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# Adding Rows in Main *e*-Abacus Diagrams

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**Abstract**
In the past years, there was an attempt to add one column to the *e*-abacus diagram, then another work followed by other researchers by adding several columns to the same diagram. At that time, a question remained, is it possible to add a row and later several rows on the same diagram? Or is this not an easy thing except under certain conditions? In this research, we will know when we can take these steps.
**Keywords**: Partition, *e*-abacus diagram, main *e*-abacus diagrams.

# اضافة صفوف على المخططات المعداد الرئيسة من النمط – *e*

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**الخلاصة:**
في السنوات الماضية كانت هناك محاولة اضافة عمود واحد على المخطط المعداد من النمط – *e* ثم لحقه عمل اخر لباحثين اخرين بإضافة عدة اعمدة على المخطط نفسه. وفي حينها بقى سؤال مطروح هل ممكن اضافة صف ولاحقا عدة صفوف على diagram نفسه؟ ام ان هذا الامر ليس سهلا الا وفق شروط معينة؟ سنتعرف في هذا البحث متى يمكننا القيام بهذه الخطوات.

## 1. Introduction:

Assume have the non-negative integers $\tau_1, \tau_2, \dots, \tau_n$ whose total equals the value *r*. A partition $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ is an arranging of these numbers that takes advantage of the fact that they are either equal or decreasing. For example, (6, 6, 3, 3, 2, 1) is a partition of 21, but (6, 6, 3, 2, 3, 1) is not. Defining, $b$ is a non-negative integer, $\beta_{\mathrm{j}} = \tau_j + b - j$, $1 \le j \le b$. The set $\{\beta_1, \beta_2, \cdots \beta_{\mathrm{b}}\}$ is said to be the set of $\beta$ - number for $\tau$, see [3]. Let *e* be a positive integer number greater than or equal to 2, we can represent numbers by a diagram called *e*-abacus diagram, see [4], as shown in Table 1:

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**Table 1:** e-Abacus Diagram

| **runner – 1** | **runner - 2** | **…** | **runner - *e*** |
|---|---|---|---|
| **0** | 1 | … | ***e* – 1** |
| ***e*** | *e* + 1 | … | **2*e* -1** |
| **2*e*** | 2*e* + 1 | … | **3*e* -1** |
| ⋮ | ⋮ | ⋮ | ⋮ |

Where every $\beta$ – numbers will be represented by (▪) which takes its location in *e*-abacus diagram and in case of nonexistence of $\beta$ - numbers, then it will be represented by (-). From the above example (6, 6, 3, 3, 2, 1):

**Table 2:** Abacus for (6, 6, 3, 3, 2, 1)

| **(6, 6, 3, 3, 2, 1)** | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **2-abacus** | | **3-abacus** | | | **4-abacus** | | | | **5-abacus** | | | | |
| - | ▪ | - | ▪ | - | - | ▪ | - | ▪ | - | ▪ | - | ▪ | - |
| - | ▪ | ▪ | - | ▪ | - | ▪ | ▪ | - | ▪ | ▪ | - | - | - |
| - | ▪ | ▪ | - | - | - | - | ▪ | ▪ | ▪ | ▪ | - | - | - |
| ▪ | - | - | ▪ | ▪ | | | | | | | | | |
| - | - | | | | | | | | | | | | |
| ▪ | ▪ | | | | | | | | | | | | |

Any partition has *e* main diagrams [5]. From the above example where (6, 6, 3, 3, 2, 1) and *e*=2 or 3, then:

**Table 3:** Main diagrams of (6, 6, 3, 3, 2, 1)

| **(6, 6, 3, 3, 2, 1) and *e* = 2** | | | | **(6, 6, 3, 3, 2, 1) and *e* = 3** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1 main diagram** | | **2 main diagram** | | **1 main diagram** | | | **2 main diagram** | | | **3 main diagram** | | |
| - | ▪ | ▪ | - | - | ▪ | - | ▪ | - | ▪ | ▪ | ▪ | - |
| - | ▪ | ▪ | - | ▪ | - | ▪ | - | ▪ | - | ▪ | - | ▪ |
| - | ▪ | ▪ | - | ▪ | - | - | ▪ | ▪ | - | - | ▪ | ▪ |
| ▪ | - | ▪ | ▪ | - | ▪ | ▪ | - | - | ▪ | - | - | - |
| - | - | - | - | | | | ▪ | - | - | ▪ | ▪ | - |
| ▪ | ▪ | - | ▪ | | | | | | | - | - | - |
| | | ▪ | - | | | | | | | | | |

**2. Adding single row in *e*-abacus diagram:**

In this section, we will discover how to add a single row in 1st main *e*-abacus diagram and the relation applied therein, also spell out the conditions for adding. Before that we need to know some of the following rules:

**Rule 2.1**: The number of rows in any partition $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ and 1st main *e*-abacus diagram is equal to:

$$(\boldsymbol{\tau_1} + \boldsymbol{n}) \div \boldsymbol{e} = \begin{cases} \boldsymbol{k} & \boldsymbol{if\ there\ is\ no\ remaining}, \\ \boldsymbol{k+1} & \boldsymbol{if\ there\ is\ any\ left}. \end{cases}$$

**Proof**: Here is also the procedure of collecting it with the number of parts of the partition, and then conducting the division by *e*, which is the method we will choose because the initial value of the partition has the most impact on the process of computing the *e*-abacus diagram. The division process will produce a row with the first partition's exact location at the conclusion. If there is a remainder, beads are present (and these will always have a number fewer than the e that was picked), thus we must add an extra row to hold the beads that are left over.

From the above example; (6, 6, 3, 3, 2, 1), if we take *e*=2 then (6+6)$\div 2 = 6$, but in *e*=5 then (6+6)$\div 5 = 2$ and the remaining 2, then we have 3 rows; see (Table 2).

**Rule 2.2**: The location of the beads in any row of the 1st main *e*-abacus diagram is as follows:

$$(\boldsymbol{\tau_m} + (\boldsymbol{n} - (\boldsymbol{m} - \boldsymbol{1}))) \div \boldsymbol{e} = \begin{cases} \boldsymbol{f} & \boldsymbol{if\ there\ is\ no\ remaining}, \\ \boldsymbol{f+1} & \boldsymbol{if\ there\ is\ any\ left}. \end{cases}$$

Where $1 \le m \le n$ and $1 \le f \le k$.
**Proof**: The same method of rule 2.1.

**Rule 2.3:** The following relation guides how to add a single row in 1st main *e*-abacus diagram:

$$\boldsymbol{l = ak + w - b}$$

where:
$\boldsymbol{l}$ is an integer greater than or equal to 0,
$\boldsymbol{a}$ is number of beads to be added at the beginning of the row ($\boldsymbol{1 \le a \le e}$),
$\boldsymbol{k}$ is number of main rows in *e*-abacus diagram,
$\boldsymbol{w}$ is row location to be added ( $\boldsymbol{1 \le w \le k}$), and $\boldsymbol{b}$ is the total number of beads.

**Rule 2.4: The conditions for adding:**
The addition is made according to several conditions, as follows:
1. After the placement of the $\boldsymbol{\beta}$ - number in the diagram they represent, the symbol $(\boldsymbol{1})$ will designate the first row, the symbol $(\boldsymbol{2})$ will designate the next row, then the third row will be designate by the symbol $(\boldsymbol{3})$ ... up till the final row, which designate by the symbol $(\boldsymbol{k})$.
2. Applying the above mentioned relation $\boldsymbol{l = ak + w - b}$ , to determine $\boldsymbol{a}$ and $\boldsymbol{w}$ values.
3. If the number of beads that must be added surpasses **e** , we will stop .
Adding this new row will obtain a new partition that will be denoted by the symbol $^{+l}\boldsymbol{\tau}$.

**Example 1.**
Assume have the partition $\boldsymbol{\tau = (5, 5, 4, 1)}$ where $\boldsymbol{b = 4}$, $\boldsymbol{k = 3}$, the abacus diagram where $\boldsymbol{e = 3}$ will be as follows:

| | | | |
|---|---|---|---|
| **(1)** | - | ▪ | - |
| **(2)** | - | - | ▪ |
| **(3)** | - | ▪ | ▪ |

by applying the relation $\boldsymbol{l = ak + w - b}$ , where ( $\boldsymbol{1 \le w \le k}$),

**Table 4:** Adding single row **of** $\boldsymbol{\tau} = (\mathbf{5}, \mathbf{5}, \mathbf{4}, \mathbf{1})$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $^{+0}\tau=$ | | ▪ | - | - | $^{+1}\tau =$ | **(1)** | - | ▪ | - |
| | **(1)** | - | ▪ | - | | | ▪ | - | - |
| | **(2)** | - | - | ▪ | | **(2)** | - | - | ▪ |
| | **(3)** | - | ▪ | ▪ | | **(3)** | - | ▪ | ▪ |
| $^{+2}\tau =$ | **(1)** | - | ▪ | - | $^{+3}\tau=$ | | ▪ | ▪ | - |
| | **(2)** | - | - | ▪ | | **(1)** | - | ▪ | - |
| | | ▪ | - | - | | **(2)** | - | - | ▪ |
| | **(3)** | - | ▪ | ▪ | | **(3)** | - | ▪ | ▪ |
| $^{+4}\tau=$ | **(1)** | - | ▪ | - | $^{+5}\tau=$ | **(1)** | - | ▪ | - |
| | | ▪ | ▪ | - | | **(2)** | - | - | ▪ |
| | **(2)** | - | - | ▪ | | | ▪ | ▪ | - |
| | **(3)** | - | ▪ | ▪ | | **(3)** | - | ▪ | ▪ |
| $^{+6}\tau =$ | | ▪ | ▪ | ▪ | $^{+7}\tau=$ | **(1)** | - | ▪ | - |
| | **(1)** | - | ▪ | - | | | ▪ | ▪ | ▪ |
| | **(2)** | - | - | ▪ | | **(2)** | - | - | ▪ |
| | **(3)** | - | ▪ | ▪ | | **(3)** | - | ▪ | ▪ |
| $^{+8}\tau=$ | **(1)** | - | ▪ | - | | | | | |
| | **(2)** | - | - | ▪ | | | | | |
| | | ▪ | ▪ | ▪ | | | | | |
| | **(3)** | - | ▪ | ▪ | | | | | |

while the following case , which follows it upward does not fulfill the equation because it does not fulfill the condition: $\mathbf{9} = [\mathbf{4}]\mathbf{3} + [\mathbf{1}] - \mathbf{4}$

$$^{+9}\boldsymbol{\tau} = \begin{array}{lcccc} & ▪ & ▪ & ▪ & ▪ \\ \textbf{(1)} & - & ▪ & - & \\ \textbf{(2)} & - & - & ▪ & \\ \textbf{(3)} & - & ▪ & ▪ & \end{array}$$

the number of beads that must be added surpasses $e$, making it impossible for us to proceed, Therefore, we shall pause at case $^{+8}\boldsymbol{\tau}$, where $a$=3 , $w$=3.

**Theorem 2.5:** Let $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ be any partition of $r$ , with $k$ rows and $e$ columns in 1st main e-abacus diagram associated to it, then following the addition of a one row, the new partition $(^{+l}\boldsymbol{\tau})$ where $(1 \le a \le e)$ and $(1 \le w \le k)$, $b^* = \sum_{t=1}^{w-1} b_t$ is the sum of the beads in row (1) to row $(w-1)$, will appear as follows:

$$^{+l}\boldsymbol{\tau} = \begin{cases} ((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_n+(e-a))) & , \ if\ w=1, \\ ((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_{(n-b^*)}+(e-a)),((w-1)e-b^*)^a,(\tau_{n-(h_{b^*}-1)})) & , \ if\ w>1\ and\ b^*>0, \\ ((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_n+(e-a)),((w-1)e)^a) & , \ if\ w>1\ and\ b^*=0. \end{cases}$$

where $h_{b^*}=b^*,\ b^*-1,\dots,1.$

**Proof:** Let be $\tau=(\tau_1,\tau_2,\dots,\tau_n)$ and if $\boldsymbol{w=1}$, then we have: If $a=1$ then

$$^{+l}\boldsymbol{\tau}=((\tau_1+(e-1)),(\tau_2+(e-1)),\dots,(\tau_n+(e-1))),$$

also if $a=2$ then

$$^{+l}\boldsymbol{\tau}=((\tau_1+(e-2)),(\tau_2+(e-2)),\dots,(\tau_n+(e-2))),$$

Now for any $(1\le a\le e)$ then $^{+l}\boldsymbol{\tau}=((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_n+(e-a)))$.

With the exception of one of the spaces where the beads will be and whose location is to the left of the row from the top, which has no bearing on the partition value, this addition will be made in the first exclusively in the order that it is a complete row of spaces. The remaining spaces in the same row, however, are those that have the greatest influence.

Now, if $\boldsymbol{w=2\ \ and\ b^*>0}$, then we have three areas make up an abacus: the top region, which is essentially the initial row, the second region, which is the row that was added by e spaces and contains a region of beads that is located to the left of this new row, and the final region, which is the remaining abacus rows. Without any problems, the partition value for the third region will be determined as if $w$ = 1. Simple computations will be made in the second region, which will have an impact on the partition value there and in the final region according to the following:

$$^{+l}\boldsymbol{\tau}=\underbrace{(((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_{(n-b_1)}+(e-a))}_{region\ 3},\underbrace{((w-1)e-1)^a}_{region\ 2},\underbrace{(\tau_{n-(h_1-1)}))}_{region\ 1}\ \dots \qquad (s_1)$$

where $b_1$ is the sum of the beads in row (1) , $h_1=b_1,b_1-1,\dots,1$.
Where $w=3$

$$^{+l}\boldsymbol{\tau}=\underbrace{(((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_{(n-b_u)}+(e-a))}_{region\ 3},\underbrace{((w-1)e-b_u)^a}_{region\ 2},\underbrace{(\tau_{n-(h_u-1)}))}_{region\ 1}\dots \qquad (s_u)$$

where $b_u$ is the sum of the beads in row ($u$) where ($u$ = 1 and 2) and $h_u=(b_u),\ (b_u-1),\dots,1$.
Now for any $1<w\le k$ and by using $(s_1)$ and $(s_u)$ then we have

$$^{+l}\boldsymbol{\tau}=((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_{(n-b^*)}+(e-a)),((w-1)e-(b_1+b_2+\dots+b_{(w-1)}))^a,(\tau_{n-(h-1)}))$$

$$=((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_{(n-b^*)}+(e-a)),((w-1)e-(\textstyle\sum_{t=1}^{w-1}b_t))^a,(\tau_{n-(h-1)}))$$

$$=\underbrace{((\tau_1+(e-a)),(\tau_2+(e-a)),\dots,(\tau_{(n-b^*)}+(e-a))}_{region\ 3},\underbrace{((w-1)e-(b^*))^a}_{region\ 2},\underbrace{(\tau_{n-(\boldsymbol{h_{b^*}}-1)}))}_{region\ 1}\ \dots \qquad (s_{b^*})$$

where $b^*$ is the sum of the beads in row (1) to row $(w-1)$ and $h_{b^*}=b^*,\ (b^*-1),\dots,1$
Finally , if $\boldsymbol{w>1\ \ and\ b^*=0}$, As the above case ($\boldsymbol{w>1\ \ and\ b^*>0}$) we have three areas make up an abacus but the top region won't have any beads, thus it won't have locations that would result its mean the $(\tau_{n-(\boldsymbol{h_{b^*}}-1)})$ term being zero which is essentially unavailable in a partition $\tau$, so it will be removed.
Let's take for example the cases $^{+3}\boldsymbol{\tau}$ and $^{+5}\boldsymbol{\tau}$ from the previous example1 mentioned above , then

to determine ${}^{+3}\tau$ , from $3 = [2]3 + [1] - 4 \;\Rightarrow a = 2,\; w = 1$ since $w = 1$ so first portion of Theorem 2.5 will be applied,

$$\begin{aligned} {}^{+l}\tau &= ((\tau_1 + (e - a)), (\,\tau_2 + (e - a)), \dots\;, (\;\tau_n + (e - a)\,)) \\ &= ((\tau_1 + (e - a)), (\tau_2 + (e - a)), (\,\tau_3 + (e - a)), (\tau_4 + (e - a))) \\ &= \Big(\big(5 + (3 - 2)\big), \big(5 + (3 - 2)\big), \big(\,4 + (3 - 2)\big), \big(1 + (3 - 2)\big)\Big) = (6, 6, 5, 2). \end{aligned}$$

To determine ${}^{+5}\tau$ , from $5 = [2]3 + [3] - 4 \;\Rightarrow a = 2,\; w = 3\; > 1$

to get the number of rows in $\tau = (5, 5, 4, 1)$ we use the following rule :

$(\tau_1 + n) \div e = \begin{cases} k & if\ there\ is\ no\ remaining, \\ k + 1 & if\ there\ is\ any\ left. \end{cases} \to (5 + 4) \div 3 = 3$ without remaining so $k = 3$ to get location of the beads in $\tau = (5, 5, 4, 1)$ we use the following rule:

$$(\tau_m + (n - (m - 1))) \div e = \begin{cases} f & if\ there\ is\ no\ remaining, \\ f + 1 & if\ there\ is\ any\ left. \end{cases}$$

Where $1 \leq m \leq n$ and $1 \leq f \leq k$.

$$\left.\begin{aligned} &(\tau_1 + (4 - (1 - 1))) \div 3 = (5 + (4 - (1 - 1))) \div 3 = 3 \\ &(\tau_2 + (4 - (2 - 1))) \div 3 = (5 + (4 - (2 - 1))) \div 3 = 2, there\ is\ any\ left \\ &(\tau_3 + (4 - (3 - 1))) \div 3 = (4 + (4 - (3 - 1))) \div 3 = 2, \\ &(\tau_4 + (4 - (4 - 1))) \div 3 = (1 + (4 - (4 - 1))) \div 3 = 0.6, there\ is\ any\ left \end{aligned}\right\} \to b_1$$

$$= 1, b_2 = 1, b_3 = 2$$

now, we will determine $b^*$ , $b^* = \sum_{t=1}^{w-1} b_t \;= \sum_{t=1}^{2} b_t \;= b_1 + b_2 = 2 > 0$ so second portion of Theorem 2.5 will be applied,

$$((\tau_1 + (e - a)), (\tau_2 + (e - a)), \dots, (\tau_{(n-b^*)} + (e - a)), ((w - 1)e - b^*)^a\,, (\,\tau_{n-(h_{b^*}-1)}\,))$$

where $h_{b^*} = b^*,\; b^* - 1, \dots\;, 1$.

$h_{b^*} = 2, 1$ , $(\tau_{(n-b^*)} = \tau_{(4-2)} = \tau_2$ ,

$\tau_{n-(h_{b^*}-1)} = \;\tau_{4-(2-1)} \;=\; \tau_3$ and $\tau_{4-(1-1)} \;=\; \tau_4$

$$\begin{aligned} {}^{+l}\tau &= ((\tau_1 + (e - a)), (\tau_2 + (e - a)), \dots, (\tau_{(n-b^*)} + (e - a)), ((w - 1)e \\ &\qquad - b^*)^a\,, (\,\tau_{n-(h_{b^*}-1)}\,)) \\ &= ((\tau_1 + (e - a)), (\tau_2 + (e - a)), \qquad ((w - 1)e - b^*)^a\,, \qquad \tau_3\,, \tau_4) \\ &= (5 + (3 - 2)), (\tau_2 + (3 - 2)), \qquad ((3 - 1)3 - 2)^2\,, \qquad 4\,, 1) \;= (6, 6, 4, 4, 4, 1) \end{aligned}$$

With the same steps, other cases are calculated ${}^{+0}\tau = (7,7,6,3)$, ${}^{+1}\tau = (7, 7, 6, 2, 1)$, ${}^{+2}\tau = (7, 7, 4, 4, 1)$, ${}^{+4}\tau = (6, 6, 5, 2, 2, ,1)$, ${}^{+6}\tau = (5, 5, 4, 1)$, ${}^{+7}\tau = (5, 5, 4, 2, 2, 2, 1)$, ${}^{+8}\tau = (5, 5, 4, 4, 4, 4,1)$.

**Example 2.**

Let $\tau = (3, 3, 3, 2, 2)$ , the abacus diagram where $e = 2$ will be as follows:

| | | |
|---|---|---|
| (1) | - | - |
| (2) | ▪ | ▪ |
| (3) | - | ▪ |
| (4) | ▪ | ▪ |

**Table 5:** Adding single row for $\boldsymbol{\tau} = (\mathbf{3}, \mathbf{3}, \mathbf{3}, \mathbf{2}, \mathbf{2})$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $^{+0}\boldsymbol{\tau}=$ | | ▪ | - | $^{+1}\boldsymbol{\tau}=$ | (1) | - | - |
| | **(1)** | - | - | | | ▪ | - |
| | **(2)** | ▪ | ▪ | | **(2)** | ▪ | ▪ |
| | **(3)** | - | ▪ | | **(3)** | - | ▪ |
| | **(4)** | ▪ | ▪ | | **(4)** | ▪ | ▪ |
| $^{+2}\boldsymbol{\tau}=$ | **(1)** | - | - | $^{+3}\boldsymbol{\tau}=$ | **(1)** | - | - |
| | **(2)** | ▪ | ▪ | | **(2)** | ▪ | ▪ |
| | | ▪ | - | | **(3)** | - | ▪ |
| | **(3)** | - | ▪ | | | ▪ | - |
| | **(4)** | ▪ | ▪ | | **(4)** | ▪ | ▪ |
| $^{+4}\boldsymbol{\tau}=$ | | ▪ | ▪ | $^{+5}\boldsymbol{\tau}=$ | **(1)** | - | - |
| | **(1)** | - | - | | | ▪ | ▪ |
| | **(2)** | ▪ | ▪ | | **(2)** | ▪ | ▪ |
| | **(3)** | - | ▪ | | **(3)** | - | ▪ |
| | **(4)** | ▪ | ▪ | | **(4)** | ▪ | ▪ |
| $^{+6}\boldsymbol{\tau}=$ | **(1)** | - | - | $^{+7}\boldsymbol{\tau}=$ | **(1)** | - | - |
| | **(2)** | ▪ | ▪ | | **(2)** | ▪ | ▪ |
| | | ▪ | ▪ | | **(3)** | - | ▪ |
| | **(3)** | - | ▪ | | | ▪ | ▪ |
| | **(4)** | ▪ | ▪ | | **(4)** | ▪ | ▪ |

to determine $^{+1}\tau$ , from $l = ak + w - b \to 0 = (1)4 + (2) - 5 \to \ a = 1, w = 2 \ > 1$
by applying Rule 2.1 and Rule 2.2 we have $k = 4$ , $b_1 = 0, b_2 = 2, b_3 = 1$ and $b_4 = 2$ .

now, we will determine $b^*$ , $b^* = \sum_{t=1}^{w-1} b_t \ = \sum_{t=1}^{1} b_t \ = b_1 = 0$ , since $b^* = 0$ so we will use the third part of Theorem 2.5,

$$\begin{aligned} ^{+l}\tau \ &= ((\tau_1 + (e-a)), (\tau_2 + (e-a)), \ldots, (\tau_n + (e-a)), ((w-1)e)^a \,) \\ &= ((\tau_1 + (e-a)), (\tau_2 + (e-a)),\ (\tau_3 + (e-a)), (\tau_4 + (e-a)), (\tau_5 \\ &\qquad + (e-a)), ((w-1)e)^a) \\ &= ((3 + (2-1)), (3 + (2-1)), (3 + (2-1)), (2 + (2-1)), (2 + (2-1)), ((2 - \\ &\quad 1)2)^1) \\ &= (4, 4, 4, 3, 3, 2). \end{aligned}$$

With the same steps, other cases are calculated

$$^{+0}\tau = (4,4,4,3,3),\ ^{+2}\tau = (4,4,4,2,2,2),\ ^{+3}\tau = (4,4,3,3,2,2),\ ^{+4}\tau = (3,3,3,2,2),$$
$$^{+5}\tau = (3,3,3,2,2,2,2),\ ^{+6}\tau = (3,3,3,2,2,2,2) \text{ and } ^{+7}\tau = (3,3,3,3,3,2,2).$$

Previous results on 1st main *e*-abacus diagram made us think about the possibility of circulating those relationships on the rest of diagrams; 2nd to last main *e*-abacus, directly to all of the above according to Rule 2.1 – Theorem 2.5 we have:

**Rule 2.6**: The location of the beads in any row of the $q$ main $e$-abacus diagrams is as follows:

$$(\boldsymbol{\tau_m} + ((\boldsymbol{n} + (\boldsymbol{q} - \mathbf{1})) - (\boldsymbol{m} - \mathbf{1}))) \div \boldsymbol{e} = \begin{cases} \boldsymbol{f} & \boldsymbol{if\ there\ is\ no\ remaining}, \\ \boldsymbol{f} + \mathbf{1} & \boldsymbol{if\ there\ is\ any\ left}. \end{cases}$$

Where $1 \leq m \leq \big(n + (q-1)\big),\ 2 \leq q \leq e$ and $1 \leq f \leq k$.

**Rule 2.7:** The number of rows in any partition $\tau = (\tau_1, \tau_2, \dots, \tau_n)$ and the $q$ main $e$-abacus diagrams is equal to:

$$(\boldsymbol{\tau_1} + (\boldsymbol{n} + (\boldsymbol{q} - \mathbf{1}))) \div \boldsymbol{e} = \begin{cases} \boldsymbol{k} & \boldsymbol{if\ there\ is\ no\ remaining}, \\ \boldsymbol{k} + \mathbf{1} & \boldsymbol{if\ there\ is\ any\ left}. \end{cases}$$

Where $2 \leq q \leq e$.

**Theorem 2.8:** The new partition $({}^{+l}\boldsymbol{\tau})$ will appear as follows for any $q$ main $e$-abacus diagrams is equal to:

$$\begin{aligned} &{}^{+l}\boldsymbol{\tau} \\ &= \begin{cases} ((\tau_1 + (e-a)), \dots, \big(\tau_n + (e-a)\big), (e-a)^{(q-1)}) & , \ if\ w = 1, \\ ((\tau_1 + (e-a)), \dots, (\tau_{(n-(b^*-(q-1))} + (e-a)), ((w-1)e - b^*)^a, (\tau_{n-(h_{b^*}-1)})) , & if\ w > 1 and\ b^* > (q-1), \\ ((\tau_1 + (e-a)), \dots, (\tau_n + (e-a)), ((w-1)e - b^*)^a) & , \ if\ w > 1\ and\ b^* = (q-1). \end{cases} \end{aligned}$$

where $h_{b^*} = \ ((b^* - (q-1)),\ ((b^* - (q-1)) - 1) \dots, 1)$ and $2 \leq q \leq e$.

## 3. Generation $( {}^{*}\boldsymbol{\tau})$ of $({}^{+l}\boldsymbol{\tau})$:

Through recent studies on this topic, we see that what was presented above can make us think that if we have a particular diagram, is it possible to get the "origin" of the diagram? and we mean here that we delete all the rows that come at the beginning of the bead(s) from without any space between them within the same row but without using the e-abacus diagram? The answer was yes, through what was presented by Satyanarayana and Prasad [6], Rosen [7] Hashim [8], as follows:

**Example 3.**

Let a partition $(\mathbf{11}, \mathbf{9}, \mathbf{7}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{4}, \mathbf{3}, \mathbf{3})$ **with** $\boldsymbol{e} = \mathbf{4}$**:**

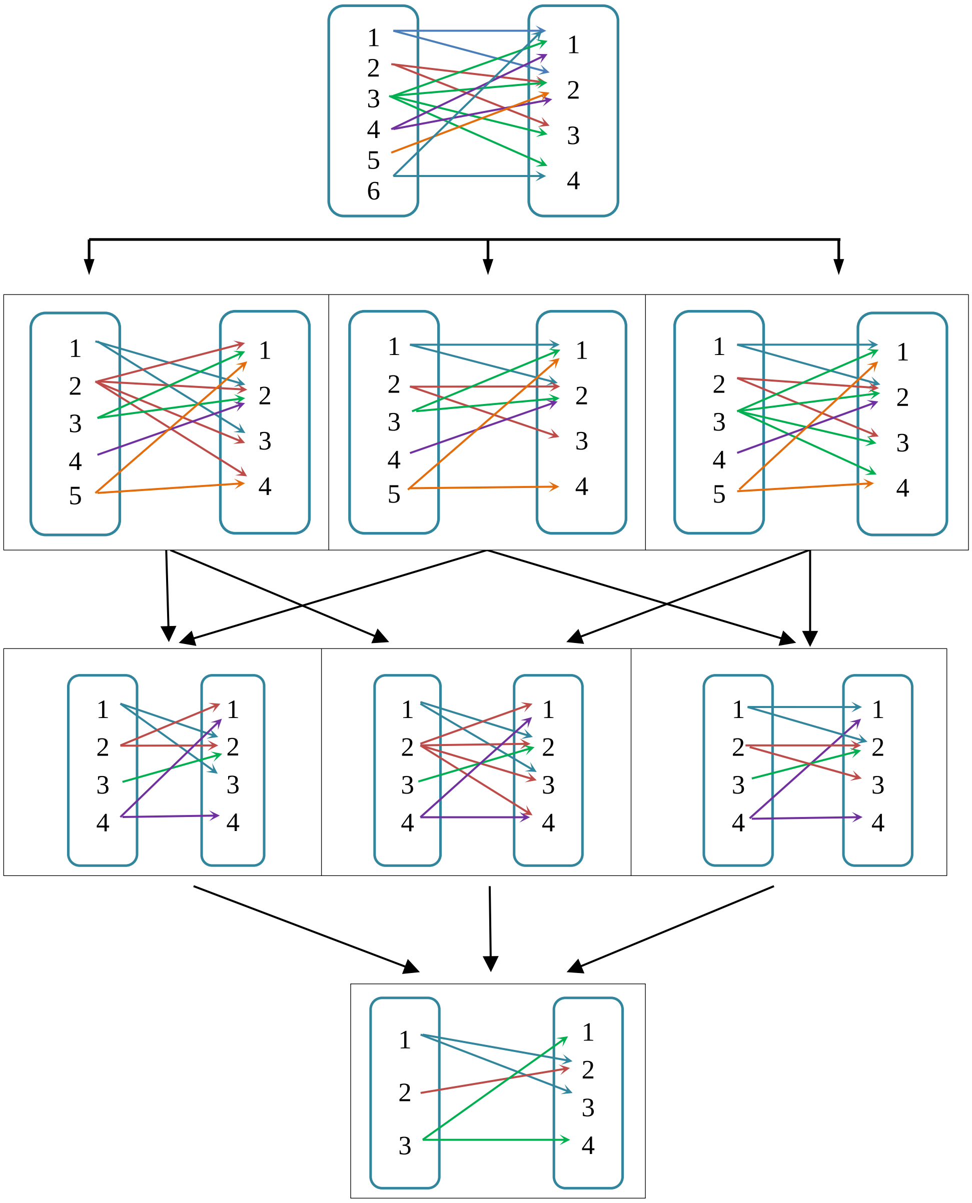


**Figure 1:** The new representation of (11,9,7,4,4,4,4,4,4,3,3) Or by graph see [9], [10] and [11]:

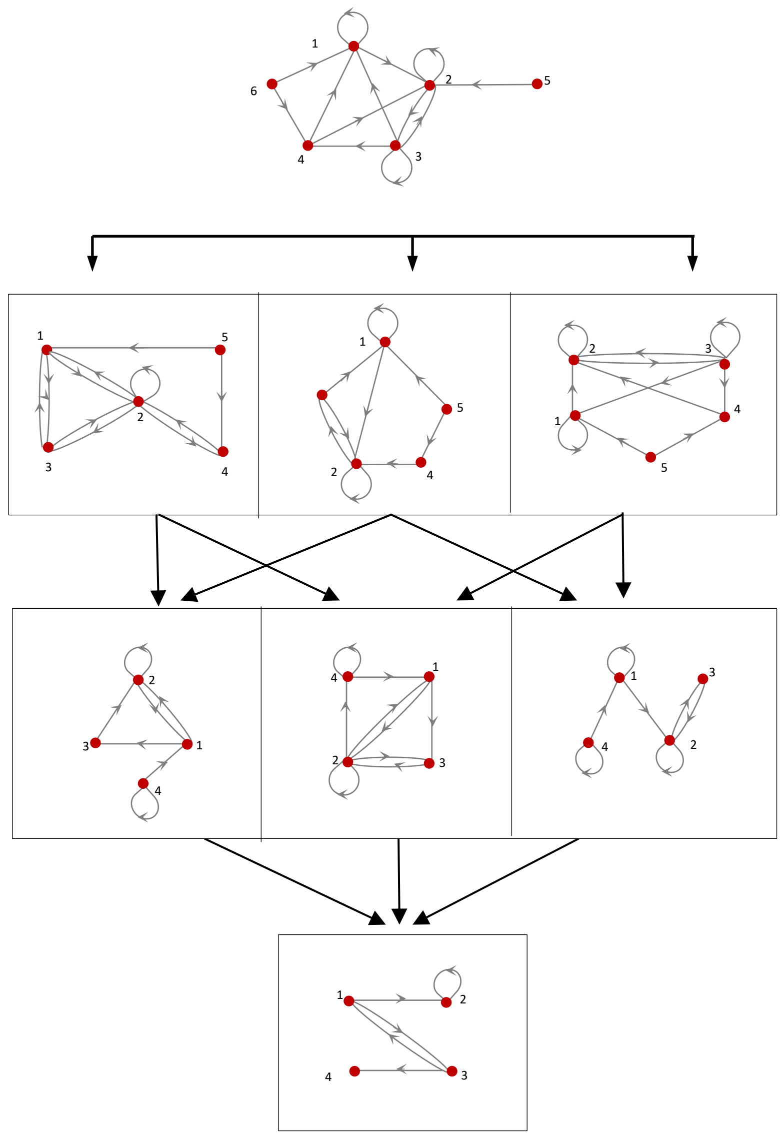

**Figure 2:** The Graph of (11,9,7,4,4,4,4,4,4,3,3)

## 4. Adding more than one row in *e*-abacus diagram:

In this part, we will stretched the above mentioned relation $\boldsymbol{l = ak + w - b}$, to adding more than one row for each of the cases of $^{+l}\boldsymbol{\tau}$ , so we get a tree of partitions denoted by ( $^{+l,\ +l',\ +l'',\ \cdots}\boldsymbol{\tau}$ ) as the following:

1.After completing the conditions of the addition mentioned in (Rules 2.4 and 2.6) , specifically the second condition and representing it in e- abacus diagram , we will assume that the new case is the initial case.

2.The values we will adopt are the $\boldsymbol{b}$ beads' total numbers, which vary with each addition and it is symbolized by the symbol $\boldsymbol{b'}$, $\boldsymbol{b' = b + a}$

where :

$\boldsymbol{b}$ is the number of beads that were mentioned before adding ,

$\boldsymbol{a}$ is the number of beads added which equals $\boldsymbol{\frac{b+l-w}{k}}$

The above provides us $\boldsymbol{b'} = \boldsymbol{\frac{(k\times b)+b+l-w}{k}}$

moreover, $((\boldsymbol{k} \times \boldsymbol{b}) + \boldsymbol{b} + \boldsymbol{l} - \boldsymbol{w})$ must be divisible by $\boldsymbol{k}$ with no residual, and k transforms into $\boldsymbol{k'} = \boldsymbol{k} + \mathbf{1}$ as a result.
3.Repeating the previous actions results in new rows, if the same situation as in step (3) of conditions of the addition (2.4 and 2.6) applies, it stops.

**Example 3.** Tree ${}^{+1,\ +4,\ +2}\,\tau$ of partition $(5, 5, 4, 1)$ is :

**Table 8:** Tree of partition $(5, 5, 4, 1)$

| $\tau$ | | | |
|---|---|---|---|
| (1) | - | ▪ | - |
| (2) | - | - | ▪ |
| (3) | - | ▪ | ▪ |

→

| ${}^{+1}\tau$ | | | |
|---|---|---|---|
| (1) | - | ▪ | - |
| | ▪ | - | - |
| (2) | - | - | ▪ |
| (3) | - | ▪ | ▪ |

→

| ${}^{+1,\ +4}\tau$ | | | |
|---|---|---|---|
| | ▪ | ▪ | - |
| (1) | - | ▪ | - |
| | ▪ | - | - |
| (2) | - | - | ▪ |
| (3) | - | ▪ | ▪ |

→

| ${}^{+1,\ +4,\ +2}\tau$ | | | |
|---|---|---|---|
| | ▪ | ▪ | - |
| (1) | - | ▪ | - |
| | ▪ | - | - |
| | ▪ | - | - |
| (2) | - | - | ▪ |
| (3) | - | ▪ | ▪ |

## Conclusion:

It is clear that this work is complementary to the works in [1] and [2], who focused on adding one column and then several columns, but this time it was adding a row and later rows. On the other hand, the association with maps, which is what made this subject take a new way of appearing, with the exception of the association with the graph, which nearly five years ago took on a new and useful reality. All of these will be important for application of this topic in any subsequent study. The only note that these techniques cannot be applied to is the partition theory notation of English letters [12] or Syriac Letters [13], most of these letters do not have adjacent bead rows without any space between them and so the process will be exposed and therefore not recommended here at all.

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