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The Structure of the Generalized Cayley Graph $Cay_m(D_{2n}, S)$ of Valency Three

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Abstract

Suppose that G is a finite group and S is a non-empty subset of G such that $e \notin S$ and $S^{-1} \subseteq S$. Suppose that $Cay(G, S)$ is the Cayley graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. In this paper,we introduce the generalized Cayley graph denoted by $Cay_m(G, S)$ that is a graph with vertex set consists of all column matrices X_m which all components are in G and two vertices X_m and Y_m are adjacent if and only if $X_m[(Y_m)^{-1}]^t \in M(S)$, where ${Y_m}^{-1}$ is a column matrix that each entry is the inverse of similar entry of Y_m and $M(S)$ is $m \times m$ matrix with all entries in S , $[Y^{-1}]^t$ is the transpose of Y^{-1} and $m \geq 1$. We aim to determine the structure of Cay_m(G, S) when G is the dihedral group of order 2n and | S |= 3 for every $m \geq 2$, n ≥ 3.

Keywords: Cayley graph , dihedral group, generalized Cayley graph, Crtisian product, Corona product.
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1 Introduction

Algebraic graph theory has been considered as one of the important topics in mathematics that specialists in algebra and graph theory have been interested in the recent years. In algebraic graph theory, every graph is associated to a group, ring, module or any other algebraic structures. One of the oldest algebraic graph theory is Cayley graph which is associated to a group and a subset of this group. The history of Cayley graph came back to many years ago. In 1878, Cayley graph was introduced by Arthur Cayley in [1]. He gave a geometrical representation of group by means of a set of generators. This translates groups into geometrical objects which can be investigated from the geometrical view. In particular, it provides a rich source of highly symmetric graphs, known as transitive graphs, which plays an important role in many graph theoretical problems and group theoretical problems. During the past ten years, some authors introduced different generalizations for the Cayley graph. For example, Marušič in [2] gave a generalization of the Cayley graph in terms of an automorphism of group G. Afterwards, Zho in [3] introduced the Cayley graph on a semigroup. Recently, the second author introduced a new generalization of Cayley graph by replacing all elements of group by all m×1 matrices with entries in the group, as a vertex set. He denoted it by $Cay_m(G,S)$ for every $m \geq 1$, and it is clear that if $m = 1$ then we will achieve the known Cayley graph Cay(G,S). In 2021, Neamah , Erfanian and others [4,5] established the structure of a generalized Cayley

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graph $\mathrm{Cay_m}$(G,S), when Cay(G,S) is a cycle graph C_n, for all $n \geq 3$.

In this paper, we are going to determine the structure of the $\mathrm{Cay_m}(D_{2n}$,S) where D_{2n} is the dihedral group of order 2n and S is a non empty subset of D_{2n} such that $e \notin S$, $S^{-1} \subseteq S$ and $1 \leq |S| \leq 3$ for every $m \geq 2$, $n \geq 3$.

We recall that for any group G and any nonempty subset S of G with $e \notin S$ and $S^{-1} \subseteq S$, the Cayley graph $\mathrm{Cay}(G, S)$ is an undirected simple graph whose vertices are all elements of G and two vertices x and y are adjacent if and only if $xy^{-1} \in S$. It is known that $\mathrm{Cay}(G, S)$ is a connected graph whenever S is a generating set of G and that it is always regular and vertex transitive (see [6] for more details). Now, we can define the generalized Cayley graph $\mathrm{Cay_m}(G, S)$ as follows.

Definition 1.1. [7] For every $m \geq 1$, the generalized Cayley graph, denoted by $\mathrm{Cay_m}(G, S)$ is an undirected simple graph with vertex set consisting all $m \times 1$ matrices $[x_1 \quad x_2 \quad \cdots \quad x_m]^t$, where $x_i \in G$, $1 \leq i \leq m$, and two vertices $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ and $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]^t$ are adjacent if and only if

$$X(Y^{-1})^t = \begin{bmatrix} x_1{y_1}^{-1} & x_1{y_2}^{-1} & \cdots & x_1{y_m}^{-1} \\ x_2{y_1}^{-1} & x_2{y_2}^{-1} & \cdots & x_2{y_m}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_m{y_1}^{-1} & x_m{y_2}^{-1} & \cdots & x_m{y_m}^{-1} \end{bmatrix} \in M_{m\times m}(S), \text{ where}$$

$$M_{m\times m}(S) = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ x_{21} & x_{22} & \cdots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mm} \end{bmatrix} \;\middle|\; x_{ij} \in S\;, \quad 1 \leq i, j \leq m \right\}.$$

In the following lemma from [2], we can find a necessary and sufficient condition for two arbitrary vertices in $\mathrm{Cay_m}(G, S)$ to be adjacent.

Lemma 1.2. [8] Let $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ and let $Y = [y_1 \quad y_2 \quad \cdots \quad y_m]^t$ be two vertices in $\mathrm{Cay_m}(G, S)$, where $x_i, y_j \in G$ for $1 \leq i, j \leq m$. Then X and Y are adjacent in $\mathrm{Cay_m}(G, S)$ if and only if x_i is adjacent to y_j in $\mathrm{Cay}(G, S)$ for all $1 \leq i, j \leq m$. The following lemma gives a formula for the degree of any vertex in the $\mathrm{Cay_m}(G, S)$ in terms of some right cosets of S.

Lemma 1.3. [8] Let $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ be a vertex in the $\mathrm{Cay_m}(G, S)$. Then $\deg(X) = |\bigcap_{i=1}^{m} Sx_i|$.

As we mentioned earlier, $\mathrm{Cay}(G, S)$ is connected (by assuming S as a generating set of G), so there is no isolated vertex. Indeed, one can easily see that $\mathrm{Cay_m}(G, S)$ is not necessary to be connected, even when S is a generating set and we may have some isolated vertices [6]. The following lemma states that under some conditions, we may have an isolated vertex in $\mathrm{Cay_m}(G, S)$.

Lemma 1.4. [8] Suppose that $X = [x_1 \quad x_2 \quad \cdots \quad x_m]^t$ is a vertex in $\mathrm{Cay_m}(G, S)$. If $d(x_i, x_j) \neq 2$ in $\mathrm{Cay}(G, S)$ for some $1 \leq i \neq j \leq m$ and the $\mathrm{Cay}(G, S)$ is triangle free. Then X is an isolated vertex in the $\mathrm{Cay_m}(G, S)$ (note that $d(x_i, x_j)$ stands for the distance between x_i and x_j, which is the length of the shortes path between x_i and x_j and triangle free means that the graph must have no cycle of lengh 3).

As we mentioned at the beginning of this paper, the structure of the $\mathrm{Cay_m}(G,S)$ whenever the $\mathrm{Cay}(G,S)$ is a complete graph K_n is investigated by Naeemah et. al. in [5], for all $n \geq 3$ and $m \geq 2$. Moreover, , the structure of the $\mathrm{Cay_m}(G,S)$ when $\mathrm{Cay}(G,S)$ is a cycle graph $C_n$by [8]. By using these structures we are going to to find the structure of $\mathrm{Cay_m}(D_{2n},S)$ with $|S| = 1,2,3$. First, let us state the structure of $\mathrm{Cay}(D_{2n}, S)$ for the case $|S| = 1,2,3$. One can see that if $|S| = 1$ then Cay(G,S) is the union of n disjoin edges. In other words, $\mathrm{Cay}(D_{2n}, S)=nK_2$. Similarly, for the case $|S| = 2$ the structure of $\mathrm{Cay}(D_{2n}, S)$ is the union of some cycles. For these two cases, the structure of $\mathrm{Cay}(D_{2n}, S)$ can be deduced directly from [8]. So, we focus on the case $|S| = 3$. We know that if $|S| = 3$, then we have two cases. The first case is $S = \{x, x^{-1}, y\}$, where $x \neq x^{-1}$ and $y^2 = e$ and the second case is when $S = \{x, y, z\}$, with $x^2 = y^2 = z^2 = e$. The following two theorems from [1] give the structure of $\mathrm{Cay}(D_{2n}, S)$ for each of the above two cases for all $n \geq 3$.

Theorem 1.5. [9] Assume that $S = \{x, x^{-1}, y\} \subseteq D_{2n}$ such that $x \neq x^{-1}$ and $y^2 = e$ and $o(x)=m$. Then $\mathrm{Cay}(D_{2n}, S) = \frac{n}{m}(K_2 \square C_n)$.

Theorem 1.6. [9] Let $S = \{x, y, z\}\} \subseteq D_{2n}$, with $x^2 = y^2 = z^2 = e$. Then $\mathrm{Cay}(D_{2n}, S) = K_2 \square C_n$.
To determine the graph structure of $\mathrm{Cay}(D_{2n}, S)$ for $|S| = 3$, it is enough to consider on of the above two cases. So, we deal with the first case and assume that $S = \{a, a^{-1}, b\} \subseteq D_{2n}$, where a and b are generatores of D_{2n} and $a^n = b^2 = e$. So, in this case by Theorem 1.5 we have $\mathrm{Cay}(D_{2n}, S) = K_2 \square C_n$. In the next section, we determine the structure of $\mathrm{Cay_m}(G,S)$ *when* $\mathbf{G} = \boldsymbol{D_6}$ and $|S|=3$ for all values $m \geq 2$
At the end of this section is necessary to remind definitions of Cartesian product and Corona product of two graphs G and H.

Definition 1.7. [7] The Cartesian product of G and H is a graph denoted by $G \square H$, whose vertex set is $V(G) \times V(H)$. Two vertices $(g,h),(g',h')$ are adjacent if ($g = g'$ and $hh' \in E(H)$) or ($gg' \in G$ and $h = h'$). Thus, $V(G \square H) = \{(g,h) \mid g \in V(G), h \in V(H)\}$
$E(G \square H) = \{(g,h)(g',h') | g = g', hh' \in E(H) \text{ or } gg' \in E(G), h = h')\}$.

Definition 1.8. [7] Suppose that G and H be graphs, then the Corona product of G and H denoted by $G \circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of $i-th$ copy of H to the $i-th$ vertex of G, where $1 \leq i \leq |V(G)|$.
Corona product is a non-commutative operation. *i.e.* $G \circ H \neq H \circ G$.
Throughout this paper, we always assume that group G is finite, S is a nonempty subset of G, $e \notin S$, $S^{-1} = S$ and S is a genetaing set for G. Moreover, all of the notations and terminologies about graphs are standard and can be found in [9].

2 The Structure of $\mathrm{Cay_m}(D_6, S)$ with |S|=3

In this section, we start with dihedral group of order 6. We investigate the graph structure of $\mathrm{Cay_m}(D_6,S)$, whenever $|S| = 3$. Let us start with the case m=2. Assume that $D_6 = \langle a,b \mid a^3 = b^2 = e, bab = a^{-1} \rangle = \{e,a,a^2,b,ab,a^2b\}$ is dihedral group of order 6 and $S = \{a,a^{-1},b\}$. Then, by Theorem 1.5 we have $\mathrm{Cay}(D_6, S) = K_2 \square C_3$ (see Figure 1). In the following, we give the structure for $\mathrm{Cay_2}(D_6, S)$.

Lemma 2.1. Let $D_6 = \langle a,b \mid a^n = b^2 = e, bab = a^{-1} \rangle$ be a dihedral group of order 6 and $S = \{a,a^{-1},b\}$. Then $\mathrm{Cay_2}(D_6, S) \cong [3K_{4,4} - 3(K_2 \square C_3)] \cup ((K_2 \square C_3) \circ \overline{K}_2) \cup \overline{K}_6$.

Proof : Suppose that $\Gamma_1 = K_2$ with vertex set $\{x_1, x_2\}$ and $\Gamma_2 = C_3$ with vertex set

$\{x_3, x_4, x_5\}$. Since Cay(D6,S) = K2 □ C3, so, its vertex set is $V(Cay(D_6,S)) = \{(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_2,x_3),(x_2,x_4),(x_2,x_5)\}$ such that $(x_1,x_3) = e, (x_1,x_4) = a, (x_1,x_5) = a^2, (x_2,x_3) = b, (x_2,x_4) = ab, (x_2,x_5) = a^2b$. Thus, $V(Cay_2(G,S)) = \left\{\begin{bmatrix}a\\b\end{bmatrix} \middle| \ a,b \in D_6\right\} = \left\{\begin{bmatrix}e\\e\end{bmatrix},\begin{bmatrix}e\\a\end{bmatrix},\begin{bmatrix}e\\a^2\end{bmatrix},\begin{bmatrix}e\\b\end{bmatrix},\begin{bmatrix}e\\ab\end{bmatrix},\begin{bmatrix}e\\a^2b\end{bmatrix},\begin{bmatrix}a\\e\end{bmatrix},\begin{bmatrix}a\\a\end{bmatrix},\begin{bmatrix}a\\a^2\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}a\\ab\end{bmatrix},\begin{bmatrix}a\\a^2b\end{bmatrix}\right.$
$,\begin{bmatrix}a^2\\e\end{bmatrix},\begin{bmatrix}a^2\\a\end{bmatrix},\begin{bmatrix}a^2\\a^2\end{bmatrix},\begin{bmatrix}a^2\\b\end{bmatrix},\begin{bmatrix}a^2\\ab\end{bmatrix},\begin{bmatrix}a^2\\a^2b\end{bmatrix},\begin{bmatrix}b\\e\end{bmatrix},\begin{bmatrix}b\\a\end{bmatrix},\begin{bmatrix}b\\a^2\end{bmatrix},\begin{bmatrix}b\\b\end{bmatrix},\begin{bmatrix}b\\ab\end{bmatrix},\begin{bmatrix}b\\a^2b\end{bmatrix}$
$\left.,\begin{bmatrix}ab\\e\end{bmatrix},\begin{bmatrix}ab\\a\end{bmatrix},\begin{bmatrix}ab\\a^2\end{bmatrix},\begin{bmatrix}ab\\b\end{bmatrix},\begin{bmatrix}ab\\ab\end{bmatrix},\begin{bmatrix}ab\\a^2b\end{bmatrix},\begin{bmatrix}a^2b\\e\end{bmatrix},\begin{bmatrix}a^2b\\a\end{bmatrix},\begin{bmatrix}a^2b\\a^2\end{bmatrix},\begin{bmatrix}a^2b\\b\end{bmatrix},\begin{bmatrix}a^2b\\ab\end{bmatrix},\begin{bmatrix}a^2b\\a^2b\end{bmatrix}\right\}$
and so $|V(Cay_2(G,S))| = 6^2 = 36$. Now, we have three independent sets

$$X = \left\{\left\{\begin{bmatrix}a^2\\b\end{bmatrix},\begin{bmatrix}b\\a^2\end{bmatrix},\begin{bmatrix}e\\a^2b\end{bmatrix},\begin{bmatrix}a^2b\\e\end{bmatrix}\right\}\right., \ Y = \left\{\left\{\begin{bmatrix}a\\a^2b\end{bmatrix},\begin{bmatrix}a^2b\\a\end{bmatrix},\begin{bmatrix}a^2\\ab\end{bmatrix},\begin{bmatrix}ab\\a^2\end{bmatrix}\right\}\right\}, \ Z = \left\{\left\{\begin{bmatrix}a\\b\end{bmatrix},\begin{bmatrix}b\\a\end{bmatrix},\begin{bmatrix}e\\ab\end{bmatrix},\begin{bmatrix}ab\\e\end{bmatrix}\right\}\right\}.$$

We have four types of vertices in terms of degrees as the following :

Type (I) of vertices: The degree of these vertices is 9. Define
$A_i = \{ [w_i \ \ w_i]^t \mid w_i \in D_6 \ and \ i = 1,2,\dots,6 \}$.

So, $|A_i|$ =1. So the number of these sets is 6. It is Clear that, the induced subgraph to the set $\bigcup_{i=1}^{6} A_i$ is the graph K2□C3. So, $A_1 = \left\{\begin{bmatrix}e\\e\end{bmatrix}\right\}$, $A_2 = \left\{\begin{bmatrix}a\\a\end{bmatrix}\right\}$, $A_3 = \left\{\begin{bmatrix}a^2\\a^2\end{bmatrix}\right\}$, $A_4 = \left\{\begin{bmatrix}b\\b\end{bmatrix}\right\}$, $A_5 = \left\{\begin{bmatrix}ab\\ab\end{bmatrix}\right\}$ & $A_6 = \left\{\begin{bmatrix}a^2b\\a^2b\end{bmatrix}\right\}$. We can see that $A_i = \begin{bmatrix}w_i\\w_i\end{bmatrix}$ is adjacent to $A_j = \begin{bmatrix}w_j\\w_j\end{bmatrix}$ such that $w_i \sim w_j$ in K2□C3 where i,j= 1,2, ... ,6.

Type (II) of vertices: The degree of these vertices is 4. Now, put

$$A_{ij} = \left\{\begin{matrix}[a_1 \quad a_2\,]^t \mid a_1, a_2 \in D_6 \ \ where\ in\ the\ \ matrix\ components \\ w_{i_1}, w_{i_2}\ is\ not\ adjacent\ together\end{matrix}\right\} - (A_i \cup A_j)$$

and $|A_{ij}| = 2 \ \ where \ \ 1 \le i < j \ \le 6$.

Assume $A_i = \begin{bmatrix}w_i\\w_i\end{bmatrix}$, $A_{jk} = \begin{bmatrix}w_j\\w_k\end{bmatrix}$ or $\begin{bmatrix}w_k\\w_j\end{bmatrix}$, thus we have A_i is adjacent to A_{jk} when w_i is adjacent to w_j and w_k such that $j < k$ and $i \ne j,k$. The number of these sets is 6 as follows, $A_{16} = \left\{\begin{bmatrix}w_1\\w_6\end{bmatrix},\begin{bmatrix}w_6\\w_1\end{bmatrix}\right\}$, Likewise, A_{34}, A_{26}, A_{35}, A_{24} and A_{15}.

It is clear that, the set A_1 is adjacent to $A_2, A_3, A_4, A_{23}, A_{24}, A_{34}$,
the set A_2 is adjacent to $A_1, A_3, A_5, A_{13}, A_{15}, A_{35}$,
the set A_3 is adjacent to $A_2, A_4, A_7, A_{27}, A_{47}, A_{47}$,
the set A_4 is adjacent to $A_1, A_3, A_8, A_{13}, A_{18}, A_{38}$,
the set A_5 is adjacent to $A_1, A_6, A_8, A_{18}, A_{16}, A_{68}$,
the set A_6 is adjacent to $A_2, A_5, A_7, A_{25}, A_{27}, A_{57}$. *So*, we define
$X_1 = \{A_{15}, A_1, A_5\}$ and $Y_1 = \{A_{24}, A_2, A_4\}$
$X_2 = \{A_{26}, A_2, A_6\}$ and $Y_2 = \{A_{35}, A_3, A_5\}$
$X_3 = \{A_{16}, A_1, A_6\}$ and $Y_3 = \{A_{34}, A_3, A_4\}$
Moreover, X_i and Y_i are disjoint and each one has four vertices where i=1,2,3. Hence, the subgraph induced by $X_i \dot\cup Y_i$ is a complete 2-bipartite graph $K_{4,4}$. So, we have $\bigcup_{i=1}^{3}(X_i \dot\cup Y_i)$ in the structure of $Cay_2(D_6,S)$. We will obtain three of the complete 2-bipartite graph $K_{4,4}$. The sets of $\{A_{15}, A_{24}\}$,$\{A_{26}, A_{35}\}$,$\{A_{34}, A_{16}\}$ are independent sets. On the other hand, it can be said that each element of these sets is adjacent to their other elements and are independent of each other.So, they can be shown as $3K_{4,4}$.

Type (III) of vertices: The degree of these vertices is one. We code it by ${A_{ij}}^1$ and $|{A_{ij}}^1| = 2 \ \ where \ \ 1 \le i < j \ \le 6$. We have A_i is adjacent to ${A_{ij}}^1$ where w_i is adjacent to w_j & w_k

such that $j < k$ *and* $i \neq j, k$. We can see that A_i is adjacent to ${A_{ij}}^1$ and the number of these sets is 6. The elements of these sets are ${A_{23}}^1, {A_{13}}^1, {A_{12}}^1, {A_{45}}^1, {A_{46}}^1$ and ${A_{65}}^1$ .Moreover, we can reach this conclusion that each element of this set is adjacent to each of vertices of the graph $K_2\square C_3$. It is obtained by corona product ($K_2\square C_3$) to $\bar{K}_2$.
Type (III) of vertices: The degree of these vertices is zero. Now, put
${A_{ij}}^* = \{[a_1 \ \ a_2 \ ]^t \mid a_1, a_2 \in \{w_i, w_j\} \subseteq D_6\} - \{A_{ij} \cup {A_{ij}}^1\}$ & $|{A_{ij}}^*| = 2$
where $1 \leq i < j \leq 6$.

We observe that the rest of the other vertices are all isolated vertices.The elements of this set are ${A_{14}}^*, {A_{25}}^*, {A_{36}}^*$ and the number of this type is 6. Therefore,
$Cay_2(D_6,S) \cong [3K_{4,4} - 3(K_2\square C_3)] \cup ((K_2\square C_3) \circ \bar{K}_2) \cup \bar{K}_6$.

By using the same method as above for $Cay_2(D_6,S)$, we can state the general structure of $Cay_m(D_6,S)$ for all $m \geq 2$.

Theorem 2.2. Let $D_6 = \langle a,b \mid a^n = b^2 = e, bab = a^{-1} \rangle$ be a dihedral group of order 3 and $S = \{a,a^{-1},b\}$. Then for all $m \geq 2$
$Cay_m(D_6,S) \cong [3K_{2^m,2^m} - 3(K_2\square C_3)] \cup \left((K_2\square C_3) \circ \bar{K}_{3^m - 2^{m+1} + 1}\right) \cup \bar{K}_{6(6^{m-1} - 3^m + 2^m - 1)}$

Proof : Suppose that $\Gamma_1 = K_2$ with vertex set $\{x_1, x_2\}$ and $\Gamma_2 = C_3$ with vertex set $\{x_3, x_4, x_5\}$. As we mentioned in Lemma 2.1, we have $Cay(D_6,S) = K_2\square C_3$ and
$V(Cay(D_6,S)) = \{(x_1,x_3),(x_1,x_4),(x_1,x_5),(x_2,x_3),(x_2,x_4),(x_2,x_5)\}$ such that $(x_1,x_3) = e, (x_1,x_4) = a, (x_1,x_5) = a^2, (x_2,x_3) = b, (x_2,x_4) = ab, (x_2,x_5) = a^2b$.
Thus, $V(Cay_m(D_6,S)) = \{ [a_1 \ \ a_2 \ldots \ a_m]^t \mid a_1, a_2, \ldots, a_m \in D_6\}$ and so $|V(Cay_m(D_6,S))| = 6^m$.
We have four types of vertices in terms of degrees here.
Type (I) of vertices: The degree of these vertices is
$(3|A_i| + 3|A_{ij}| + |A_{ijk}|) = 3 + 3(2^m - 2) + 3(3^{m-1} - 2^m + 1) = 3(3^{m-1}) = 3^m$. Define
$A_i = \{ [w_i \ \ w_i \ \ldots \ w_i]^t \mid w_i \in D_6$ *and* $i = 1,2,\ldots,6 \}$. So, $|A_i| = 1$ and the number of these sets is 6. It is Clear that, the induced subgraph to the set $\bigcup_{i=1}^{6} A_i$ is the graph $K_2 \square C_3$. We can see that $A_i = [w_i \ \ w_i \ \ldots \ w_i]^t$ is adjacent to $A_j = [w_j \ \ w_j \ \ldots \ w_j]^t$ such that w_i is adjacent to w_j in $K_2 \square C_3$ where i,j= 1,2, ... ,6.
Type (II) of vertices: The degree of these vertices is $(|A_{ij}| + 2) = 2^m$. Now, put
$$A_{ij} = \left\{ \begin{matrix} [a_1 \ \ a_2 \ \ldots \ a_m]^t \mid a_1, a_2, \ldots, a_m \in \{w_i, w_j\} \subseteq D_6 \ \textit{where in the matrix} \\ \textit{components} \ w_i \ \textit{is not adjacent to} \ w_j \end{matrix} \right\} - (A_i \cup A_j)$$
and $|A_{ij}| = 2^m - 2$ *where* $1 \leq i < j \leq 6$. They are $A_{16}, A_{34}, A_{26}, A_{35}, A_{24}$ and A_{15}.
In $Cay_m(D_6,S)$, A_i is adjacent to A_{jk} where w_i is adjacent to w_j *and* w_k in $Cay(D_6,S)$ such that $j < k$ *and* $i \neq j, k$. The number of these sets is 6 .
It is clear that, the set A_1 is adjacent to $A_2, A_3, A_4, A_{23}, A_{24}, A_{34}$, A_{234}
the set A_2 is adjacent to $A_1, A_3, A_5, A_{13}, A_{15}, A_{35}, A_{135}$
the set A_3 *is adjacent to* $A_2, A_4, A_7, A_{27}, A_{47}, A_{47}$, A_{247}
the set A_4 *is adjacent to* $A_1, A_3, A_8, A_{13}, A_{18}, A_{38}, A_{138}$
the set A_5 *is adjacent to* $A_1, A_6, A_8, A_{18}, A_{16}, A_{68}, A_{168}$,
the set A_6 *is adjacent to* $A_2, A_5, A_7, A_{25}, A_{27}, A_{57}, A_{257}$. So, we define
$X_1 = \{A_{15}, A_1, A_5\}$ and $Y_1 = \{A_{24}, A_2, A_4\}$
$X_2 = \{A_{26}, A_2, A_6\}$ and $Y_2 = \{A_{35}, A_3, A_5\}$

$X_3 = \{A_{16}, A_1, A_6\}$ and $Y_3 = \{A_{34}, A_3, A_4\}$
Moreover, X_i and Y_i are disjoint and each one has 2^m vertices where i=1,2,3. Hence, the subgraph induced by $X_i \dot{\cup} Y_i$ is a complete 2-bipartite graph $K_{2^m,2^m}$. So, we have $\bigcup_{i=1}^{3}(X_i \dot{\cup} Y_i)$ in the structure of $Cay_m(G,S)$. We will obtain 3 of the complete 2-bipartite graph $K_{2^m,2^m}$. The sets of $\{A_{15}, A_{24}\}$,$\{A_{26}, A_{35}\}$,$\{A_{34}, A_{16}\}$ are independent sets. On the other hand, it can be said that each element of these sets is adjacent to their other elements and are independent of each other. So, they can be shown as follows $3K_{2^m,2^m}$.
Type (III) of vertices: The degree of these vertices is one. We code it by ${A_{ij}}^1$and $|{A_{ij}}^1| = 2^m - 2$ *and* $|A_{ijk}| = 3(3^{m-1} - 2^m + 1)$ *where* $1 \le i < j < k \le 6$. We have A_i is adjacent to ${A_{ij}}^1$ where w_i is adjacent to w_j & w_k such that $j < k$ *and* $i \neq j,k$. We can see that A_i is adjacent to ${A_{ij}}^1$and the number of these sets is 6. They are ${A_{23}}^1$, ${A_{13}}^1$, ${A_{12}}^1$, ${A_{45}}^1$, ${A_{46}}^1$ and ${A_{65}}^1$. Also, We have A_i is adjacent to A_{jkl} where w_i is adjacent to w_j, w_k *and* w_k where $i \neq j,k,l$ & $j < k < l$. They are A_{234},A_{135},A_{247}, A_{138},A_{168}, A_{257}.

Moreover, we can reach this conclusion that each element of this set is adjacent to each of vertices of the graph $K_2 \Box C_3$.

It is obtained by corona product ($K_2 \Box C_3$) to $\overline{K}_{|{A_{ij}}^1| + |A_{ijk}|} = \overline{K}_{3^m - 2^{m+1} + 1}$.
Type (IV) of vertices: The degree of these vertices is zero. Put

$${A_{i_1 i_1}}^* = \left\{[a_1\ a_2\ \dots\ a_m]^t \middle| a_1, a_2, \dots, a_m \in \{w_{i_1}, w_{i_1}\}\right\} - \{A_{i_1 i_2} \cup {A_{i_1 i_2}}^1\}$$

$${A_{i_1 i_2 i_3}}^* = \left\{[a_1\ a_2\ \dots\ a_m]^t \middle| a_1, a_2, \dots, a_m \in \{w_{i_1}, w_{i_2}, w_{i_3}\}\right\} - \{A_{i_1 i_2 i_3}\}, \dots,$$

$${A_{i_1 i_2 i_3 \dots i_m}}^* = \left\{[a_1\ a_2\ \dots\ a_m]^t \middle| a_1, a_2, \dots, a_m \in \{w_{i_1}, w_{i_2}, \dots, w_{i_3}\}\right\} \text{ such that}$$

$$1 \le i_1 < i_2 < \dots < i_6 \le 6.$$

We observe that the rest of the other vertices are all isolated vertices. The elements of this set are ${A_{14}}^*$, ${A_{25}}^*$, ${A_{36}}^*$ and all triple sets (${A_{i_1 i_2 i_3}}^*$) except $A_{i_1 i_2 i_3}$.
The number of this type is

$$\left[\binom{6}{2} - 12\right] |{A_{i_1 i_2}}^*| + \left[\binom{6}{3} - 6\right] |{A_{i_1 i_2 i_3}}^*| + |{A_{i_1 i_2 \dots i_4}}^*| + |{A_{i_1 i_2 \dots i_5}}^*| + |{A_{i_1 i_2 \dots i_6}}^*|$$

$$= 3|{A_{i_1 i_2}}^*| + 14|{A_{i_1 i_2 i_3}}^*| + \sum_{q=4}^{6} \left|{A_{i_1 i_2 \dots i_q}}^*\right|.$$

On the other hand, the number of the isolated vertices is

$$6^m - 6|A_i| - 6|A_{ij}| - 6|{A_{ij}}^1| - 6|A_{ijk}| = 6(6^{m-1} - |A_i| - |A_{ij}| - |{A_{ij}}^1| - |A_{ijk}|)$$

$$= 6(6^{m-1} - 3^m + 2^m - 1)$$

Therefore, $Cay_m(D_6,S) = [\bigcup_{i=1}^{3} K_{2^m,2^m} - 3(K_2 \Box C_3)] \cup \left((K_2 \Box C_3) \circ \overline{K}_{3^m - 2^{m+1} + 1}\right) \cup \overline{K}_{6(6^{m-1} - 3^m + 2^m - 1)} = [3K_{2^m,2^m} - 3(K_2 \Box C_3)] \cup \left((K_2 \Box C_3) \circ \overline{K}_{3^m - 2^{m+1} + 1}\right) \cup \overline{K}_{6(6^{m-1} - 3^m + 2^m - 1)}$
As required.

Remark 2.3 One can easily see that we may state the above formula in terms of size of sets as the following:

$$Cay_m(D_6,S) \cong [3K_{|A_{ij}|+2,|A_{ij}|+2} - 3(K_2 \Box C_3)] \cup \left((K_2 \Box C_3) \circ \overline{K}_{|{A_{ij}}^1| + |A_{ijk}|}\right) \cup \overline{K}_{6(6^{m-1} - |A_i| - |A_{ij}| - |{A_{ij}}^1| - |A_{ijk}|)}$$

The graphs $Cay(D_6,S), Cay_2(D_6,S), Cay_3(D_6,S)$ and $Cay_m(D_6,S)$ are shown in following figures.

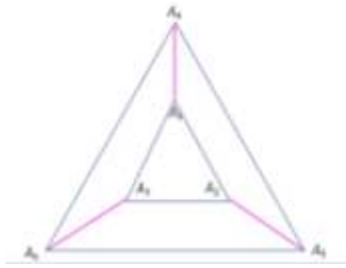

Figure 1: Cay(D_6,S)

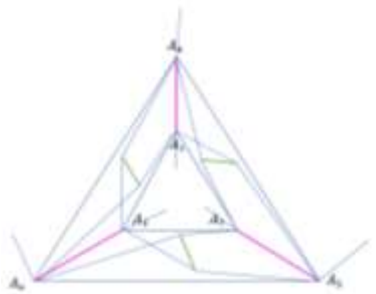

Figure 2: The component of $Cay_2(D_6,S)$

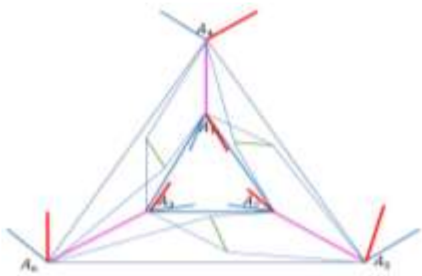

Figure 3: The component of $Cay_3(D_6,S)$

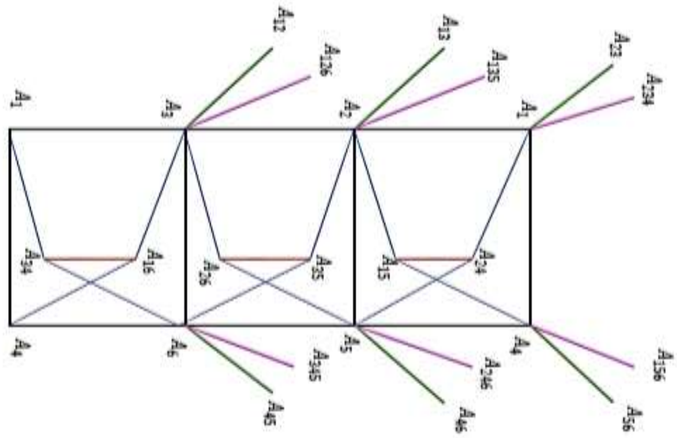

Figure 4: The generalized Cayley graph when m$\geq$ **3** and |S|=3 $(\boldsymbol{Cay_m(D_6,S)})$ and has $\boldsymbol{6(6^{m-1}-3^m+2^m-1)}$ isolated vertices.

3 The Structure of $\mathbf{Cay_m}(\boldsymbol{D_{2n}},\mathbf{S})$ with $|\mathbf{S}|=3$

In this section, we investigate the graph structure of $Cay_m(D_{2n},S)$, whenever $|S|=3$, $m\geq 2$ and $n\geq 3$. Let us remind that $Cay(D_{2n},S)=K_2\square C_n$ by Theorem 1.5. So, suppose that $\Gamma_1=K_2$ with vertex set $\{x_1,x_2\}$ and $\Gamma_2=C_n$ with vertex set $\{x_3,x_4,\dots,x_{n+2}\}$. Then we have

$$V\big(Cay(D_{2n},S)\big)=\{(x_1,x_3),(x_1,x_4),\dots,(x_1,x_{n+2}),(x_2,x_3),(x_2,x_4),\dots,(x_2,x_{n+2})\}$$

We put $w_1=(x_1,x_3), w_2=(x_1,x_4),\dots, w_n=(x_1,x_{n+2})$ and $w_{n+1}=(x_2,x_3), w_7=(x_2,x_4),\dots,w_{2n}=(x_2,x_{2n+2})$. So, $\big|V\big(Cay(D_{2n},S)\big)\big|=2n$.

Consider the subsets X and Y of $V\big(Cay(D_{2n},S)\big)$ as follows:

$X=\{w_{1,},w_3,\dots,w_{2n-1}\}$, $Y=\{w_{2,},w_4,\dots,w_{2n}\}$.

It is clear that both sets X, Y are independent and that each vertex in X is adjacent to each vertex in Y and vice versa in Cay(D_{2n},S).

Now, we are going to state the main results. The case m=2 is stated as the follow lemma.

Lemma 3.1 Let $D_{2n}=\{a,b \mid a^n=b^2=e, bab=a^{-1}\}$ be a dihedral group of order 2n and $S=\{a,a^{-1},b\}$ where $n\geq 3$. Then

$$\boldsymbol{Cay_2(D_{2n},S)\cong[nK_{4,4}-3(K_2\square C_n)]\cup((K_2\square C_n)\circ\bar{K}_2)\cup\bar{K}_{2n(2n-5)}}$$

Proof. We define $V(Cay_2(D_{2n},S))=\{[a_1\quad a_2]^t \mid a_1,a_2\in D_{2n}\}$. So, $\big|V\big(Cay_2(G,S)\big)\big|=(2n)^2=4n^2$.

We have three types of vertices in terms of degrees. They are:
Type (I) of vertices: The degree of these vertices is $|A_i| + 3|A_{ij}| = 3(2^{\mathrm{m}} - 1)$ =9. We define $A_i = \{ [w_i \ \ w_i]^t \mid w_i \in V(Cay(D_{2\mathrm{n}}, S)) \ and \ i = 1,2, \dots, 2\mathrm{n} \}$. So, $|A_i|$ =1. So the number of these se ts is 2n. It is Clear that, the induced subgraph to the set $\bigcup_{i=1}^{2\mathrm{n}} A_i$ is the $K_2 \square C_{\mathrm{n}}$. We can see that $A_i = \begin{bmatrix} w_i \\ w_i \end{bmatrix}$ is adjacent to $A_j = \begin{bmatrix} w_j \\ w_j \end{bmatrix}$ such that $w_i \sim w_j \ in \ Cay(D_{2\mathrm{n}}, S)$ where i,j = $1,2, \dots, 2$n.
Type (II) of vertices: The degree of these vertices is $2|A_i| + |A_{ij}| = 2^{\mathrm{m}} = 4$. Now, put $A_{ij} = \left\{ \begin{matrix} [a_1 \quad a_2\ ]^t \mid a_1, a_2 \in \{w_i, w_j\} \ where \ in \ the \ \ matrix \\ components \ \ w_i, w_j \ is \ not \ adjacent \ together \end{matrix} \right\} - (A_i \cup A_j)$ and $|A_{ij}| = 2$ such that $1 \le i < j \le 2\mathrm{n}$. We have A_i is adjacent to A_{jk} where w_i is adjacent to $w_j \ and \ w_k$ in $Cay(D_{2\mathrm{n}}, S)$ such that $j < k \ and \ i \ne j, k$. They are $A_{1(2\mathrm{n})}$, $A_{\mathrm{n(n+1)}}$, $A_{1(2n-4)}, A_{2(n+1)}, A_{3(2n-2)}$, $A_{4(2\mathrm{n}-3)}$, $A_{2(2\mathrm{n}-3)}, A_{3(2\mathrm{n}-4)}$, $A_{(\mathrm{n}-1)(2\mathrm{n})}$, $A_{n(n-1)}, \dots and \ A_{1(2\mathrm{n})}$, $A_{\mathrm{n(n+1)}}$. Also, $X_1 = \{A_{1(\mathrm{n}+2)}, A_1, A_{\mathrm{n}+2}\}$ and $Y_1 = \{A_{2(\mathrm{n}+1)}, A_2, A_{\mathrm{n}+1}\}$
$X_2 = \{A_{2(n+3)}, A_2, A_{\mathrm{n}+3}\}$ and $Y_2 = \{A_{3(\mathrm{n}+2)}, A_3, A_{\mathrm{n}+2}\}, \dots,$
$X_{\mathrm{n}-2} = \{A_{(n-2)(2n-1)}, A_{n-1}, A_{2\mathrm{n}-1}\}$ and $Y_{\mathrm{n}-2} = \{A_{(\mathrm{n}-1)(2\mathrm{n}-2)}, A_{\mathrm{n}-1}, A_{2n-2}\}$
$X_{\mathrm{n}-1} = \{A_{(n-1)(2n)}, A_{\mathrm{n}-1}, A_{2n}\}$ and $Y_{\mathrm{n}-1} = \{A_{\mathrm{n}(2\mathrm{n}-1)}, A_{\mathrm{n}}, A_{2\mathrm{n}-1}\}$
$X_{\mathrm{n}} = \{A_{1(2n)}, A_1, A_{2\mathrm{n}}\}$ and $Y_{\mathrm{n}} = \{A_{\mathrm{n(n+1)}}, A_{\mathrm{n}}, A_{\mathrm{n}+1}\}$.

Moreover, X_i and Y_i are disjoint and each one has 4 vertices where i=1,2,..,n. Hence, the subgraph induced by $X_i \dot\cup Y_i$ is a complete 2-bipartite graph $K_{4,4}$. So, we have $\bigcup_{i=1}^{n} (X_i \dot\cup Y_i)$ in the structure of $Cay_2(D_{2n}, S)$. We will obtain (n) of the complete 2-bipartite graph $K_{4,4}$. The sets of $\{A_{1(\mathrm{n}+2)}, A_{2(\mathrm{n}+1)}\}$,{ $A_{2(n+3)}, A_{3(\mathrm{n}+2)}$ },{ $A_{(n-2)(2n-1)}, A_{(\mathrm{n}-1)(2\mathrm{n}-2)}$ },…, $\{A_{(n-1)(2n)}, A_{(\mathrm{n}-1)(2\mathrm{n}-2)}\}$, $\{A_{(n-1)(2n)}, A_{\mathrm{n}(2\mathrm{n}-1)}\}$ and $\{A_{1(2n)}, A_{\mathrm{n(n+1)}}\}$ are independent sets.
Type (III) of vertices: The degree of these vertices is one. We code it by ${A_{ij}}^1$ and $|{A_{ij}}^1| = 2 \ \ where \ \ 1 \le i < j \le 2\mathrm{n}$. We have A_i is adjacent to ${A_{ij}}^1$ where w_i is adjacent to w_j & w_k such that $j < k \ and \ i \ne j, k$. We can see that A_i is adjacent to ${A_{ij}}^1$and the number of the elements of this sets is 2n. Moreover, we can reach this conclusion that each element of this set is adjacent to each of vertices of the graph $K_2 \square C_n$. It is obtained by corona product ($K_2 \square C_n$) to $\overline{K}_2$.
Type (IV) of vertices: The degree of these vertices is zero. Now, put ${A_{ij}}^* = \{[a_1 \ \ a_2\ ]^t \mid a_1, a_2 \in \{w_i, w_j\} \subseteq D_{2n}\} - \{A_{ij}\} \ \ and \ \ |{A_{ij}}^*| = 2$ such that $1 \le i < j \le 2\mathrm{n}$. So, the number of these isolated vertices is $2\left(\binom{2\mathrm{n}}{2} - 24\right)$.

Assume $A_{ij} = \begin{bmatrix} w_i \\ w_j \end{bmatrix} \ or \begin{bmatrix} w_j \\ w_i \end{bmatrix}, A_k = \begin{bmatrix} w_k \\ w_k \end{bmatrix}$, thus we have A_{ij} is not adjacent to A_k where w_i and w_k is not adjacent to w_k such that $i < j \ and \ k \ne i, j$. We observe that this type of vertices are isolated vertices. So, the rest of the vertices outside of $\bigcup_{i=1}^{\mathrm{n}} (X_i \dot\cup Y_i)$ and $\bigcup_{i=1}^{2\mathrm{n}} {A_{ij}}^1$ are all isolated vertices. The number of isolated vertices is $(2\mathrm{n})^2 - 2\mathrm{n}|A_i| - 2\mathrm{n}|A_{ij}| - 2\mathrm{n}|{A_{ij}}^1| = 2\mathrm{n}(2\mathrm{n} - 1 - 2|A_{ij}|) = 2n(2n-5)$.It is clear that,
the set A_1 is adjacent to $A_2, \ A_{\mathrm{n}+1}, \ A_{\mathrm{n}}, A_{2(n+1)}, A_{2(\mathrm{n})}, A_{\mathrm{n(n+1)}}$,
the set A_2 is adjacent to $A_1, \ A_3, \ A_{\mathrm{n}+2}, A_{13}, A_{1(n+2)}, A_{3(n+2)}$,
the set A_3 $is \ adjacent \ to$ $A_2, \ A_4, \ A_{\mathrm{n}+3}, A_{24}, A_{2(n+3)}, A_{4(n+3)}, \dots,$

the set A_i *is adjacent to* $A_{i-1}, A_{i+1}, A_{n+i}, A_{(i-1)(i+1)}, A_{(i-1)(n+i)}, A_{(i+1)(n+i)}, \ldots,$
the set A_n *is adjacent to* $A_{n-1}, A_{n+1}, A_{2n}, A_{(n-1)(n+1)}, A_{(n-1)(2n)}, A_{(n+1)(2n)}, \ldots,$
and the set A_{2n} *is adjacent to* $A_{2n-1}, A_{n+1}, A_n, A_{(2n-1)(n+1)}, A_{(2n-1)(n)}, A_{(n)(n+1)}.$
Thus ,the structure of generalized Cayley graph of $K_2\square C_5$ when m=2 has five faces and each face is the complete 2-bipartite graph $K_{4,4}$ such that these faces have some common vertices. Therefore, $Cay_2(D_{2n},S) \cong [nK_{4,4} - 3(K_2\square C_n)] \cup ((K_2\square C_n) \circ \overline{K}_2) \cup \overline{K}_{2n(2n-5)}.$

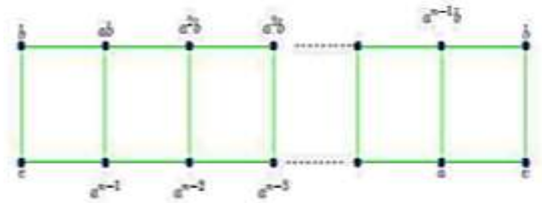

Figure 9: The graph of Cay(D_{2n},S)

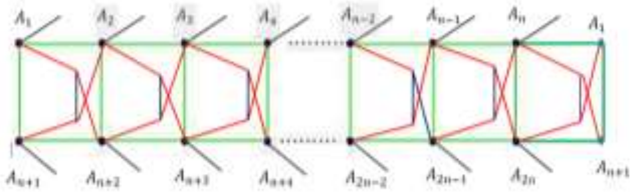

Figure10: The component of the graph of $Cay_2(D_{2n}$,S)

Lemma 3.2 Let $D_{2n} = \{ a,b \mid a^n = b^2 = e, bab = a^{-1} \}$ be a dihedral group of order 2n and $S = \{a, a^{-1}, b\}$ where $n \geq 3$. Then

$$Cay_3(D_{2n},S) \cong [n\boldsymbol{K_{8,8}} - 3(K_2\square C_n)] \cup [(K_2\square C_n) \circ \overline{K}_{12}] \cup \overline{K}_{2n[(2n)^2-25]}$$

Proof : We define $V(Cay_3(D_{2n},S)) = \{[a_1 \ \ a_2 \ \ a_3]^t \mid a_1, a_2, a_3 \in D_{2n}\}$ So,$|V(Cay_3(D_{2n},S))| = (2n)^3 = 8n^3$.We have four types of vertices in terms of degrees.They are: Type (I) of vertices: The degree of these vertices is
$|A_i| + 3|A_{ij}| + |A_{ijk}| = 3 + 3(2^m - 2) + 3(3^{m-1} - 2^m + 1) = 3^m = 3^3 = 27.$
We Define $A_i = \{[w_i \ \ w_i \ \ w_i]^t \mid w_i \in D_{2n}\}$. $|A_i|$ =1 where $i = 1,2,\ldots,2n$. So the number of these sets is 2n. It is Clear that, the induced subgraph to the set $\bigcup_{i=1}^{2n} A_i$ is the graph $K_2\square C_n$. We can see that $A_i = [w_i \ \ w_i \ \ w_i]^t$is adjacent to $A_j = [w_j \ \ w_j \ \ w_j]^t$ such that w_i is adjacent to w_j in Cay(D_{2n},S) where i, j= $1,2,\ldots,2n$.
Type (II) of vertices:The degree of these vertices is $2|A_i| + |A_{ij}| = 2^m$=8. Now, put

$$A_{ij} = \left\{ \begin{matrix} [a_1 \ \ a_2 \ \ a_3]^t \mid a_1, a_2, a_3 \in \{w_i, w_j\} \subseteq D_{2n} \ \ where \ \ in \\ the\ matrix\ components\ w_i, w_j\ isnot\ adjacent\ together \end{matrix} \right\} - (A_i \cup A_j)$$

and $|A_{ij}| = 6$ *such that* $1 \leq i < j \leq 2n$. In $Cay_3(D_{2n},S)$, we have A_i is adjacent to A_{jk} where $w_i \sim w_j$ *and* $w_i \sim w_k$such that $j < k$ and $i \neq j, k$.
The induced subgraph to the sets A_{ij} is complete 2-bipartite graph $K_{8,8}$. They are
Theyare
$A_{1(2n)}$, $A_{n(n+1)}$, $A_{1(2n-4)}, A_{2(n+1)}, A_{3(2n-2)}$, $A_{4(2n-3)}$, $A_{2(2n-3)}, A_{3(2n-4)}$, $A_{(n-1)(2n)}$, $A_{n(n-1)}, \ldots$ *and* $A_{1(2n)}$, $A_{n(n+1)}$ .We put $X_1 = \{A_{1(n+2)}, A_1, A_{n+2}\}$ and $Y_1 = \{A_{2(n+1)}, A_2, A_{n+1}\}$
$X_2 = \{A_{2(n+3)}, A_2, A_{n+3}\}$ and $Y_2 = \{A_{3(n+2)}, A_3, A_{n+2}\}$,…,
$X_{n-2} = \{A_{(n-2)(2n-1)}, A_{n-1}, A_{2n-1}\}$ and $Y_{n-2} = \{A_{(n-1)(2n-2)}, A_{n-1}, A_{2n-2}\}$
$X_{n-1} = \{A_{(n-1)(2n)}, A_{n-1}, A_{2n}\}$ and $Y_{n-1} = \{A_{n(2n-1)}, A_n, A_{2n-1}\}$
$X_n = \{A_{1(2n)}, A_1, A_{2n}\}$ and $Y_n = \{A_{n(n+1)}, A_n, A_{n+1}\}$.

Moreover, X_i and Y_i are disjoint and each one has 8 vertices where i=1,2,..,n. Hence, the subgraph induced by $X_i \dot{\cup} Y_i$ is a complete 2-bipartite graph $K_{8,8}$. So, we have $\bigcup_{i=1}^{n}(X_i \dot{\cup} Y_i)$ in the structure of $Cay_2(D_{2n},S)$.

We will obtain (n) of the complete 2-bipartite graph $K_{8,8}$. The sets of $A_{1(n+2)}, A_{2(n+1)}\}$,{ $A_{2(n+3)}, A_{3(n+2)}$ },{ $A_{(n-2)(2n-1)}, A_{(n-1)(2n-2)}$ },…, $\{A_{(n-1)(2n)}, A_{(n-1)(2n-2)}\}$, $\{A_{(n-1)(2n)}, A_{n(2n-1)}\}$ and $\{A_{1(2n)}, A_{n(n+1)}\}$ are independent sets.

Type (III) of vertices: The degree of these vertices is one. We code it by ${A_{ij}}^1$ and $|{A_{ij}}^1| = 2$ $where$ $1 \le i < j \le 2n$. We have A_i is adjacent to ${A_{ij}}^1$ where w_i is adjacent to w_j & w_k such that $j < k$ and $i \ne j, k$. We can see that A_i is adjacent to ${A_{ij}}^1$and the number of the elements of this sets is 2n. Moreover, we can reach this conclusion that each element of this set is adjacent to each of vertices of the graph $K_2\square C_n$. It is obtained by corona product ($K_2\square C_n$) to $\overline{K}_2$.

Type (III) of vertices: The degree of these vertices is one. We code it by ${A_{ij}}^1$ and $|{A_{ij}}^1| = 2$ $where$ $1 \le i < j \le 2n$. We have A_i is adjacent to ${A_{ij}}^1$ where w_i is adjacent to w_j & w_k such that $j < k$ and $i \ne j, k$. We can see that A_i is adjacent to ${A_{ij}}^1$and the number of the elements of this sets is 2n. We define

$${A_{ij}}^1 = \left\{ \begin{array}{l} [a_1 \quad a_2 \quad a_3]^t \mid a_1, a_2, a_3 \in \{w_i, w_j\} \; where \;\; in \; the \; matrix \\ components \; w_i, w_j \; is \; not \; adjacent \; together \; in \; Cay(D_{2n}, S) \end{array} \right\} - (A_i \cup A_j)$$

& $|{A_{ij}}^1| = 6$. They are ${A_{25}}^1, {A_{13}}^1, {A_{24}}^1, {A_{35}}^1, {A_{14}}^1, {A_{7(10)}}^1, {A_{68}}^1, {A_{79}}^1, {A_{8(10)}}^1, {A_{69}}^1$.

$$A_{ijk} = \left\{ \begin{array}{l} [a_1 \quad a_2 \quad a_3]^t \mid a_1, a_2, a_3 \in \{w_i, w_j, w_k\} \; where \;\; in \; the \; matrix \\ components \; w_i, w_j, w_k \; isnot \; adjacent \; together \; in \; Cay(D_{2n}, S) \end{array} \right\} - (A_i \cup A_j \cup A_k \cup A_{ij} \cup A_{jk} \cup A_{ik}).$$

So, $|A_{ijk}| = 6$ $such \; that$ $1 \le i < j < k \le 2n$.

It is easy to see that A_i is adjacent to A_{jkl} where w_i is adjacent to w_j , w_k and w_l in $Cay(D_{2n}, S)$ such that $i \ne j, k, l$, and $1 \le j < k < l \le 2n$.

The number of these vertices is $10|{A_{ij}}^1| + 10|A_{ijk}| = 10(|{A_{ij}}^1| + |A_{ijk}|) = 120$

Moreover, we can reach this conclusion that each element of this set is adjacent to each of vertices of the graph $K_2\square C_n$. It is obtained by corona product ($K_2\square C_n$) to $\overline{K}_{|{A_{ij}}^1|+|A_{ijk}|} = \overline{K}_{12}$.

Type (IV) of vertices: The degree of these vertices is zero. Now, put

${A_{ij}}^* = \{[a_1 \quad a_2 \quad a_3]^t \mid a_1, a_2, a_3 \in \{w_i, w_j\}\} - \{A_{ij} \cup {A_{ij}}^1\}$

${A_{ijk}}^* = \{[a_1 \quad a_2 \quad a_3]^t \mid a_1, a_2, a_3 \in \{w_i, w_j, w_k\}\} - A_{ijk}$.

So, $|{A_{ij}}^*| = 6$ and $|{A_{ijk}}^*| = 6$ where $1 \le i < j < k \le 2n$.

We observe that the rest of the other vertices are all isolated vertices.

The number of these isolated vertices is $(\binom{2n}{2} - 4n)|{A_{ij}}^*|) + (\binom{2n}{3} - 2n)|{A_{ijk}}^*|$

In $Cay_3(D_{2n}, S)$, we have ${A_{ij}}^*$ is not adjacent to A_k where w_i and w_k is not adjacent to w_k in $Cay(D_{2n}, S)$ such that $i < j$ and $k \ne i, j$.

We can express that the rest of the vertices outside of $\bigcup_{i=1}^{n}(X_i \dot\cup Y_i)$, $\bigcup_{i=1}^{2n} A_{ijk}$ and $\bigcup_{i=1}^{2n} {A_{ij}}^1$ are all isolated vertices. The number of isolated vertices is

$$(2n)^3 - 2n|A_i| - 2(2n)|A_{ij}| - 2n|A_{ij}| - 2n|A_{ijk}| =$$
$$2n[(2n)^2 - |A_i| - 3|A_{ij}| - |A_{ijk}|] = 2n[(2n)^2 - 25].$$

It is clear that, the set A_1 is adjacent to $A_2,\ A_{n+1},\ A_n, A_{2(n+1)}, A_{2(n)}, A_{n(n+1)}, A_{2(n)(n+1)}$

the set A_2 is adjacent to $A_1,\ A_3,\ A_{n+2}, A_{13}, A_{1(n+2)}, A_{3(n+2)}, A_{1(3)(n+2)}$

the set A_3 *is adjacent to* $A_2,\ A_4,\ A_{n+3}, A_{24}, A_{2(n+3)}, A_{4(n+3)}, A_{2(4)(n+3)}$,…,

the set

A_i *is adjacent to* $A_{i-1},\ A_{i+1},\ A_{n+i}, A_{(i-1)(i+1)}, A_{(i-1)(n+i)}, A_{(i+1)(n+i)}, A_{(i-1)(i+1)(n+i)}$, …,

the set

A_n *is adjacent to* $A_{n-1},\ A_{n+1},\ A_{2n}, A_{(n-1)(n+1)}, A_{(n-1)(2n)}, A_{(n+1)(2n)}, A_{(n-1)(n+1)(2n)}$,…

and the set
$A_{2\mathrm{n}}$ *is adjacent to* $A_{2\mathrm{n}-1}, A_{\mathrm{n}+1}, A_{\mathrm{n}}, A_{(n+1)(2n-1)}, A_{(n)(2n-1)}, A_{(n)(n+1)}, A_{(n)(n+1)(2n-1)}$
Thus , the structure of generalized Cayley graph of $K_2\square C_n$ when m =3 has (n) faces and each face is Complete 2-bipartite graph $K_{8,8}$ such that these faces have some common vertices. Therefore, $Cay_3(\mathrm{D}_{2n}, S) \cong [n\boldsymbol{K}_{\mathbf{8,8}} - 3(K_2\square C_n)] \cup [(K_2\square C_n) \circ \overline{K}_{12}] \cup \overline{K}_{2\mathrm{n}[(2\mathrm{n})^2-25]}$.
By using the above lemmas, we may state the main result here which cover all cases.

Theorem 3.3 Let $\mathrm{D}_{2n} = \{a,b \mid a^n = b^2 = e, bab = a^{-1}\}$ be a dihedral group of order 2n and $S = \{a, a^{-1}, b\}$ where $n \geq 3$. Then for all $m \geq 2$

$$Cay_{\mathrm{m}}(\mathrm{D}_{2n}, S) \cong \left[n\boldsymbol{K}_{2^{\mathrm{m}},2^{\mathrm{m}}} - 3(K_2\square C_n)\right] \cup \left[(K_2\square C_n) \circ \overline{K}_{2\mathrm{n}(3^{\mathrm{m}}-2^{\mathrm{m}+1}+1)}\right] \cup \overline{K}_{2\mathrm{n}[(2\mathrm{n})^{\mathrm{m}-1}-3^{\mathrm{m}}+2]}.$$

Proof. We define $V(Cay_{\mathrm{m}}(D_{2n}, S)) = \{[a_1 \quad a_2 \ldots \quad a_{\mathrm{m}}]^t \mid a_1, a_2, \ldots, a_{\mathrm{m}} \in D_{2\mathrm{n}}\}$
So, $\left|V\left(Cay_{\mathrm{m}}(D_{2\mathrm{n}}, S)\right)\right| = (2\mathrm{n})^{\mathrm{m}}$.We have four types of vertices in terms of degrees. They are :
Type (I) of vertices: The degree of these vertices is $|A_i| + 3|A_{ij}| + |A_{ijk}| = 3^m$.
Define $A_i = \{[w_i \quad w_i \quad \ldots \quad w_i]^t \mid w_i \in D_{2n}\}$. So, $|A_i| = 1$ where $i = 1,2, \ldots, 2\mathrm{n}$. So the number of these sets is 2n. It is Clear that, the induced subgraph to the set $\bigcup_{i=1}^{2n} A_i$ is the graph $K_2\square C_n$. We can see that $A_i = [w_i \quad w_i \quad \ldots \quad w_i]^t$ is adjacent to $A_j = \left[w_j \quad w_j \quad \ldots \quad w_j\right]^t$ such that $w_i \sim w_j$ in Cay(D_{2n}, S)where i, j= 1,2, … ,2n.
Type (II) of vertices:The degree of these vertices is $2|A_i| + |A_{ij}| = 2^{\mathrm{m}}$. Now, put

$$A_{ij} = \left\{\begin{matrix}[a_1 \quad a_2 \quad \ldots \quad a_{\mathrm{m}}]^t \mid a_1, a_2, \ldots, a_{\mathrm{m}} \in \{w_i, w_j\} \subseteq D_{2n} \ \ where \ \ in \\ the \ matrix \ components \ w_i, w_j \ isnot \ adjacent \ together\end{matrix}\right\} - \left(A_i \cup A_j\right)$$

and $|A_{ij}| = 2^{\mathrm{m}} - 2$ *such that* $1 \leq i < j \leq 2\mathrm{n}$. In $Cay_{\mathrm{m}}(D_{2n}, S)$, we have A_i is adjacent to $A_{j\mathrm{k}}$ where $w_i \sim w_j$ *and* $w_i \sim w_k$ such that $j < k$ and $i \neq j, k$.
The induced subgraph to the sets A_{ij} is complete 2-bipartite graph $K_{2^{\mathrm{m}},2^{\mathrm{m}}}$. They are
They are $A_{1(2\mathrm{n})}, A_{\mathrm{n}(\mathrm{n}+1)}, A_{1(2n-4)}, A_{2(n+1)}, A_{3(2n-2)}, A_{4(2\mathrm{n}-3)}, A_{2(2\mathrm{n}-3)}, A_{3(2\mathrm{n}-4)}, A_{(\mathrm{n}-1)(2\mathrm{n})}, A_{n(n-1)}, \ldots$ *and* $A_{1(2\mathrm{n})}$, $A_{\mathrm{n}(\mathrm{n}+1)}$ .We put $X_1 = \{A_{1(\mathrm{n}+2)}, A_1, A_{\mathrm{n}+2}\}$ and $Y_1 = \{A_{2(\mathrm{n}+1)}, A_2, A_{\mathrm{n}+1}\}$
$X_2 = \{A_{2(n+3)}, A_2, A_{\mathrm{n}+3}\}$ and $Y_2 = \{A_{3(\mathrm{n}+2)}, A_3, A_{\mathrm{n}+2}\}, \ldots,$.
$X_{\mathrm{n}-2} = \{A_{(n-2)(2n-1)}, A_{n-1}, A_{2\mathrm{n}-1}\}$ and $Y_{\mathrm{n}-2} = \{A_{(\mathrm{n}-1)(2\mathrm{n}-2)}, A_{\mathrm{n}-1}, A_{2n-2}\}$
$X_{\mathrm{n}-1} = \{A_{(n-1)(2n)}, A_{\mathrm{n}-1}, A_{2n}\}$ and $Y_{\mathrm{n}-1} = \{A_{\mathrm{n}(2\mathrm{n}-1)}, A_{\mathrm{n}}, A_{2\mathrm{n}-1}\}$
$X_{\mathrm{n}} = \{A_{1(2n)}, A_1, A_{2\mathrm{n}}\}$ and $Y_{\mathrm{n}} = \{A_{\mathrm{n}(\mathrm{n}+1)}, A_{\mathrm{n}}, A_{\mathrm{n}+1}\}$.
Moreover, X_i and Y_i are disjoint and each one has 8 vertices where i=1,2,..,n. Hence, the subgraph induced by $X_i \dot{\cup} Y_i$ is a complete 2-bipartite graph $K_{8,8}$. So, we have $\bigcup_{i=1}^{n}(X_i \dot{\cup} Y_i)$ in the structure of $Cay_2(D_{2n}, S)$. We will obtain (n) of the complete 2-bipartite graph $K_{2^{\mathrm{m}},2^{\mathrm{m}}}$. The sets of $\{A_{1(\mathrm{n}+2)}, A_{2(\mathrm{n}+1)}\}$,{ $A_{2(n+3)}, A_{3(\mathrm{n}+2)}$ },{ $A_{(n-2)(2n-1)}, A_{(\mathrm{n}-1)(2\mathrm{n}-2)}$ },…, $\{A_{(n-1)(2n)}, A_{(\mathrm{n}-1)(2\mathrm{n}-2)}\}$, $\{A_{(n-1)(2n)}, A_{\mathrm{n}(2\mathrm{n}-1)}\}$ and $\{A_{1(2n)}, A_{\mathrm{n}(\mathrm{n}+1)}\}$ are independent set.
Type (III) of vertices: The degree of these vertices is one. We code it by ${A_{ij}}^1$ and $|{A_{ij}}^1| = 2^{\mathrm{m}} - 2$ *where* $1 \leq i < j \leq 2\mathrm{n}$. We can see that A_i is adjacent to ${A_{ij}}^1$and the number of the elements of this sets is 2n.We define

$${A_{ij}}^1 = \left\{\begin{matrix}[a_1 \quad a_2 \ldots \quad a_{\mathrm{m}}]^t \mid a_1, a_2, \ldots, a_m \in \{w_i, w_j\} \ where \ \ in \ the \ matrix \\ components \ w_i, w_j \ isnot \ adjacent \ together \ in \ Cay(D_{2n}, S)\end{matrix}\right\} - \left(A_i \cup A_j\right) \ .$$

So, $|A_{ij}{}^{1}| = 2^{\mathrm{m}} - 2$ ..

$A_{ijk} = \left\{ \begin{matrix} [a_1 \quad a_2 \quad \dots \ a_{\mathrm{m}}]^t \mid a_1, a_2, \dots, a_{\mathrm{m}} \in \{w_i, w_j, w_k\} \ where \ \ in \ the \ matrix \\ components \ w_i, w_j, w_k \ isnot \ adjacent \ together \ in \ Cay(D_{2\mathrm{n}}, S) \end{matrix} \right\} - (A_i \cup A_j \cup A_k \cup A_{ij} \cup A_{\mathrm{jk}} \cup A_{ik})$. So, $|A_{ijk}| = 3(3^{\mathrm{m}-1} - 2^{\mathrm{m}} + 1)$ $such \ that \ 1 \leq i < j < k \leq 2\mathrm{n}$.

It is easy to see that A_i is adjacent to A_{jkl} where w_{i} is adjacent to w_j, w_k and w_l $in \ Cay(D_{2n}, S)$ such that $i \neq j, k, l$, and $1 \leq j < k < l \leq 2\mathrm{n}$. The number of these vertices is $2\mathrm{n}|A_{ij}{}^{1}| + 2\mathrm{n}|A_{ijk}| = 2\mathrm{n}(|A_{ij}{}^{1}| + |A_{ijk}|) = 2\mathrm{n}(3^{\mathrm{m}} - 2^{\mathrm{m}+1} + 1)$.

Moreover, we can reach this conclusion that each element of this set is adjacent to each of vertices of the graph $K_2\square C_n$.It is obtained by corona product ($K_2\square C_n$) to $\overline{K}_{|A_{ij}{}^{1}|+|A_{ijk}|} = \overline{K}_{(3^{\mathrm{m}}-2^{\mathrm{m}+1}+1)}$.

Type (IV) of vertices: The degree of these vertices is zero. Now, put

$A_{ij}{}^{*} = \{[a_1 \quad a_2 \ \dots \ a_{\mathrm{m}}]^t \mid a_1, a_2, \dots, a_{\mathrm{m}} \in \{w_i, w_j\}\} - \{A_{ij} \cup A_{ij}{}^{1}\}$

$A_{ijk}{}^{*} = \{[a_1 \quad a_2 \dots \ a_{\mathrm{m}}]^t \mid a_1, a_2, \dots, a_{\mathrm{m}} \in \{w_i, w_j, w_k\}\} - A_{ijk}$.

So, $|A_{ij}{}^{*}| = 2^{\mathrm{m}} - 2$ and $|A_{ijk}{}^{*}| = 3(3^{\mathrm{m}-1} - 2^{\mathrm{m}} + 1)$ where $1 \leq i < j < k \leq 2\mathrm{n}$.

We observe that the rest of the other vertices are all isolated vertices.

The number of these isolated vertices is $(\binom{2\mathrm{n}}{2} - 4n)|A_{ij}{}^{*}| + (\binom{2\mathrm{n}}{3} - 2n)|A_{ijk}{}^{*}|$

In $Cay_{\mathrm{m}}(D_{2n}, S)$, we have $A_{ij}{}^{*}$ is not adjacent to A_k where w_i and w_k is not adjacent to w_k in $Cay(D_{2n}, S)$ such that $i < j \ and \ k \neq i, j$.

We can express that the rest of the vertices outside of $\bigcup_{i=1}^{\mathrm{n}}(X_i \dot{\cup} Y_i)$, $\bigcup_{i=1}^{2\mathrm{n}} A_{ijk}$ and $\bigcup_{i=1}^{2\mathrm{n}} A_{ij}{}^{1}$ are all isolated vertices. The number of isolated vertices is

$|V|^{\mathrm{m}} - 2\mathrm{n}|A_i| - 2(2\mathrm{n})|A_{ij}| - 2\mathrm{n}|A_{ij}{}^{1}| - 2\mathrm{n}|A_{ijk}| = 2\mathrm{n}[(2\mathrm{n})^{\mathrm{m}-1} - |A_i| - 3|A_{ij}| - |A_{ijk}|]$
$= 2\mathrm{n}[(2\mathrm{n})^{\mathrm{m}-1} - 3^{\mathrm{m}} + 2]$.

It is clear that, the set A_1 is adjacent to $A_2, A_{\mathrm{n}+1}, A_{\mathrm{n}}, A_{2(n+1)}, A_{2(\mathrm{n})}, A_{\mathrm{n}(\mathrm{n}+1)}, A_{2(\mathrm{n})(\mathrm{n}+1)}$

the set A_2 is adjacent to $A_1, A_3, A_{\mathrm{n}+2}, A_{13}, A_{1(n+2)}, A_{3(n+2)}, A_{1(3)(n+2)}$

the set A_3 is adjacent to $A_2, A_4, A_{\mathrm{n}+3}, A_{24}, A_{2(n+3)}, A_{4(n+3)}, A_{2(4)(n+3)}, \dots,$

the set A_{i} is adjacent to

$A_{\mathrm{i}-1}, A_{\mathrm{i}+1}, A_{\mathrm{n}+\mathrm{i}}, A_{(i-1)(i+1)}, A_{(i-1)(n+i)}, A_{(i+1)(n+i)}, A_{(i-1)(i+1)(n+i)}, \dots,$

the set A_{n} is adjacent to $A_{\mathrm{n}-1}, A_{\mathrm{n}+1}, A_{2\mathrm{n}}, A_{(n-1)(n+1)}, A_{(n-1)(2n)}, A_{(n+1)(2n)}, A_{(n-1)(n+1)(2n)}, \dots$ and

the set $A_{2\mathrm{n}}$ is adjacent to

$A_{2\mathrm{n}-1}, A_{\mathrm{n}+1}, A_{\mathrm{n}}, A_{(n+1)(2n-1)}, A_{(n)(2n-1)}, A_{(n)(n+1)}, A_{(n)(n+1)(2n-1)}$

Thus , the structure of generalized Cayley graph of $K_2\square C_n$ when m ≥ 3 has (n) faces and each face is Complete 2-bipartite graph $K_{2^{\mathrm{m}},2^{\mathrm{m}}}$ such that these faces have some common vertices. Therefore,

$$Cay_{\mathrm{m}}(D_{2n}, S) \cong [n\boldsymbol{K}_{2^{\mathrm{m}},2^{\mathrm{m}}} - 3(K_2\square C_n)] \cup [(K_2\square C_n) \circ \overline{K}_{2\mathrm{n}(3^{\mathrm{m}}-2^{\mathrm{m}+1}+1)}] \cup \overline{K}_{2\mathrm{n}[(2\mathrm{n})^{\mathrm{m}-1}-3^{\mathrm{m}}+2]}.$$

Corollary 3.4. The general formula as given in Theorem 3.3 can be also presented in terms of defined sets of vertices as the following. The proof come directly from Theorem 3.3 and we omit here.

$$Cay_m(D_{2n}, S) \cong \left[\mathrm{n}\boldsymbol{K}_{(|A_{i_1i_2}|+2),(|A_{i_1i_2}|+2)} - 3(K_2\square C_n)\right] \cup \left((K_2\square C_n) \circ \overline{K}_{|A_{i_1i_2}|+|A_{i_1i_2i_3}|}\right) \cup \overline{K}_{|V|[(|V|)^{\mathrm{m}-1}-|A_i|-3|A_{ij}|-|A_{ijk}|]}$$ for all m ≥ 3.

The graph $Cay_{\mathrm{m}}(D_{2\mathrm{n}}, S)$ is shown in Figure 11.

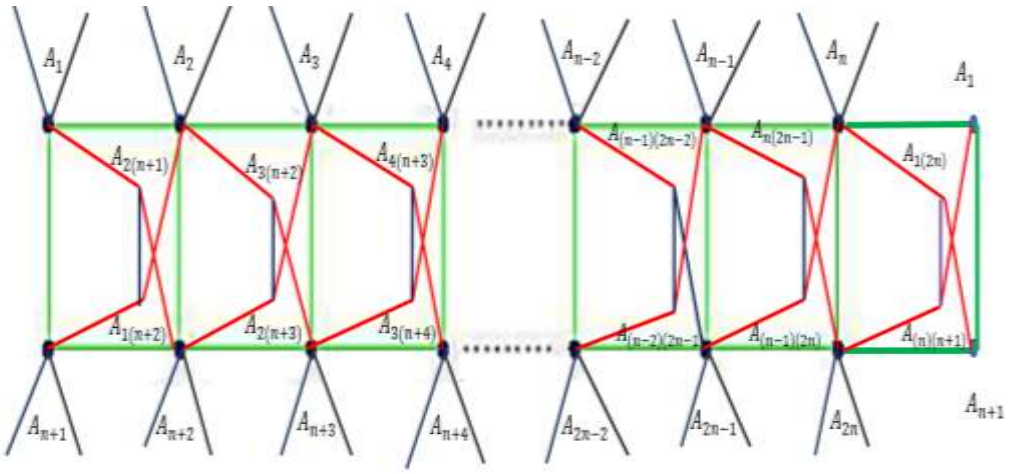

Figure 11: The component of the graph of Caym(D2n,S)

4- Conclusions

In this paper we determined the graph structure of the generalized Cayley graph $\boldsymbol{Cay_m(D_{2n}, S)}$ for given dihedral group $\boldsymbol{D_{2n}}$ of order 2n and subset S of $\boldsymbol{D_{2n}}$ such that $\mathbf{e} \notin \mathbf{S}$, $\mathbf{S^{-1}} \subseteq \mathbf{S}$ and $\mathbf{1} \leq |\,\mathbf{S}\,| \leq \mathbf{3}$ for every $\boldsymbol{m} \geq \mathbf{2},\ \mathbf{n} \geq \mathbf{3}$

References

[1] Cayley, P, "Desiderata and suggestions, the theory of groups: graphical representation". *American Journal of Mathematics 1*, vol. 2, pp. 174–176, 1878.

[2] Marušič, D., Scapellato, R., and Salvi, N. Z., "Generalized Cayley graphs". *Discrete Mathematics*, vol.102, no.3, pp. 279–285, 1992.

[3] Zhu, Y. "Generalized Cayley graphs of semigroups I". In *Semigroup Forum*, vol. 84, pp. 131–143,(2012).

[4] Neamah A A, Majeed A. H. and Erfanian A, "The generalized Cayley graph of complete graph $\mathbf{K_n}$ and complete multipartite graphs $\mathbf{K_{n,n}}$ and $\mathbf{K_{n,n,n}}$", *Iraqi Journal of Science*, vol. 63, no. 7, pp. 3103-3110, 2020.

[5] Neamah, A, A., Erfanian, A, Majeed A H, *On a generalized Cayley graph of column matrices of elements of a finite group*, *Journal of Mathematica*, vol. 64 (87), no 2, pp. 247-254, 2022.

[6] Godsil, C. and Royle, G. F., *Algebraic Graph Theory*, *vol. 207*. *Springer Science & Business Media*, 2001.

[7] Sabidussi. G, "Graph multiplication*"*, *Math. Z.*, vol. 72, pp. 446-457, 1969.

[8] Mohammadi, S., "Some generalizations of Cayley graph and Intersection graph", *Ph.D Thesis, Ferdowsi University of Mashhad*, *Mashhad*, *Iran* (March 2020).

[9] Al-Kaseabeh. S and Erfanian. A, "The structure of Cayley graphs of dihedral groups of valencies 1,2 and 3*." Journal of Proycciones*, vol. 40, no. 6, 2021.