 0$, $(1+2r)e_i^1 - r\,[e_{i+1}^1 + e_{i-1}^1] = e_i^0$
Let $|e_p^1| = Max_{1\le i\le I-1}|e_i^1|$, we have
$|e_p^1| = (1+2r)|e_p^1| - r\,[|e_p^1| + |e_p^1|] \le (1+2r)|e_p^1| - r\,[|e_{p+1}^1| + |e_{p-1}^1|]$
$\le |(1+2r)e_p^1 - r\,(e_{p+1}^1 + e_{p-1}^1)| = |e_p^0 + \delta[f(x_p,t_1,\tilde{u}_p^0) - f(x_p,t_1,u_p^0)]|$.
$\le |e_p^0| + \delta|f(x_p,t_1,\tilde{u}_p^0) - f(x_p,t_1,u_p^0)|$
$\le |e_p^0| + \delta L|\tilde{u}_p^0 - u_p^0| = (1+\delta L)|e_p^0| \le (1+\delta L)\|E^0\|_\infty$.
Hence

$$\|E^1\| \le (1+\delta L)\|E^0\|_\infty$$

New, supposes that $\|E^s\|_\infty \le (1+\delta L)^s\,\|E^0\|_\infty$, $s = 0,1,2,\ldots,n$,
Let $|e_p^{n+1}| = Max_{1\le i\le I-1}|e_i^{n+1}|$, we have

$$|e_p^{n+1}| = (1+2r)|e_p^{n+1}| - r\,[|e_p^{n+1}| + |e_p^{n+1}|]$$

$\le (1+2r)|e_p^{n+1}| - r\,[|e_{p+1}^{n+1}| + |e_{p-1}^{n+1}|]$
$\le |(1+2r)e_p^{n+1} - r\,(e_{p+1}^{n+1} + e_{p-1}^{n+1})|$
$= |(b_0 - b_1)e_p^n + \sum_{s=1}^{n-1}(b_s - b_{s+1})e_p^{n-s} + b_n\, e_p^0 + \delta f(x_p,t_{n+1},\tilde{u}_p^n) - \delta\, f(x_p,t_{n+1},u_p^n)|$
$\le (b_0 - b_1)|e_p^n| + \sum_{s=1}^{n-1}(b_s - b_{s+1})|e_p^{n-s}| + b_n\,|e_p^0| + \delta L|e_p^n|$
$\le (b_0 - b_1)\|E^n\|_\infty + \sum_{s=1}^{n-1}(b_s - b_{s+1})\|E^{n-s}\|_\infty + b_n\,\|E^0\|_\infty + \delta L\|E^n\|_\infty$.
Thus
$\|E^{n+1}\| \le (b_0 - b_1)\|E^n\|_\infty + \sum_{s=1}^{n-1}(b_s - b_{s+1})\|E^{n-s}\|_\infty + b_n\,\|E^0\|_\infty + \delta L\|E^n\|_\infty$

$\leq (b_0 - b_1)(1+\delta L)^n\|E^0\|_\infty + \sum_{s=1}^{n-1}(b_s - b_{s+1})\,(1+\delta L)^{n-s}\|E^0\|_\infty + (1+\delta L)^n b_n\,\|E^0\| + (1+\delta L)^n\|E^0\|_\infty(\delta L)$

$\leq (b_0 - b_1)(1+\delta L)^n\|E^0\|_\infty + \sum_{s=1}^{n-1}(b_s - b_{s+1})\,(1+\delta L)^{n}\|E^0\|_\infty + (1+\delta L)^n b_n\,\|E^0\| + (1+\delta L)^n\|E^0\|_\infty(\delta L)$

$$= \left\{(b_0 - b_1) + \sum_{s=1}^{n-1}(b_s - b_{s+1}) + b_n\right\}(1+\delta L)^n\|E^0\|_\infty + (\delta L)(1+\delta L)^n\|E^0\|_\infty$$

$=(1+\delta L)^n[1+\delta L]\|E^0\|_\infty = (1+\delta L)^{n+1}\|E^0\|_\infty$

Thus $\|E^{n+1}\| \leq (1+k^\alpha\Gamma(2-\alpha)L)^{n+1}\|E^0\|_\infty \leq (1+T^\alpha\Gamma(2-\alpha)L)^{n+1}\|E^0\|_\infty$

Hence, there exists $C > 0$, such that $\|E^{n+1}\| \leq C\|E^0\|_\infty$

3.1.2 Consistency and Convergence Analysis of the fully implicit Euler formula

Let $u(x_i,t_n)$ be the exact solution to equation (1) at mesh point (x_i,t_n),

$$i = 0,1,2,\ldots,I-1\ ,\ n = 0,1,2,\ldots,N\,.$$

Definition 3.2:

$e_i^n = u(x_i,t_n) - u_i^n\ ,\ n = 0,1,2,\ldots,N\ ,\ i = 0,1,2,\ldots,I-1$

$$E^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{I-1}^n \end{bmatrix}\ ,\quad E^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Substitution e_i^n in the implicit formulas (10) and (11) yields that

$(1+2r)e_i^1 - r\,[e_{i+1}^1 + e_{i-1}^1] = e_{i,j}^0 + \delta f\left(x_i,t_1,u(x_i,t_0) - \delta f(x_i,t_1,u_i^0)\right) + T_i^1$

$(1+2r)e_i^{n+1} - r\,[e_{i+1}^{n+1} + e_{i-1}^{n+1}] =$

$(2-2^{1-\alpha})e_i^n + \sum_{s=1}^{n-1}(b_{s-1} - b_s)e_i^{n-s} + b_n\,e_i^0 + \delta f(x_i,t_{n+1},u(x_i,t_{n+1})) - \delta f(x_i,t_{n+1},u_i^n) + T_i^{n+1}$

where T_i^{n+1} is the local truncation error.

Theorem 3.3 The linearly implicit Euler formulas (10) and (11) are consistent.

Proof:

$T_i^{n+1} = \sum_{s=0}^{n} b_s\,[u(x_i,t_{n+1-s}) - u(x_i,t_{n-s})]$

$-r\,[u(x_{i+1},t_{n+1}) + u(x_{i-1},t_{n+1}) - 2u(x_i,t_{n+1})] - \delta f(x_i,t_{n+1},u(x_i,t_n))$

$= \Gamma(2-\alpha)k^\alpha\,[u_t^\alpha(x_i,t_{n+1}) + O(k)] - \Gamma(2-\alpha)k^\alpha\,[u_{xx}(x_i,t_{n+1}) + O(h^2)]$

$-\Gamma(2-\alpha)k^\alpha\left(f\left(x_i,t_{n+1},u_i^{n+1}\right) + O(k)\right) = O(k+h^2)\,.$

Hence, $|T_i^n| \leq C(k+h^2)\ ,\ i = 0,1,2,\ldots,I-1\ ,\ n = 0,1,2,\ldots,N$

Clearly $|T_i^n| \to 0$ *as* $h,k \to 0$, so the implicit Euler method is consistent

Theorem 3.4 There exists $C > 0$ such that:

$|e_i^n| \leq C(k+h^2)\ ,\ i = 0,1,2,\ldots,I-1\ ,\ n = 0,1,2,\ldots,N\,.$

Proof:

Define $\|E^n\|_\infty = Max_{1\leq i\leq I-1}|e_i^n|$

We prove this theorem using the mathematical induction method

For $n = 1$, let $\|E^1\|_\infty = |e_p^1| = Max_{1\leq i\leq I-1}|e_i^n|$, we have

$|e_p^1| = (1+2r)|e_p^1| - r\left[|e_p^1| + |e_p^1|\right] \leq (1+2r)|e_p^1| - r\left[|e_{p+1}^1| + |e_{p-1}^1|\right]$

$\leq \left|(1+2r)e_p^1 - r\left(e_{p+1}^1 + e_{p-1}^1\right)\right|$

$= \left|\delta f\left(x_p,t_1,u(x_i,t_0)\right) - \delta f\left(x_p,,t_1,u_p^0\right) + T_p^1\right|$

$\leq \left|\delta f\left(x_p,t_1,u(x_i,t_0)\right) - \delta f\left(x_p,t_1,u_p^0\right)\right| + \left|T_p^1\right|$

$\leq \delta L|e_p^0| + |T_p^1| = |T_p^1| \leq C(k+h^2)\,.$

Hence, $\|E^1\|_\infty \le b_0^{-1} C_1(k + h^2),\ C_1 = C$
Now, suppose that $\|E^s\|_\infty \le C_s b_{s-1}^{-1}(k + h^2)$, $s = 0, 1, 2, \ldots, n$
Let $|e_p^{n+1}| = Max_{1\le i\le I-1} |e_i^{n+1}|$

$$|e_p^{n+1}| = (1 + 2r)|e_p^{n+1}| - r\left[|e_p^{n+1}| + |e_p^{n+1}|\right]$$
$$\le (1 + 2r)|e_p^{n+1}| - r\left[|e_{p+1}^{n+1}| + |e_{p-1}^{n+1}|\right]$$
$$\le \left|(1 + 2r)e_p^{n+1} - r\left(e_{p+1}^{n+1} + e_{p-1}^{n+1}\right)\right|$$
$$= \left|\sum_{s=0}^{n-1}(b_s - b_{s+1})e_p^{n-s} + \delta f\left(x_p, t_{n+1}, u(x_p, t_n)\right) - \delta f\left(x_p, t_{n+1}, u_p^n\right) + T_p^{n+1}\right|$$
$$\le \sum_{s=0}^{n-1}(b_s - b_{s+1})|e_p^{n-s}| + \delta L|e_p^n| + |T_p^{n+1}|$$
$$\le \sum_{s=0}^{n-1}(b_s - b_{s+1})\|E^{n-s}\|_\infty + \delta L\|E^n\|_\infty + |T_p^{n+1}|$$
$$\le \sum_{s=0}^{n-1}(b_s - b_{s+1})\, C_{n-s} b_{n-s-1}^{-1}(k + h^2) + \delta L[C_n b_{n-1}^{-1}(k + h^2)] + C(k + h^2)$$
$$\le \sum_{s=0}^{n-1}(b_s - b_{s+1})\, C_s b_n^{-1}(k + h^2) + \delta L[C_n b_n^{-1}(k + h^2)] + C(k + h^2)$$

Let $C^* = \max\{C, C_1, C_2, \ldots\ldots C_n\}$
Thus

$$|e_p^{n+1}| \le \left[\sum_{s=0}^{n-1}(b_s - b_{s+1}) + b_n + \delta L\right] b_n^{-1} C^*(k + h^2)$$
$$= (1 + \delta L) b_n^{-1} C^*(k + h^2)$$

Thus, there exists $C_{n+1} > 0$ such that
$\|E^{n+1}\|_\infty \le C_{n+1} b_n^{-1}(k + h^2)$.

This means the implicit Euler approximation formulas (10) and (11) are convergent with the order of accuracy: $O(k + h^2)$.

3.2 Crank Nicolson Method

The goal of this section is to derive the Crank-Nicolson scheme for equation (1), which is one of the most popular methods in practice. Moreover, it has a high order of convergence in both space and time.

In order to approximate equation (1) and (5) at the mush point $\left(x_i, t_{n+\frac{1}{2}}\right)$, we use the following approximations [10,13]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}\Big|_i^{n+\frac{1}{2}} = \left[\widetilde{w}_1 u_i^n + \sum_{s=1}^{n-1}(\widetilde{w}_{n-s+1} - \widetilde{w}_{n-s})\, u_i^s - \widetilde{w}_n\, u_i^0 + \sigma\left(\frac{u_i^{n+1} - u_i^n}{2^{1-\alpha}}\right)\right] + O(k^{(2-\alpha)}) .$$
$$\frac{\partial^2 u}{\partial x^2}\Big|_i^{n+\frac{1}{2}} = \left[\left(\frac{u_{i+1}^{n+1} + -2u_i^{n+1} + u_{i-1,}^{n+1}}{2h^2}\right)\right] + \left[\left(\frac{u_{i+1}^n + -2u_i^n + u_{i-1}^n}{2h^2}\right)\right] + O(h^2) .$$
$$\sigma = \frac{1}{k^\alpha \Gamma(2-\alpha)},\ \widetilde{w}_s = \sigma\left[\left(s + \frac{1}{2}\right)^{(1-\alpha)} - \left(s - \frac{1}{2}\right)^{(1-\alpha)}\right]$$

The nonlinear term can be approximated using Taylor expansion [19]:

$$f\left(x_i, t_{n+\frac{1}{2}}, u\left(x_i, t_{n+\frac{1}{2}}\right)\right) = f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u(x_i, t_n) - \frac{1}{2}u(x_i, t_{n-1})\right) + O(k^2)$$

Thus $f\left(x_i, t_{n+\frac{1}{2}}, u\left(x_i, t_{n+\frac{1}{2}}\right)\right) = f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right) + O(k^2)$.

From above forms, we can get the Crank – Nicolson approximated formula for equation (1) as follows:

$$\left[\widetilde{w}_1 u_i^n + \sum_{s=1}^{n-1} (\widetilde{w}_{n-s+1} - \widetilde{w}_{n-s})\, u_i^s - \widetilde{w}_n\, u_i^0 + \sigma\left(\frac{u_i^{n+1} - u_i^n}{2^{1-\alpha}}\right)\right]$$
$$= \frac{1}{2h^2}[(u_{i+1}^{n+1} + -2u_i^{n+1} + u_{i-1}^{n+1}) + (u_{i+1}^n + -2u_i^n + u_{i-1}^n)]$$
$$+ f(x_i, t_{n+\frac{1}{2}}, \tfrac{3}{2}u_i^n - \tfrac{1}{2}u_i^{n-1}).$$

Multiplying both sides of the last equation by $\delta = k^\alpha \Gamma(2-\alpha)2^{1-\alpha}$, yields that:

$$(1+r)u_i^{n+1} - \frac{r}{2}\,[u_{i+1}^{n+1} + u_{i-1}^{n+1}] = (1 - 2^{1-\alpha}w_1 - r)u_i^n + \frac{r}{2}\,[u_{i+1}^n + u_{i-1}^n] +$$
$$2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})u_i^s + 2^{1-\alpha}\, w_n u_i^0 + \delta f\left(x_i, t_{n+\frac{1}{2}}, \tfrac{3}{2}u_i^n - \tfrac{1}{2}u_i^{n-1}\right) \quad (14)$$

Where $\delta = k^\alpha \Gamma(2-\alpha)2^{1-\alpha}$, $r = \frac{\delta}{h^2}$, $w_s = \left[\left(s+\frac{1}{2}\right)^{(1-\alpha)} - \left(s-\frac{1}{2}\right)^{(1-\alpha)}\right]$

For $n = 0$, this formula becomes as follows:

$$(1+r)u_i^1 - \frac{r}{2}\,[u_{i+1}^1 + u_{i-1}^1] = (1-r)u_i^0 + \frac{r}{2}\,[u_{i+1}^0 + u_{i-1}^0] + \delta f\left(x_i, t_{n+\frac{1}{2}}, \tfrac{3}{2}u_i^0\right)$$

Lemma 3.2 [13]

For $s = 0,1,2,\dots,n$

- $w_s > 0$
- $w_{s+1} < w_s$, $w_{s+1}^{-1} > w_s^{-1}$
- $\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1}) + w_n = w_1.$

The Crank – Nicolson difference formula (14) can be written in matrix form as follows:

$$\begin{Bmatrix} A\,U^1 = BU^0 + \delta F^0 + Z^n \\ AU^{n+1} = BU^n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})U^s + 2^{1-\alpha}w_n U^0 + \delta F^n + Z^n \end{Bmatrix}$$
(15)

$$A = \begin{bmatrix} 1+r & -\frac{r}{2} & 0 & \cdots & & \cdots & 0 \\ -\frac{r}{2} & 1+r & -\frac{r}{2} & 0 & & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & & -\frac{r}{2}I & 1+r \end{bmatrix}_{(I-1)\times(I-1)}$$

$$B = \begin{bmatrix} (1-2^{1-\alpha}w_1 - r) & \frac{r}{2} & 0 & \cdots & & \cdots & 0 \\ \frac{r}{2} & (1-2^{1-\alpha}w_1 - r) & \frac{r}{2} & 0 & & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & & \frac{r}{2} & (1-2^{1-\alpha}w_1 - r) \end{bmatrix}_{(I-1)\times(I-1)}$$

$U^n = (u_1^n, u_2^n, u_3^n, \dots, u_{I-1}^n)$, $F^n = \left(f_1^n,\, f_2^n\,, f_3^n, \dots, f_{I-1}^n\right)$, $f_i^0 = f\left(x_i, t_{\frac{1}{2}}, \frac{3}{2}u_i^0\right)$,

$f_i^n = f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right)$, $Z^n = \frac{r}{2}(u_0^{n+1} + u_0^n, 0 \dots .0, u_I^{n+1} + u_I^n)$.

Theorem 3.5 At each time level, $(n+1)$, the linear system (15) is uniquely salable.

Proof: Since A is diagonally dominant with positive real diagonal entries, then A is positive definite and nonsingular [27].

Hence, the linear system (15) is uniquely solvable.

3.2.1 Stability Analysis for C.N. Method

Suppose that $\tilde{u}_i^n$ is the approximate solution of equation (1)

Set $e_i^n = \tilde{u}_i^n - u_i^n$, $i = 0, 1, 2, \dots, I-1$, $n = 0, 1, 2, \dots, N$.

Define $\|E^n\|_\infty = Max_{1\le i\le I-1}\,|e_i^n|$, For $n = 0$,

$(1+r)e_i^1 - \frac{r}{2}\,[e_{i+1}^1 + e_{i-1}^1] = (1-2r)e_i^0 + \frac{r}{2}\,[e_{i+1}^0 + e_{i-1}^0] + \delta\left[f\left(x_i, y_j, t_{\frac{1}{2}}, \frac{3}{2}\tilde{u}_{i,j}^0\right) - f\left(x_i, t_{\frac{1}{2}}, \frac{3}{2}u_i^0\right)\right]$

For $n > 0$,

$(1+r)e_i^{n+1} - \frac{r}{2}\,[e_{i+1}^{n+1} + e_{i-1}^{n+1}]$

$= (1 - 2^{1-\alpha}\,w_1 - r)\,e_i^n + \frac{r}{2}\,[e_{i+1}^n + e_{i-1}^n] + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,e_i^s + 2^{1-\alpha}w_n e_i^0 + \delta\left[f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}\tilde{u}_i^n - \frac{1}{2}\tilde{u}_i^{n-1}\right) - f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right)\right].$

Based on (15), the above error formula can be written in matrix form as follows:

$$\left\{\begin{array}{c} AE^1 = BE^0 + \delta G^0 \\ AE^{n+1} = BE^n + 2^{1-\alpha}\displaystyle\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})E^s + 2^{1-\alpha}w_n E^0 + \delta G^n \end{array}\right\}$$

E^0 is given, where $E^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{I-1}^n \end{bmatrix}$, $G = \begin{bmatrix} g_1^n \\ g_2^n \\ \vdots \\ g_{I-1}^n \end{bmatrix}$

$$g_i^n = f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}\tilde{u}_i^n - \frac{1}{2}\tilde{u}_i^{n-1}\right) - f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right)$$

$$i = 0,1,2,\ldots,I-1\ , \qquad n = 0,1,2,\ldots,N$$

Theorem 3.6 The C.N. finite difference approximation is stable, if

$$(1 - 2^{1-\alpha}\,w_1 - r) \geq 0$$

Proof: To prove this theorem, we use the Mathematical induction

For $n = 0$, set $\left|e_p^1\right| = Max_{1\leq i\leq I-1}\left|e_i^1\right|$, we have

$\left|e_p^1\right| = (1+r)\left|e_p^1\right| - \frac{r}{2}\left(2\left|e_p^1\right|\right)$

$\leq (1+r)\left|e_p^1\right| - \frac{r}{2}\left(\left|e_{p+1}^1\right| + \left|e_{p-1}^1\right|\right)$

$\leq \left|(1+r)e_p^1 - \frac{r}{2}\left(e_{p+1}^1 + e_{p-1}^1\right)\right|$

$= \left|(1-r)e_p^0 + \frac{r}{2}\left(e_{p+1}^0 + e_{p-1}^0\right) + \delta\left(f\left(x_p, t_{\frac{1}{2}}, \frac{3}{2}\tilde{u}_p^0\right) - f\left(x_p, t_{\frac{1}{2}}, \frac{3}{2}u_p^0\right)\right)\right|$

$\leq (1-r)\|E^0\|_\infty + r\,(\|E^0\|_\infty) + \delta L\left|\frac{3}{2}\left(\tilde{u}_p^0 - u_p^0\right)\right|.$

$\leq \|E^0\|_\infty + \frac{3}{2}\,\delta L\|E^0\|_\infty = \left(1 + \frac{3}{2}\delta L\right)\|E^0\|_\infty \leq (1+2\delta L)\|E^0\|_\infty.$

Thus, $\|E^1\|_\infty \leq (1+2\delta L)\|E^0\|_\infty$

New, suppose that $\|E^s\|_\infty \leq C^s\,\|E^0\|_\infty$, where $C = (1+2\delta L)$, $s = 0,1,2,\ldots,n$.

For $n+1$, let $\left|e_p^{n+1}\right| = Max_{1\leq i\leq I-1}\left|e_i^{n+1}\right|$, we have

$\left|e_p^{n+1}\right| = (1+r)\left|e_{p,q}^{n+1}\right| - \frac{r}{2}\left[2\left|e_p^{n+1}\right|\right].$

$\leq (1+r)\left|e_p^{n+1}\right| - \frac{r}{2}\left(\left|e_{p+1}^{n+1}\right| + \left|e_{p-1}^{n+1}\right|\right)$

$\leq \left|(1+r)e_p^{n+1} - \frac{r}{2}\left(e_{p+1}^{n+1} + e_{p-1}^{n+1}\right)\right|$

$\leq \left|(1 - 2^{1-\alpha}\,w_1 - r)e_p^n + \frac{r}{2}\left(e_{p+1}^n + e_{p-1}^n\right)\right| + \left|2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,e_p^s\right| + 2^{1-\alpha}w_n\left|e_p^0\right| + \delta\left|f\left(x_p, t_{n+\frac{1}{2}}, \frac{3}{2}\tilde{u}_p^n - \frac{1}{2}\tilde{u}_p^{n-1}\right) - f\left(x_p, t_{n+\frac{1}{2}}, \frac{3}{2}u_p^n - \frac{1}{2}u_p^{n-1}\right)\right|.$

$\leq (1 - 2^{1-\alpha}\,w_1 - r)\left|e_p^n\right| + \frac{r}{2}\left(\left|e_{p+1}^n\right| + \left|e_{p-1}^n\right|\right) + 2^{1-\alpha}w_n\left|e_p^0\right| + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\left|e_i^s\right| + \delta L\left|\frac{3}{2}\left(\tilde{u}_p^n - u_p^n\right) + \frac{1}{2}\left(u_p^{n-1} - \tilde{u}_p^{n-1}\right)\right|.$

$\leq (1- 2^{1-\alpha} w_1 - r) \|E^n\| + r\|E^n\| + 2^{1-\alpha} w_n \left|e_p^0\right| + 2^{1-\alpha} \sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1}) |e_i^s| + \delta L \left[\frac{3}{2}\left|e_p^n\right| + \frac{1}{2}\left|e_p^{n-1}\right|\right]$.
$\leq (1 - 2^{1-\alpha}w_1)\, C^n \|E^0\| + 2^{1-\alpha}w_n \|E^0\| + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\, C^s \|E^0\| + \delta L \left[\frac{3}{2} C^n \|E^0\| + \frac{1}{2} C^{n-1}\|E^0\|\right]$.
$\leq (1 - 2^{1-\alpha}w_1)\, C^n \|E^0\| + 2^{1-\alpha}w_n C^n \|E^0\| + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\, C^n \|E^0\| + 2\,\delta L\; C^n \|E^0\|$.
$= [1 - 2^{1-\alpha}w_1 + 2^{1-\alpha}w_n + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1}) + 2\delta L\,]C^n\|E^0\|$.
$= (1 + 2\delta L)\, C^n \|E^0\| = (1 + 2\delta L)^{n+1} \|E^0\|$.
Thus
$\|E^{n+1}\|_\infty \leq (1 + 2\delta L)^{n+1}\|E^0\|$.
So, $\|E^{n+1}\| \leq (1 + 2^{2-\alpha}k^\alpha \Gamma(2 - \alpha)L)^{n+1}\|E^0\|_\infty$
$\leq (1 + 2^{2-\alpha}T^\alpha \Gamma(2 - \alpha)L)^{n+1}\|E^0\|_\infty$
Hence, there exists $C > 0$, such that $\|E^{n+1}\| \leq C\|E^0\|_\infty$

3.2.2 Convergence Analysis of Crank – Nicolson Method

Let $u(x_i, t_n)$ and u_i^n be the exact and numerical solutions of problem (1) at mesh point (x_i, t_n), respectively, $i = 0, 1, 2, \dots, I-1$, $n = 0, 1, 2, \dots, N$.

Definition 3.3

$e_i^n = u(x_i, t_n) - u_i^n$, $n = 0, 1, 2, \dots, N$, $i = 0, 1, 2, \dots, I-1$

$$E^n = \begin{bmatrix} e_1^n \\ e_2^n \\ \vdots \\ e_{I-1}^n \end{bmatrix}, \quad E^0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

By Substitution e_i^n in the C.N. equation, it follows that
For $n = 0$,

$$(1 + 2r)e_i^1 - \frac{r}{2}\,[e_{i+1}^1 + e_{i-1}^1] = \delta\left[f\left(x_i, t_{\frac{1}{2}}, \frac{3}{2}u(x_i, t_0)\right) - f\left(x_i, t_{\frac{1}{2}}, \frac{3}{2}u_i^0\right)\right] + T_i^{\frac{1}{2}} .$$

For $n > 0$

$$(1 + r)e_i^{n+1} - \frac{r}{2}\,[e_{i+1}^{n+1} + e_{i-1}^{n+1}]$$
$$= (1 - 2^{1-\alpha} w_1 - r)\, e_i^n + \frac{r}{2}\,[e_{i+1}^n + e_{i-1}^n] + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\, e_i^s + \delta\left[f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u(x_i, t_n) - \frac{1}{2}u(x_i, t_{n-1})\right) - f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right)\right] + T_i^{n+1} .$$

Theorem 3.7 There exists $C > 0$ such that:
$|T_i| \leq C(k^{2-\alpha} + h^2)$

Proof: For simplicity, we denote $u_i^n = u(x_i, t_n)$ as the exact solution to the problem (1).

$$T_{i,}^{n+\frac{1}{2}} = (u_i^{n+1} - u_i^n) + 2^{1-\alpha} w_1 u_i^n - 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\, u_i^s - 2^{1-\alpha} w_n - \frac{r}{2}\left[u_{i+1}^{n+1} + u_{i-1}^{n+1} - 2\,u_{i,j}^{n+1}\right] - \frac{r}{2}\left[u_{i+1}^n + u_{i-1}^n - 2\,u_{i,j}^n\right] - \delta f\left(x_i, t_{n+\frac{1}{2}}, \frac{3}{2}u_i^n - \frac{1}{2}u_i^{n-1}\right).$$

$$= k^\alpha \Gamma(2 - \alpha)\, 2^{1-\alpha}\left[u_t^\alpha\left(x_i, t_{n+\frac{1}{2}}\right) + O(k^{2-\alpha})\right]$$
$$- k^\alpha \Gamma(2 - \alpha)\, 2^{1-\alpha}\left[u_{xx}\left(x_i, t_{n+\frac{1}{2}}\right) + o(h^2)\right]$$
$$- k^\alpha \Gamma(2 - \alpha)\, 2^{1-\alpha}\left[f\left(x_i, t_{n+\frac{1}{2}}, u_i^{n+\frac{1}{2}}\right) - O(k^2)\right]$$
$$= k^\alpha \Gamma(2 - \alpha)\, 2^{1-\alpha}[u_t^\alpha - u_{xx} - f]_i^{n+\frac{1}{2}} + k^\alpha \Gamma(2 - \alpha)\, 2^{1-\alpha}\,[O(k^{2-\alpha}) + O(h^2) + O(k^2)]$$

Thus $\left|T_i^{n+\frac{1}{2}}\right| \le C(k^{2-\alpha}+h^2)$

Remark 3.1 Based on Theorem 3.7, the C.N. formula (14) is consistent.

i.e. $\left|T_i^{n+\frac{1}{2}}\right| \to 0\ ,\ h\,,k\ \to 0\,.$

Theorem 3.8 If $(1-r-\ 2^{1-\alpha}\,w_1)\ge 0$, there exists $c>0$ such that
$\left|e_{i,j}^{n+1}\right|\ \le C(k^{2-\alpha}+h^2)\,, i=1,2\,,\dots\dots,I-1\,,\ n=0,1\,,\dots\dots\dots.,N$

Proof: Define $\|E^n\|_\infty=\ Max_{1\le i\le I-1}\,|e_i^n|$.

By using mathematical induction, we can prove this theorem as follows

For $n\ =\ 0$, let $\|E^1\|_\infty = Max_{1\le i\le I-1}\left|e_i^1\right| = \left|e_p^1\right|$.

$$\left|e_i^1\right| \le\ \left|e_p^1\right| = (1+r)\left|e_p^1\right| - \frac{r}{2}\left(2\left|e_p^1\right|\right).$$

$$\le (1+r)\left|e_p^1\right| - \frac{r}{2}\left(\left|e_{p+1}^1\right| + \left|e_{p-1}^1\right|\right).$$

$$\le \left|(1+r)\,e_p^1 - \frac{r}{2}\left(e_{p+1}^1 + e_{p-1}^1\right)\right|$$

$$= \left|\delta\left[f\left(x_p, t_{\frac{1}{2}}, \frac{3}{2}u(x_p,t_0)\right) - f\left(x_p, t_{\frac{1}{2}}, \frac{3}{2}u_p^0\right)\right] + T_i^{\frac{1}{2}}\right|$$

$$\le \delta\,L\frac{3}{2}\left|u(x_p,t_0) - u_p^0\right| + \left|T_i^{\frac{1}{2}}\right| = \delta\,L\,\frac{3}{2}\left|e_i^0\right| + \left|T_i^{\frac{1}{2}}\right| = \left|T_i^{\frac{1}{2}}\right| \le C(k^{2-\alpha}+h^2)$$

Thus $\left|e_i^1\right| \le \frac{2C}{(2^{1-\alpha}\,w_0)}\,(k^{2-\alpha}+h^2)$.

$$\|E^1\| \le \left(\frac{C_1}{2^{1-\alpha}\,w_0}\right)(k^{2-\alpha}+h^2)\ ,\qquad C_1 = 2C$$

Now suppose that

$\|E^s\|_\infty \le \left(\frac{C_s}{2^{1-\alpha}\,w_{s-1}}\right)(k^{2-\alpha}+h^2)\ ,\ C_s\ >0$ for $s = 1,2,\dots\dots,n$

Set $M_s = \frac{C_s}{2^{1-\alpha}\,w_{s-1}}$, and let $M\ =\ Max\,\{C, C_1\,, C_2, \dots\dots\dots, C_n\}$.

For $n+1$, Let $\|E^{n+1}\|_\infty = Max_{1\le i\le I-1}\left|e_i^{n+1}\right| = \left|e_p^{n+1}\right|$

$$\left|e_i^{n+1}\right| \le \left|e_p^{n+1}\right| = (1+r)\left|e_p^{n+1}\right| - \frac{r}{2}\left[2\left|e_p^{n+1}\right|\right].$$

$$\le (1+r)\left|e_p^{n+1}\right| - \frac{r}{2}\left[\left|e_{p+1}^{n+1}\right| + \left|e_{p-1}^{n+1}\right|\right]$$

$$\le \left|(1+r)e_p^{n+1} - \frac{r}{2}\left(e_{p+1}^{n+1} + e_{p-1}^{n+1}\right)\right|.$$

$$= \Bigg|(1-\ 2^{1-\alpha}\,w_1 - r)e_p^n + \frac{r}{2}\left(e_{p+1}^n + e_{p-1}^n\right) + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,e_i^s +$$

$$\delta\,f\left(x_p, t_{n+\frac{1}{2}}, \frac{3}{2}u(x_p,t_n) - \frac{1}{2}\,u(x_p,t_{n-1})\right) - \delta\,f\left(x_p, t_{n+\frac{1}{2}}, \frac{3}{2}u_p^n - \frac{1}{2}\,u_p^{n-1}\right) + T_i^{n+\frac{1}{2}}\Bigg|\ .$$

$$\le\ (1-\ 2^{1-\alpha}\,w_1 - r)\|E^n\|_\infty + r\,\|E^n\|_\infty + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,\|E^s\|_\infty +$$

$$\delta L\left[\frac{3}{2}\left|e_p^n\right| + \frac{1}{2}\left|e_p^{n-1}\right|\right] + \left|T_i^{n+\frac{1}{2}}\right|\ .$$

$$\le\ (1-\ 2^{1-\alpha}\,w_1)\,M_n\,(k^{2-\alpha}+h^2) +\ 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1})\,M_s\,(k^{2-\alpha}+h^2) +$$

$$\delta L\left[\frac{3}{2}M_n + \frac{1}{2}M_{n-1}\right](k^{2-\alpha}+h^2) +\ C(k^{2-\alpha}+h^2)\,.$$

$$\le [1-\ 2^{1-\alpha}\,w_1 + 2^{1-\alpha}\sum_{s=1}^{n-1}(w_{n-s} - w_{n-s+1}) + 2\,\delta L +\ 2^{1-\alpha}\,w_n]\left(\frac{M}{2^{1-\alpha}\,w_n}\right)(k^{2-\alpha}+h^2)\,.$$

$$= [1+2\delta\,L]\left(\frac{1}{2^{1-\alpha}\,w_n}\right)M(k^{2-\alpha}+h^2)\,.$$

Thus $\|E^{n+1}\|_\infty \le C_{n+1}\left(\frac{1}{2^{1-\alpha}\,w_n}\right)(k^{2-\alpha}+h^2)\,,\quad C_{n+1} = [1+2\delta\,L]M$.

4. Numerical Results and Discussion

In this section, two numerical experiments are presented to show the efficacy and accuracy of the proposed methods, using Matlab (R2020a) software. For each example, we take different size-meshes: ($I = 5,10,20,40$). In order to make sure that the stability condition of Crank-Nicolson is satisfied and to increase the order of convergence, the time steps are chosen according to the following formula:

$$k \leq \left[\frac{h^2\left(1-2^{(1-\alpha)}w_1\right)}{2^{(1-\alpha)}\Gamma(2-\alpha)}\right]^{\frac{1}{\alpha}}, \quad \text{where } w_1 = \left[\left(\frac{3}{2}\right)^{(1-\alpha)} - \left(\frac{1}{2}\right)^{(1-\alpha)}\right] \quad (16)$$

Clearly, with this time-stepping technique, the order of convergence for both IEM and CN methods is $O(h^2)$. However, with this formula, we cannot take a small value to α, because in this case the time step becomes too small and that leads to a very large mesh-size with respect to time.

Moreover, we present the maximum absolute errors that arise from using the proposed numerical schemes, using the formula: $E_{h,k} = \max_{\substack{1\leq i\leq I-1\\ 1\leq n\leq N-1}} |u(x_i,t_n) - u_i^n|$. In addition, the numerical order of convergence (NOC) is computed using the formula [19]:

$$S_{h,k} = \frac{\log\left(\frac{E_{2h,k}}{E_{h,k}}\right)}{\log(2)}$$

4.1 Numerical Experiments

Example 4.1

Consider the following one-dimensional time fractional semilinear diffusion equation:

$$u_t^{\alpha} = u_{xx} + \left(\Gamma(2+\alpha)t - t^{(1+\alpha)}\right)e^x - e^{2x}t^{2(1+\alpha)} + u^2$$
$$0 < x < 1 \;, \qquad 0 < t < 1$$

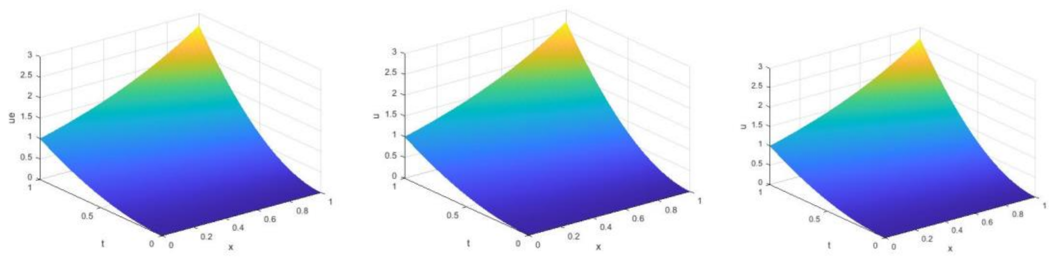
$u(0,t) = t^{1+\alpha}$, $u(1,t) = e^1 t^{1+\alpha}$, $u(x,0) = 0$,

with the exact solution: $u(x,t) = e^x t^{1+\alpha}$.

In Table 1, we present the maximum absolute errors (MAE) and numerical order of convergence (NOC), and the central processing unit times (CPUTs) in seconds that arise from using the linearly implicit Euler method and Crank-Nicolson method for example 4.1, by taking $\alpha = 0.9$ and different space and time steps. To support the numerical findings, Figure 1 shows the numerical simulations of the exact, IEM and CNM solutions for example 4.1, with $h = 1/40$, and $\alpha = 0.9$.

Table 1: Maximum absolute errors, numerical order of convergence and CPUTs, obtained from using linearly implicit Euler and C.N. methods for example 4.1, $\alpha = 0.9$

Method		IEM			CNM		
h	k	$E_{h,k}$	$S_{h,k}$	CPUT	$E_{h,k}$	$S_{h,k}$	CPUT
1/5	0.0239	0.0031	……..	0.0564	3.7610e-04	……..	0.1857
1/10	0.0051	6.1825e-04	2.3260	0.1106	1.1217e-04	1.7454	0.1899
1/20	0.0011	1.2448e-04	2.3123	0.2194	3.0727e-05	1.8681	0.2649
1/40	2.3492e-04	2.5316e-05	2.2978	4.2914	8.0831e-06	1.9265	4.4344



Exact **IEM** **CNM**

Figure 1: Exact, IEM, CNM**,** solutions for Example 4.1, with $\alpha = 0.9$**,** $h = 1/40$

Example 4.2

Consider the following one-dimensional time fractional semilinear diffusion equation:

$$u_t^{\alpha} = u_{xx} + \left(\frac{2t^{(2-\alpha)}}{\Gamma(3-\alpha)} - t^2\right)e^x - t^3e^{1.5x} + u^{3/2}, \quad 0 < x < 1, \;\; t > 0$$

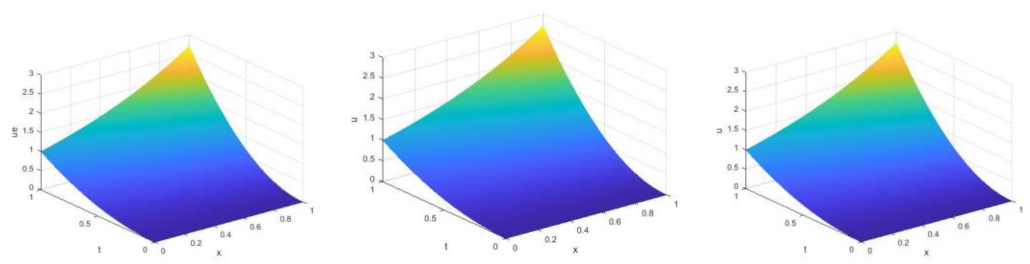
$$u(0,t) = t^2, \quad u(1,t) = e^1t^2, \quad u(x,0) = 0$$

With the exact solution: $u(x,t) = e^x t^2$

In Table 2, we present the maximum absolute errors (MAE) and numerical order of convergence (NOC), and the central processing unit times (CPUTs) in seconds that arise from using the linearly implicit Euler method and Crank-Nicolson method for example 4.2, by taking $\alpha = 0.85$ and different space and time steps. To support the numerical findings, Figure 2 shows the numerical simulations of the exact, IEM and CNM solutions for example 4.2, with $h = 1/40$, and $\alpha = 0.85$.

Table 2: Maximum absolute errors, numerical order of convergence and CPUTs, obtained from using linearly implicit Euler and C.N. methods for example 4.2, with $\alpha = 0.85$

Method		IEM			CNM		
h	k	$E_{h,k}$	$S_{h,k}$	CPUT	$E_{h,k}$	$S_{h,k}$	CPUT
1/5	0.0172	0.0021	……..	0.0787	3.9276e-04	……	0.0903
1/10	0.0034	3.8605e-04	2.4435	0.0796	1.2024e-04	1.7077	0.0849
1/20	6.6096e-04	7.3954e-05	2.3841	0.4256	3.2835e-05	1.8726	.05558
1/40	1.2938e-04	1.4894e-05	2.3119	12.4110	8.5879e-06	1.9349	14.3438



Exact **IEM** **CNM**

Figure 2: Exact, IEM, CNM**,** solutions for Example 4.2, with $\alpha = 0.85$**,** $h = 1/40$

4.2 Discussions of Numerical Results

Tables 1 and 2 show that for each method, the MAEs decrease, as we refine the space (time)-steps. Moreover, for each fixed value of α and time (space) steps, the MAEs of CNM are less than the MAEs of IEM. In addition, the NOC of both IEM and CNM are in a good agreement with the theoretical one. Furthermore, the required CPUTs for both IEM and CNM increase, as we refine the space (time)-steps. On the other hand, at any fixed space (time) step, the required CPUT for both IEM is less than the required one for CNM. In addition, from Figures 1 and 2, it can be easily noticed that the IEM and CNM solutions are in good agreement with the exact one.

5. Conclusions

In this paper, two numerical finite difference schemes: the implicit Euler scheme and the Crank-Nicolson scheme are proposed to solve a one-dimensional time-fractional order semilinear parabolic equation with homogeneous Dirichlet boundary conditions. The consistency, stability and convergence of the proposed methods are studied. In addition, two particular test cases are considered. We prove that the IEM is unconditionally stable, while CNM is conditionally stable.

The numerical results show that the two proposed methods are found to be in agreement with the theoretical computational analysis. It is also observed that with the proposed time-stepping formula (16) the CNM provides a more accurate strategic solution with less errors than the ones obtained from IEM. However, the numerical orders of convergence for both methods are close to 2, and that confirms the theoretical results. Moreover, the required CPUTs for both IEM and CNM increase, as we refine the space (time)-steps. Moreover, at any fixed space (time) step, the required CPUT for IEM is less than the required one for CNM.

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