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Oscillation and Asymptotic Behavior of a Three-Dimensional Half-Linear System of Third-Order of Neutral Type

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Abstract

In this paper, the oscillatory properties and asymptotic behaviour of a third-order three-dimensional neutral system are discussed. Some sufficient conditions are obtained to ensure that all bounded positive solutions of the system are oscillatory or non-oscillatory. On the other hand, the non-oscillatory solutions either converge or diverge when t goes to infinity. A special technique is adopted to include all possible cases. The obtained results include illustrative examples.

Keywords: Oscillation, Positive solution, Three-dimensional, Half linear neutral system, Third order.

التذبذب والسلوك المحاذي لأنظمة نصف خطية ثلاثية الأبعاد من الرتبة الثالثة من النوع المحايد

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الخلاصة

في هذه الورقة، تمت مناقشة الخصائص التذبذبية والسلوك المقارب لنظام محايد ثلاثي الأبعاد من الرتبة الثالثة. يتم الحصول على بعض الشروط الكافية لضمان أن تكون جميع الحلول الموجبة المقيدة للنظام متذبذبة أو غير متذبذبة. من ناحية أخرى، فإن الحلول غير المتذبذبة إما تتقارب أو تتباعد عندما يذهب t إلى اللانهاية. تم اعتماد تقنية خاصة لتشمل جميع الحالات الممكنة. تضمنت النتائج التي تم الحصول عليها أمثلة توضيحية.

1. Introduction

Differential equations are one of the most important topics in mathematics due to their many applications, see [1]. The oscillation theory of functional differential equations differs from that of ordinary differential equations. In fact, the former reveals the oscillation or non-oscillation of solutions caused by the appearance of deviating arguments in the differential equation. The paper of A.M. Kareem et al. [2] is the first work on the oscillation of functional differential equations.

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The oscillation and asymptotic behaviour of the solutions to the delay differential equations have extensively attracted the attention of many Mathematicians in recent years due to the realization that many number of applications are in use to delay differential equations. An important feature of the equations system solutions is the oscillation property as well as the convergence of non-oscillating solutions.

In recent years, the theory of neutral differential equations has become an independent area of research. Investigation of the oscillation and non-oscillation of neutral differential equations had already been initiated in the sixties and became popular in the eighties [3-11]. A few of them have been investigated in the system of neutral differential equations [12-14]. One of the researchers studied the oscillating of neutral differential equations system solutions. Ladas et al. studied the oscillation of solutions of the system of differential equations and neutral equations of the first order in the form

$$\frac{d}{dt}\left(X(t)-\sum_{i=1}^{n} P_i\,X(t-\tau_i)\right)=\sum_{i=1}^{n} Q_i\,X(t-\sigma_i)\,, \tag{1}$$

where the coefficients P_i andQ_i are real $m \times m$ matrices and the delays τ_i, σ_i, are non-negative real numbers. The authors obtained some necessary and sufficient conditions for oscillating of a linear system and other conditions for converging non-oscillatory solutions to zero.

Ladde and Zhang [15] discussed the oscillation and non-oscillation for systems of two first-order linear differential equations with a delay in the form

$$\begin{cases} x_1'(t) = a_{11}x_1 + a_{12}x_2(t-\tau_1) \\ x_1'(t) = a_{21}x_1(t-\tau_2) + a_{22}x_2 \end{cases} \tag{2}$$

Where a_{11}, a_{12}, a_{21} and a_{22} are constant coefficients. Also, Ladde and Zhang have considered variable coefficients.

Agarwal et al. [16] discussed the systems of delay nonlinear differential equations in the form

$$\begin{cases} \dot{x}(t) = a(t)f\big(y(t-\tau)\big) \\ \dot{y}(t) = -b(t)g\big(x(t-\tau)\big) \end{cases} \tag{3}$$

They obtained sufficient conditions for the existence of a non-oscillatory solution for this system and established some sufficient conditions to insure the oscillations of the system solutions.

Mohamad, H.A. and Abdulkareem, N. A.[17] discussed the almost oscillatory solutions of the system of differential equations of the form:

$$\begin{cases} [r_1(t)([x(t)+p_1(t)x(\tau_1(t))]')^{\alpha}]' + q_1(t)f_1\big(y(\sigma_1(t))\big) = 0 \\ [r_2(t)([y(t)+p_2(t)y(\tau_2(t))]')^{\alpha}]' + q_2(t)f_2\Big(x\big(\sigma_2(t)\big)\Big) = 0' \end{cases} \tag{4}$$

Where α is a quotient of a positive odd integer. They discussed and obtained some necessary and sufficient conditions to ensure the oscillation for every bounded solution of this system or that every non-oscillatory solution converges to zero as $t \to \infty$.

Akin et al. in[18] considered the system of the form:

$$\begin{cases} x^{\triangle}(t) = a(t)y^{\alpha_1}(t) \\ y^{\triangle}(t) = b(t)z^{\alpha_2}(t) \\ z^{\triangle}(t) = -c(t)x^{\alpha_3}(t) \end{cases} , \qquad t \geq t_0 > 0. \tag{5}$$

They classify the non-oscillatory solutions of the system (7) under the conditions

$$\int_T^{\infty} a(s)\, ds = \int_T^{\infty} b(s)\, ds = \infty.$$

In [19], Spanikova et al. investigated the oscillatory properties of neutral three-dimensional differential systems of the type

$$\begin{cases} [x(t) - a_1(t)x(g(t))]' = p_1(t)f_1(y(h_2(t)) \\ y'(t) = p_2(t)f_2(z(h_3(t)) \\ z'(t) = -p_3(t)f_1(x(h_1(t)) \end{cases}, \tag{6}$$

We consider the following system

$$\begin{cases} (\zeta_1(t)(\omega_1''(t))^{\alpha_1})' = \lambda q_1(t)y_2^{\alpha_1}(\sigma_1(t)) \\ \left(\zeta_2(t)\left(\omega_2''(t)\right)^{\alpha_2}\right)' = \lambda q_2(t)y_3^{\alpha_2}(\sigma_2(t)) \\ \left(\zeta_3(t)\left(\omega_3''(t)\right)^{\alpha_3}\right)' = \lambda q_3(t)y_1^{\alpha_3}(\sigma_3(t)) \end{cases}, \quad t \geq t_0 > 0. \tag{E}$$

Where

$$\begin{cases} \omega_1(t) = y_1(t) + p_1(t)y_1(\tau_1(t)), \\ \omega_2(t) = y_2(t) + p_2(t)y_2(\tau_2(t)), \\ \omega_3(t) = y_3(t) + p_3(t)y_3(\tau_3(t)). \end{cases} \tag{7}$$

The following hypotheses are assumed to be satisfied for all $i = 1,2,3$:

(H_1) $\lambda \in \{1, -1\}$,

(H_2) $\zeta_i, q_i \in C([t_0, \infty), \mathbb{R}^+), p_i(t) \in C([t_0, \infty), [0,1])$

(H_3) $\tau_i, \sigma_i \in C([t_0, \infty), \mathbb{R})$, $\sigma_i(t) \leq t$, $\tau_i(t) \leq t$, and $\lim_{t\to\infty} \tau_i(t) = \lim_{t\to\infty} \sigma_i(t) = \infty$,

(H_4) $\alpha_i > 0$ is the ratio of two odd integers.

By a solution to the system (E), we mean a vector function $Y(t) = [y_1(t), y_2(t), y_3(t)]^T$, which has the properties, $\zeta_1(t)\left(\omega_1''(t)\right)^{\alpha_1}, \zeta_2(t)\left(\omega_2''(t)\right)^{\alpha_2}, \zeta_3(t)\left(\omega_3''(t)\right)^{\alpha_3} \in C^1([t_0, \infty), \mathbb{R})$, and satisfies the system (E). We consider only those solutions to the system (E) which satisfy conditions $\sup\{|y_i(t)|: t \geq T\} > 0$.

We also consider only those solutions $Y(t) = [y_1(t), y_2(t), y_3(t)]^T$ to the system (E) are positive.

Definition 1.1 [5] A proper solution $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ of (E) is said to oscillate if it is eventually trivial or if at least one component does not have an eventual constant sign.

Definition 1.2 [5] A solution $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ of (E) is said to be bounded if every of its component is bounded. Otherwise, the solution is called unbounded. That is, a solution is said unbounded if at least one component is unbounded.

This paper consists of five sections; in the second section, we present two lemmas that we will rely on to get the main results. In the third and fourth sections, the non-oscillatory solutions (NOS) to the system (E) are studied with certain conditions. In the fifth section, the system (E) oscillation is studied with certain conditions, and we give some examples that illustrate the results.

2. Some basic lemmas

In this section, we present some lemmas that we will rely on to get the main results.

Lemma 2.1. [9]

Let $f \in C^n(R,R)$ and $f^{(n)}(t)f^{(n-1)}(t) > 0,\ t \geq t_0,\ t_0 \in (-\infty,\infty)$, then the following statements hold:
1. If $f^{(n)}(t) > 0$ for $t \geq t_0$, then $f^{(i)}(t)$ is increasing for $t \geq t_0$ and $\lim_{t\to\infty} f^{(i)}(t) = \infty$, where $i = n-1, n-2, \dots, 0$,
2. If $f^{(n)}(t) < 0$ for $t \geq t_0$, then $f^{(i)}(t)$ is decreasing for $t \geq t_0$ and $\lim_{t\to\infty} f^{(i)}(t) = -\infty$, where $i = n-1, n-2, \dots, 0$.

Lemma 2.2 [5] Let $\omega, y, \mathcal{P}: [t_0,\infty) \to \mathbb{R}$ and $\tau \in \mathbb{R}$ be such that,

$$\omega(t) = y(t) + \mathcal{P}(t)y(t-\tau), t \geq t_0 + \max\{0,\tau\}. \qquad (8)$$

Assume that $0 \leq \mathcal{P}(t) \leq \mathcal{P}_0 < 1$, and $y(t) > 0$ for $t \geq t_0$, $\lim_{t\to\infty}\inf y(t) = 0$ and that $\lim_{t\to\infty} \omega(t) \equiv L \in \mathbb{R}$ exists. Then $L = 0$. □

Remark 2.1: It is clear that Lemma 2.2 remains true if the delay is variable, whcih means $t-\tau$ is replaced by $\tau(t)$, where $\tau(t) < t,\ \tau(t) \to \infty$, as $t \to \infty$, also it remains true if the solution is negative.

Remark 2.2. For simplicity, we will assume that non-oscillatory solutions satisfy $y_1, y_2, y_3 > 0$, when they exist.

3. Non-oscillatory Solutions (NOS) of System (E), case $\lambda = 1$

In this section, we study the asymptotic behaviour of NOS with $\lambda = 1$, which we use in the following sections.

Lemma 3.1 Assume that $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ is the positive NOS of the system (E), and

$$\lim_{t\to\infty}\sup \int_{t_2}^{t}\int_{s}^{\delta(s)} \left(\frac{1}{\zeta_i(v)}\right)^{\frac{1}{\alpha_i}} dv\, ds = \infty,\quad i = 1,2,3. \qquad (9)$$

Then there are only the following possible classes from $K_1 - K_8$:

Table 1: The classes of all possible NOS of the system $(E), \lambda = 1$.

Classes	Sign of $\omega_i^{(j)}$						behaviour as $t \to \infty$
n	ω_1'	ω_2'	ω_3'	ω_1''	ω_2''	ω_3''	ω_i
K_1	+	+	+	+	+	+	$\omega_i \to \infty$
K_2	+	+	+	−	−	−	$\zeta_i(\omega_i''(t))^{\alpha_i} \to 0$
K_3	+	+	+	−	−	+	$\omega_3 \to \infty$
K_4	+	+	+	−	+	−	$\omega_2 \to \infty$
K_5	+	+	+	+	−	−	$\omega_1 \to \infty$
K_6	+	+	+	+	+	−	$\omega_{1,2} \to \infty$
K_7	+	+	+	−	+	+	$\omega_{2,3} \to \infty$
K_8	+	+	+	+	−	+	$\omega_{1,3} \to \infty$

Proof: Suppose that $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ be an eventually positive solution of system (E), for all $t \geq t_0$, then from (E), it follows
$(\zeta_1(t)(\omega_1''(t))^{\alpha_1})' \geq 0, (\zeta_2(t)(\omega_2''(t))^{\alpha_2})' \geq 0, (\zeta_3(t)(\omega_3''(t))^{\alpha_3})' \geq 0$
That means $\zeta_1(t)(\omega_1''(t))^{\alpha_1}$, $\zeta_2(t)(\omega_2''(t))^{\alpha_2}$, $\zeta_3(t)(\omega_3''(t))^{\alpha_3}$, nondecreasing; hence, there exists $t_1 \geq t_0$ such that $\zeta_1(t)(\omega_1''(t))^{\alpha_1}, \zeta_2(t)(\omega_2''(t))^{\alpha_2}$ and $\zeta_3(t)(\omega_3''(t))^{\alpha_3}$, are eventually positive or eventually negative. So eight cases can be discussed, which are:

Table 2: The eight possible cases can occur in the system (1), $\boldsymbol{y_i(t) > 0, i = 1, 2, 3}$.

i.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} > 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} > 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0$	
ii.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} < 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} < 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} < 0$	
iii.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} < 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} < 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0$	
iv.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} < 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} > 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} < 0$	
v.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} > 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} < 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} < 0$	
vi.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} > 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} > 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} < 0$	$t \geq t_1$
vii.	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} < 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} > 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0$	
viii	$\zeta_1(t)(\omega_1''(t))^{\alpha_1} > 0$	$\zeta_2(t)(\omega_2''(t))^{\alpha_2} < 0$	$\zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0$	

Now, we discuss the cases in Table 2 successively**:**
i. Since $\zeta_i(t)(\omega_i''(t))^{\alpha_i} > 0$ and $(\zeta_i(t)(\omega_i''(t))^{\alpha_i})' \geq 0, i = 1,2,3,$ then $\zeta_i(t)(\omega_i''(t))^{\alpha_i}$, is positive non-decreasing, then there exists $b_i > 0, t_2 \geq t_1$ such that $\zeta_i(t)(y_i''(t))^{\alpha_i} \geq b_i$

$$\omega_i''(t) \geq b_i^{\frac{1}{\alpha_i}} \left(\frac{1}{\zeta_i(t)}\right)^{\frac{1}{\alpha_i}}, t \geq t_2. \tag{10}$$

Integrating (10) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, we obtain

$$\omega_i'(\delta(t)) - \omega_i'(t) \geq b_i^{\frac{1}{\alpha_i}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}} ds, \tag{11}$$

We claim that $\omega_i'(t) > 0$ for $t \geq t_3 \geq t_2$. Otherwise, if $\omega_i'(t) < 0$, for $t \geq t_3 \geq t_2$, then (11) becomes

$$\omega_i'(t) \leq -b_i^{\frac{1}{\alpha_i}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}} ds. \tag{12}$$

Integrating (12) from t_3 to , we get,

$$\omega_i(t) - \omega_i(t_3) \leq -b_i^{\frac{1}{\alpha_i}} \int_{t_3}^{t} \int_s^{\delta(s)} \left(\frac{1}{\zeta_i(v)}\right)^{\frac{1}{\alpha_i}} dvds$$

Letting $t \to \infty$, then the last inequality leads to $\lim_{t\to\infty} \omega_i(t) = -\infty$, a contradiction. Hence, the claim is verified and $\omega_i'(t) > 0$ and $\omega_i''(t) > 0$, by Lemma (2.1); this case leads to $\lim_{t\to\infty} \omega_i(t) = \infty$. That is $(y_1, y_2, y_3) \in K_1$.
ii. Since $\zeta_i(t)(\omega_i''(t))^{\alpha_i} < 0$ and $(\zeta_i(t)(\omega_i''(t))^{\alpha_i})' \geq 0, i = 1,2,3.$
That is $\zeta_i(t)(\omega_i''(t))^{\alpha_i}$ is negative non-decreasing. So there are $\mathscr{b}_i \leq 0, i = 1,2,3,$ such that $\lim_{t\to\infty} \zeta_i(t)(\omega_i''(t))^{\alpha_i} = \mathscr{b}_i \leq 0$, hence $\zeta_i(t)(\omega_i''(t))^{\alpha_1} \leq \mathscr{b}_i, t \geq t_2$ and so

$$\omega_i''(t) \leq \mathscr{b}_i^{\frac{1}{\alpha_i}} \left(\frac{1}{\zeta_i(t)}\right)^{\frac{1}{\alpha_i}}, t \geq t_2. \tag{13}$$

Integrating (13) from t to $\delta(t)$, this yields

$$\omega_i'(\delta(t)) - \omega_i'(t) \leq \mathscr{b}_i^{\frac{1}{\alpha_i}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}} ds$$

We claim that $\omega_i'(t) > 0,\ t \geq t_3 \geq t_2, i = 1,2,3$. Otherwise, $\omega_i'(t) < 0,\ t \geq t_3 \geq t_2$ and $\omega_i''(t) < 0$ this implies to $\omega_i(t) < 0$ and $\lim_{t\to\infty} \omega_i(t) = -\infty$, a contradiction. Hence, $\omega_i(t) > 0, \omega_i'(t) > 0$, and $\omega_i''(t) < 0$, which means $(y_1, y_2, y_3) \in K_2$.

iii. Since, $\zeta_1(t)(\omega_1''(t))^{\alpha_1} < 0, \zeta_2(t)(\omega_2''(t))^{\alpha_2} < 0$ and $(\zeta_i(t)(\omega_i''(t))^{\alpha_i})' \geq 0$, that is $\zeta_j(t)(\omega_j''(t))^{\alpha_j}$ are negative non-decreasing, $j = 1,2$, there exists $\mathscr{b}_j \leq 0$, such that, $\lim_{t\to\infty} \zeta_j(t)(\omega_j''(t))^{\alpha_j} = \mathscr{b}_j \leq 0$. Then $\zeta_j(t)(\omega_j''(t))^{\alpha_j} \leq \mathscr{b}_j,\ t \geq t_2$ thus

$$\omega_j''(t) \leq \mathscr{b}_j^{\frac{1}{\alpha_j}} \left(\frac{1}{\zeta_j(t)}\right)^{\frac{1}{\alpha_j}}, t \geq t_2,\ j = 1,2. \tag{14}$$

Integrating (14) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, we obtain

$$\omega_j'(\delta(t)) - \omega_j'(t) \leq \mathscr{b}_j^{\frac{1}{\alpha_j}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_j(s)}\right)^{\frac{1}{\alpha_j}} ds$$

We claim that $\omega_j'(t) > 0,\ t \geq t_3 \geq t_2,\ j = 1,2$. Otherwise, if $\omega_j'(t) < 0,\ t \geq t_3 \geq t_2$ and $\omega_j''(t) < 0$ this implies to $\omega_j(t) < 0$ and $\lim_{t\to\infty} \omega_j(t) = -\infty$, a contradiction. Hence, $\omega_j(t) > 0, \omega_j'(t) > 0$, and $\omega_j''(t) < 0$. Now, $\zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0$ and $(\zeta_3(t)(\omega_3''(t))^{\alpha_3})' \geq 0$. So $\zeta_3(t)(\omega_3''(t))^{\alpha_3}$ is positive non-decreasing, then there exists $b_3 > 0$ and $t_2 \geq t_1$ such that

$$\omega_3''(t) \geq b_3^{\frac{1}{\alpha_3}} \left(\frac{1}{\zeta_3(t)}\right)^{\frac{1}{\alpha_3}}, t \geq t_2. \tag{15}$$

Integrating (15) from t to $\delta(t)$, we obtain

$$\omega_3'(\delta(t)) - \omega_3'(t) \geq b_3^{\frac{1}{\alpha_3}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_3(s)}\right)^{\frac{1}{\alpha_3}} ds$$

We claim that $\omega_3'(t) > 0$ for $t \geq t_3 \geq t_2$ Otherwise, if $\omega_3'(t) < 0$ for $t \geq t_3 \geq t_2$, then the last inequality becomes

$$\omega_3'(t) \leq -b_3^{\frac{1}{\alpha_3}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_3(s)}\right)^{\frac{1}{\alpha_3}} ds\,. \tag{16}$$

Integrating (16) from t_3 to t

$$\omega_3(t) - \omega_3(t_3) \leq -b_3^{\frac{1}{\alpha_3}} \int_{t_3}^{t} \int_s^{\delta(s)} \left(\frac{1}{\zeta_3(v)}\right)^{\frac{1}{\alpha_3}} dv ds. \tag{17}$$

Letting $t \to \infty$, then inequality (17) leads to $\lim_{t\to\infty} \omega_3(t) = -\infty$, a contradiction. Hence, $\omega_3'(t) > 0$ and $\omega_3''(t) > 0$, this case leads to $\lim_{t\to\infty} \omega_3(t) = \infty$, and so $(y_1, y_2, y_3) \in K_3$.

Analogously from the subcases (iv-viii), one can get$(y_1, y_2, y_3) \in K_n,\ n = 4,5, \dots, 8$, respectively.

4. Non-oscillatory Solutions of System (E), case $\lambda = -1$

In this section, we study the asymptotic behaviour of NOS with $\boldsymbol{\lambda} = -1$, which we use in the following sections.

Lemma 4.1 Assume that $Y = (y_1(t), y_2(t), y_3(t))$ is eventually positive NOS of (E), with $\lambda = -1$, if (9) holds. Then there are only the following possible classes $L_1 - L_8$:

Table 3: The classes of all non-oscillatory solutions of $(\boldsymbol{E}), \boldsymbol{\lambda} = -\mathbf{1}, \boldsymbol{y_i(t)} > \mathbf{0}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$.

Classes	Sign of $\omega_i^{(j)}(t)$						behaviour as $t \to \infty$
n	ω_1'	ω_2'	ω_3'	ω_1''	ω_2''	ω_3''	$\omega_i,\ i = 1,2,3$
L_1	+	+	+	+	+	+	$\omega_i \to \infty$
	−	−	−				$\zeta_i(t)(\omega_i''(t))^{\alpha_i} \to 0$
L_2	+	+	+	−	−	−	$\omega_i \to \infty$
L_3	+	+	+	−	−	+	$\omega_i \to \infty$
	+	+	−				$\omega_j \to \infty, j = 1,2,$ and $\zeta_3(t)(\omega_3''(t))^{\alpha_3} \to 0.$
L_4	+	+	+	−	+	−	$\omega_i \to \infty$
	+	−	+				$\omega_j \to \infty, j = 1,3,$ and $\zeta_2(t)(\omega_2''(t))^{\alpha_2} \to 0$
L_5	+	+	+	+	−	−	$\omega_i \to \infty$
	−	+	+				$y_j \to \infty, j = 2,3,$ and $\zeta_1(t)(\omega_1''(t))^{\alpha_1} \to 0$
L_6	+	+	+	+	+	−	$\omega_i \to \infty$
	−	−	+				$\omega_3 \to \infty$ $\zeta_{1,2}(t)(\omega_{1,2}''(t))^{\alpha_{1,2}} \to 0$
L_7	+	+	+	−	+	+	$\omega_i \to \infty$
	+	−	−				$\omega_1 \to \infty$ $\zeta_{2,3}(t)(\omega_{2,3}''(t))^{\alpha_{2,3}} \to 0$
L_8	+	+	+	+	−	+	$\omega_i \to \infty$
	−	+	−				$\omega_2 \to \infty$ $\zeta_{1,3}(t)(\omega_{1,3}''(t))^{\alpha_{1,3}} \to 0$

Proof: Suppose that $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ be an eventually positive solution of (1) then, $(\zeta_i(t)(\omega_i''(t))^{\alpha_i})' \leq 0\,, i = 1,2,3\,,\ t \geq t_1 \geq t_0$.

This means that $\zeta_i(t)(\omega_i''(t))^{\alpha_i}$ are non-increasing, so from Table 2, eight subcases can be discussed successively**:**

i. $\zeta_1(t)(\omega_1''(t))^{\alpha_1} > 0, \zeta_2(t)(\omega_2''(t))^{\alpha_2} > 0, \zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0, t \geq t_1$**.**

Since, $\zeta_i(t)(\omega_i''(t))^{\alpha_i}$ is positive non-increasing, there are $\mathscr{b}_i \geq 0, i = 1,2,3.$ such that $\lim_{t\to\infty} \zeta_i(t)(\omega_i''(t))^{\alpha_i} = \mathscr{b}_i \geq 0$, then there exists $t_2 \geq t_1$ such that $\zeta_i(t)(\omega_i''(t))^{\alpha_i} \geq \mathscr{b}_i,\ t \geq t_2$. Therefore,

$$\omega_i''(t) \geq \mathscr{b}_i^{\frac{1}{\alpha_i}} \left(\frac{1}{\zeta_i(t)}\right)^{\frac{1}{\alpha_i}},\quad t \geq t_2. \tag{18}$$

Integrating (18) from t to $\delta(t)$ for some continuous function $\delta(t) > t$, we obtain

$$\omega_i'(\delta(t)) - \omega_i'(t) \geq \mathscr{b}_i^{\frac{1}{\alpha_i}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}} ds, \tag{19}$$

We have two cases for $\omega_i'(t)$: a. $\omega_i'(t) < 0$, b. $\omega_i'(t) > 0$ for $t \geq t_3 \geq t_2$.

a. If $\omega_i'(t) < 0$, for $t \geq t_3 \geq t_2$, in that case, we claim that $\mathscr{b}_i = 0$. Otherwise, $\mathscr{b}_i > 0$, then (19) becomes

$$\omega_i'(t) \leq -\mathscr{b}_i^{\frac{1}{\alpha_i}} \int_t^{\delta(t)} \left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}} ds. \tag{20}$$

Integrating (20) from t_3 to t

$$\omega_i(t)-\omega_i(t_3)\leq -\mathscr{b}_i^{\frac{1}{\alpha_i}}\int_{t_3}^{t}\int_{s}^{\delta(s)}\left(\frac{1}{\zeta_i(v)}\right)^{\frac{1}{\alpha_i}}dvds.$$

As $t\to\infty$ it follows that $\lim_{t\to\infty}\omega_i(t)=-\infty$, a contradiction, hence $\mathscr{b}_i=0$.

b. If $\omega_i'(t)>0$, and $\omega_i''(t)>0$, then it follows that $\lim_{t\to\infty}\omega_i(t)=\infty$. Thus $(y_1,y_2,y_3)\in L_1$.

ii. $\zeta_1(t)(\omega_1''(t))^{\alpha_1}<0,\ \zeta_2(t)(\omega_2''(t))^{\alpha_2}<0,\ \zeta_3(t)(\omega_3''(t))^{\alpha_3}<0,\ t\geq t_1$.
then $\omega_i''(t)<0, i=1,2,3$. Since $\mathscr{p}_i(t)(\omega_i''(t))^{\alpha_i}$ is negative non-increasing. Then there exists $b_i<0$, and $t_2\geq t_1$ such that $\mathscr{p}_i(t)(\omega_i''(t))^{\alpha_i}\leq b_i$ for $t\geq t_2$. Therefore,

$$\omega_i''(t)\leq b_i^{\frac{1}{\alpha_i}}\left(\frac{1}{\zeta_i(t)}\right)^{\frac{1}{\alpha_i}}, t\geq t_2. \tag{21}$$

integrating (21) from t to $\delta(t)$, we obtain

$$\omega_i'(\delta(t))-\omega_i'(t)\leq b_i^{\frac{1}{\alpha_i}}\int_{t}^{\delta(t)}\left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}}ds, \tag{22}$$

We claim that $\omega_i'(t)>0$ for $t\geq t_3\geq t_2$. Otherwise, if $\omega_i'(t)<0$ for $t\geq t_3\geq t_2$, and $y_i''(t)<0$ implies that $\lim_{t\to\infty}\omega_i(t)=-\infty$, a contradiction, thus $\omega_i'(t)>0$
then (22) becomes

$$\omega_i'(t)\geq -b_i^{\frac{1}{\alpha_i}}\int_{t}^{\delta(t)}\left(\frac{1}{\zeta_i(s)}\right)^{\frac{1}{\alpha_i}}ds. \tag{23}$$

Integrating (23) from t_3 to t

$$\omega_i(t)-\omega_i(t_3)\geq -b_i^{\frac{1}{\alpha_i}}\int_{t_3}^{t}\int_{s}^{\delta(s)}\left(\frac{1}{\zeta_i(v)}\right)^{\frac{1}{\alpha_i}}dvds.$$

As $t\to\infty$ it follows that $\lim_{t\to\infty}\omega_i(t)=\infty$. Thus $(y_1,y_2,y_3)\in L_2$.

iii. $\zeta_1(t)(\omega_1''(t))^{\alpha_1}<0,\ \zeta_2(t)(y_2''(t))^{\alpha_2}<0,\ \zeta_3(t)(y_3''(t))^{\alpha_3}>0,\ t\geq t_1$.
then $y_j''(t)<0, j=1,2$ and $\omega_3''(t)>0$. Since $\zeta_j(t)(\omega_j''(t))^{\alpha_i}$ are negative and non-increasing. Then there exists $b_j<0$, and $t_2\geq t_1$ such that $\zeta_j(t)(\omega_j''(t))^{\alpha_i}\leq b_j$ for $t\geq t_2$. Therefore,

$$\omega_j''(t)\leq b_j^{\frac{1}{\alpha_i}}\left(\frac{1}{\zeta_j(t)}\right)^{\frac{1}{\alpha_j}}, t\geq t_2. \tag{24}$$

Integrating (24) from t to $\delta(t)$, we obtain

$$\omega_j'(\delta(t))-\omega_j'(t)\leq b_j^{\frac{1}{\alpha_j}}\int_{t}^{\delta(t)}\left(\frac{1}{\zeta_j(s)}\right)^{\frac{1}{\alpha_j}}ds, \tag{25}$$

We claim that $\omega_j'(t)>0$ for $t\geq t_3\geq t_2$. Otherwise, if $\omega_j'(t)<0$ for $t\geq t_3\geq t_2$, and $\omega_j''(t)<0$ implies that $\lim_{t\to\infty}\omega_j(t)=-\infty$, a contradiction, thus $\omega_j'(t)>0$
then (25) becomes

$$\omega_j'(t)\geq -b_j^{\frac{1}{\alpha_j}}\int_{t}^{\delta(t)}\left(\frac{1}{\zeta_j(s)}\right)^{\frac{1}{\alpha_j}}ds.$$

Integrating the last inequality from t_3 to t yields

$$\omega_j(t)-\omega_j(t_3)\geq -b_j^{\frac{1}{\alpha_j}}\int_{t_3}^{t}\int_{t}^{\delta(s)}\left(\frac{1}{\zeta_j(v)}\right)^{\frac{1}{\alpha_j}}dvds$$

As $t\to\infty$ it follows that $\lim_{t\to\infty}\omega_j(t)=\infty\,, j=1,2$.

Concerning $\zeta_3(t)(\omega_3''(t))^{\alpha_3} > 0$ and nonincreasing, so there exists $\ell_3 \geq 0$, such that $\lim_{t\to\infty}\zeta_3(t)(\omega_3''(t))^{\alpha_3} = \ell_3 \geq 0$, then $\zeta_3(t)(\omega_3''(t))^{\alpha_3} \geq \ell_3,\ \ t \geq t_2 \geq t_1$. Therefore,

$$\omega_3''(t) \geq \ell_3^{\frac{1}{\alpha_3}}\left(\frac{1}{\zeta_3(t)}\right)^{\frac{1}{\alpha_3}}, t \geq t_2.$$

Integrating the last inequality from t to $\delta(t)$ leads to

$$\omega_3'(\delta(t)) - \omega_3'(t) \geq \ell_3^{\frac{1}{\alpha_3}}\int_t^{\delta(t)}\left(\frac{1}{\zeta_3(s)}\right)^{\frac{1}{\alpha_3}} ds, \tag{26}$$

We have two cases for $\omega_3'(t)$: a. $\omega_3'(t) > 0$; b. $\omega_3'(t) < 0$ for $t \geq t_3 \geq t_2$

a. If $\omega_3'(t) > 0$ for $t \geq t_3 \geq t_2$, and $\omega_3''(t) > 0$, it follows that $\lim_{t\to\infty}\omega_3(t) = \infty$,

b. If $\omega_3'(t) < 0$ for $t \geq t_3 \geq t_2$, we claim that $\ell_3 = 0$, otherwise $\ell_3 > 0$, then (26) is reduced to

$$\omega_3'(t) \leq -\ell_3^{\frac{1}{\alpha_3}}\int_t^{\delta(t)}\left(\frac{1}{\zeta_3(s)}\right)^{\frac{1}{\alpha_3}} ds. \tag{27}$$

Integrating (27) from t_3 to t

$$\omega_3(t) - \omega_3(t_3) \leq -\ell_3^{\frac{1}{\alpha_3}}\int_{t_3}^{t}\int_s^{\delta(s)}\left(\frac{1}{\zeta_3(v)}\right)^{\frac{1}{\alpha_3}} dvds.$$

As $t \to \infty$ it follows that $\lim_{t\to\infty} y_3(t) = -\infty$, which is a contradiction; hence, $(y_1, y_2, y_3) \in L_3$. Analogously from the subcases (**iv-viii**), one can get $(y_1, y_2, y_3) \in L_n,\ n = 4,5, \dots ,8$, respectively.

5. Oscillation conditions

In this section, some theorems and corollaries are established, which ensure that all bounded solutions to system (E) are either oscillatory or non-oscillatory and converge to zero, and all unbounded solutions to (E) are either oscillatory or diverge as $t \to \infty$. Moreover, we discussed the asymptotic behaviour of (E).

Theorem 5.1 Suppose that $\lambda = 1$ and (9) holds in addition to

$$\begin{aligned}
&\int_{t_2}^{\infty}\int_s^{\delta(s)}\left(\frac{1}{\zeta_1(s)}\int_\xi^{\delta(\xi)} q_1(\theta)\left(1 - p_2(\sigma_1(\theta))\right)^{\alpha_1} d\theta\right)^{\frac{1}{\alpha_1}} d\xi ds = \infty \\
&\int_{t_2}^{\infty}\int_s^{\delta(s)}\left(\frac{1}{\zeta_2(s)}\int_\xi^{\delta(\xi)} q_2(\theta)\left(1 - p_3(\sigma_2(\theta))\right)^{\alpha_2} d\theta\right)^{\frac{1}{\alpha_2}} d\xi ds = \infty \\
&\int_{t_2}^{\infty}\int_s^{\delta(s)}\left(\frac{1}{\zeta_3(s)}\int_\xi^{\delta(\xi)} q_3(\theta)\left(1 - p_1(\sigma_3(\theta))\right)^{\alpha_3} d\theta\right)^{\frac{1}{\alpha_3}} d\xi ds = \infty
\end{aligned} \tag{28}$$

Then every bounded solution of the system (E) oscillates.

Proof: Suppose that (E) has a non-oscillatory solution $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ so by Lemma 3.1, there is only the class $\boldsymbol{K_2}$ in Table 1, that can occur for $t \geq t_1 \geq t_0$, that is:

$$\omega_i(t) > 0, \omega_i'(t) > 0, \omega_i''(t) < 0, \quad i = 1,2,3.$$

Since

$$\omega_i(t) = y_i(t) + p_i(t)y_1(\tau_i(t)),$$
$$\omega_i(t) \geq y_i(t)\,,\ \ t \geq t_2 \geq t_1\,,$$

$$y_1(\sigma_1(t)) \geq \left(1 - p_1(\sigma_3(t))\right)\omega_1\left(\tau_2(\sigma_3(t))\right) \qquad (a)$$
$$y_2(\sigma_1(t)) \geq \left(1 - p_2(\sigma_1(t))\right)\omega_2\left(\tau_2(\sigma_1(t))\right) \qquad (b) \qquad (29)$$
$$y_3(\sigma_2(t)) \geq \left(1 - p_3(\sigma_2(t))\right)\omega_3\left(\tau_2(\sigma_2(t))\right) \qquad (c)$$

Integrating the first equation of system (E), from t to $\delta(t)$, we get

$$\zeta_1(\delta(t))\left(\omega_1''(\delta(t))\right)^{\alpha_1} - \zeta_1(t)\left(\omega_1''(t)\right)^{\alpha_1} = \int_t^{\delta(t)} q_1(s) y_2^{\alpha_1}(\sigma_1(s))\, ds, \qquad (30)$$

Using (29b) in (30), we get

$$-\zeta_1(t)\left(\omega_1''(t)\right)^{\alpha_1} \geq \int_t^{\delta(t)} q_1(s)\left(1 - p_2(\sigma_1(s))\right)^{\alpha_1} \omega_2^{\alpha_1}\left(\tau_2(\sigma_1(s))\right) ds,$$

$$\zeta_1(t)\left(\omega_1''(t)\right)^{\alpha_1} \leq -\int_t^{\delta(t)} q_1(s)\left(1 - p_2(\sigma_1(s))\right)^{\alpha_1} \omega_2^{\alpha_1}\left(\tau_2(\sigma_1(s))\right) ds,$$

$$\zeta_1(t)\left(\omega_1''(t)\right)^{\alpha_1} \leq -\omega_2^{\alpha_1}\left(\tau_2(\sigma_1(t))\right)\int_t^{\delta(t)} q_1(s)\left(1 - p_2(\sigma_1(s))\right)^{\alpha_1} ds,$$

$$\omega_1''(t) \leq -\omega_2\left(\tau_2(\sigma_1(t))\right)\left(\frac{1}{\zeta_1(t)}\int_t^{\delta(t)} q_1(s)\left(1 - p_2(\sigma_1(s))\right)^{\alpha_1} ds\right)^{\frac{1}{\alpha_1}},$$

Integrating the last inequality from t to $\delta(t)$, we get:

$$\omega_1'(\delta(t)) - \omega_1'(t) \geq \omega_2\left(\tau_2(\sigma_1(t))\right)\int_t^{\delta(t)}\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)} q_1(s)(1 - p_2(\sigma_1(\xi)))^{\alpha_1}\, d\xi\right)^{\frac{1}{\alpha_1}} ds.$$

Integrating the last inequality from t_2 to t we get:

$$\omega_1(t) - \omega_1(t_2) \geq \omega_2\left(\tau_2(\sigma_1(t_2))\right)\int_{t_2}^{t}\int_s^{\delta(s)}\left(\frac{1}{\zeta_1(\xi)}\int_\xi^{\delta(\xi)} q_1(\theta)(1 - p_2(\sigma_1(\theta)))^{\alpha_1}\, d\theta\right)^{\frac{1}{\alpha_1}} d\xi\, ds,$$

As $t \to \infty$, $\omega_1(t) \to \infty$. This implies $y_1(t) \to \infty$ a contradiction. Similarly, it can be shown that $\lim_{t\to\infty} y_2(t) = \infty$, $\lim_{t\to\infty} y_3(t) = \infty$, a contradiction.
This leads to the solution $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ oscillates.

Corollary 5.1 Suppose that $\lambda = 1$, and $(2), (21)$ are hold. Then every solution of the system (E) is either oscillatory or $\lim_{t\to\infty} y_i(t) = \infty$, $i = 1,2,3$.

Proof. Suppose that system (E) has a non-oscillatory solution $Y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}}$ so by Lemma 3.2, table 1, there are only the possible classes $\boldsymbol{K}_1 - \boldsymbol{K}_8$ to consider for $t \geq t_1 \geq t_0$. If $Y(t)$ is bounded, then by Theorem 3.1, it follows that $Y(t)$ is oscillatory. Otherwise, $Y(t)$ is unbounded. There are the following cases to consider:
Case 1. Suppose that $Y(t) \in \boldsymbol{K}_1$. By Lemma 3.1, it follows $\lim_{t\to\infty} y_i(t) = \infty$, $i = 1,2,3$. **Case 2.** Suppose that $Y(t) \in \boldsymbol{K}_2$. By Theorem 3.1, $\lim_{t\to\infty} y_i(t) = \infty$, $i = 1,2,3$..
Case 3. Suppose that $Y(t) \in \boldsymbol{K}_3$. By Lemma 3.1, it follows $\lim_{t\to\infty} y_3(t) = \infty$.
Integrating the first equation of system (E), from t to $\delta(t)$, we get:

$$\zeta_1(\delta(t))\big(\omega_1''(\delta(t))\big)^{\alpha_1}-\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1}=\int_t^{\delta(t)}q_1(s)y_2^{\alpha_1}\big(\sigma_1(s)\big)\,ds,$$

$$-\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1}\geq\int_t^{\delta(t)}q_1(s)y_2^{\alpha_1}\big(\sigma_1(s)\big)\,ds, \tag{31}$$

Using (29b) in (31), we get

$$-\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1}\geq\int_t^{\delta(t)}q_1(s)\left(1-p_2\big(\sigma_1(s)\big)\right)^{\alpha_1}\omega_2^{\alpha_1}\left(\tau_2\big(\sigma_1(s)\big)\right)ds,$$

$$\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1}\leq-\int_t^{\delta(t)}q_1(s)\left(1-p_2\big(\sigma_1(s)\big)\right)^{\alpha_1}\omega_2^{\alpha_1}\left(\tau_2\big(\sigma_1(s)\big)\right)ds,$$

$$\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1}\leq-\omega_2^{\alpha_1}\left(\tau_2\big(\sigma_1(t)\big)\right)\int_t^{\delta(t)}q_1(s)\left(1-p_2\big(\sigma_1(s)\big)\right)^{\alpha_1}ds,$$

$$\omega_1''(t)\leq-\omega_2\left(\tau_2\big(\sigma_1(t)\big)\right)\left(\frac{1}{\zeta_1(t)}\int_t^{\delta(t)}q_1(s)\left(1-p_2\big(\sigma_1(s)\big)\right)^{\alpha_1}ds\right)^{\frac{1}{\alpha_1}},$$

Integrating the last inequality from t to $\delta(t)$, we get:

$$\omega_1'\big(\delta(t)\big)-\omega_1'(t)\geq\omega_2\left(\tau_2\big(\sigma_1(t)\big)\right)\int_t^{\delta(t)}\left(\frac{1}{\zeta_1(s)}\int_s^{\delta(s)}q_1(\xi)(1-p_2\big(\sigma_1(\xi)\big))^{\alpha_1}\,d\xi\right)^{\frac{1}{\alpha_1}}ds.$$

Integrating the last inequality from t_2 to t we get:

$$\omega_1(t)-\omega_1(t_2)\geq\omega_2\left(\tau_2\big(\sigma_1(t_2)\big)\right)\int_{t_2}^{t}\int_s^{\delta(s)}\left(\frac{1}{\zeta_1(\xi)}\int_\xi^{\delta(\xi)}q_1(\theta)(1-p_2\big(\sigma_1(\theta)\big))^{\alpha_1}\,d\theta\right)^{\frac{1}{\alpha_1}}d\xi\,ds,$$

As $t\to\infty$, $\omega_1(t)\to\infty$. This implies $y_1(t)\to\infty$. Similarly, it can be shown that $\lim_{t\to\infty}y_2(t)=\infty$.this leads to $\lim_{t\to\infty}y_1(t)=\infty=\lim_{t\to\infty}y_2(t)=\infty=\lim_{t\to\infty}y_3(t)=\infty=0$. The proof of the cases when $Y(t)\in\boldsymbol{K}_4,\boldsymbol{K}_5$ is similar to the proof the **case 3**.

Case 4. Suppose that $Y(t)\in\boldsymbol{K}_6$. By Lemma 3.1, it follows $\lim_{t\to\infty}y_{1,2}(t)=\infty$.

Integrating the third equation of system (E), from t to $\delta(t)$, we get:

$$\zeta_3(\delta(t))\big(\omega_3''(\delta(t))\big)^{\alpha_3}-\zeta_3(t)\big(\omega_3''(t)\big)^{\alpha_3}=\int_t^{\delta(t)}q_3(s)y_1^{\alpha_1}\big(\sigma_3(s)\big)\,ds,$$

$$-\zeta_3(t)\big(\omega_3''(t)\big)^{\alpha_3}\geq\int_t^{\delta(t)}q_3(s)y_1^{\alpha_1}\big(\sigma_3(s)\big)\,ds, \tag{32}$$

Using (29a) in (32), we get

$$-\zeta_3(t)\big(\omega_3''(t)\big)^{\alpha_3}\geq\int_t^{\delta(t)}q_3(s)\left(1-p_1\big(\sigma_3(s)\big)\right)^{\alpha_3}\omega_1^{\alpha_1}\left(\tau_1\big(\sigma_3(s)\big)\right)ds,$$

$$\zeta_3(t)\big(\omega_3''(t)\big)^{\alpha_3}\leq-\int_t^{\delta(t)}q_3(s)\left(1-p_1\big(\sigma_3(s)\big)\right)^{\alpha_3}\omega_1^{\alpha_1}\left(\tau_1\big(\sigma_3(s)\big)\right)ds,$$

$$\zeta_3(t)\big(\omega_3''(t)\big)^{\alpha_3}\leq-\omega_1^{\alpha_3}\left(\tau_1\big(\sigma_3(t)\big)\right)\int_t^{\delta(t)}q_3(s)\left(1-p_1\big(\sigma_3(s)\big)\right)^{\alpha_3}ds,$$

$$\omega_3''(t)\leq-\omega_1\left(\tau_1\big(\sigma_3(t)\big)\right)\left(\frac{1}{\zeta_3(s)}\int_t^{\delta(t)}q_3(s)\left(1-p_1\big(\sigma_3(s)\big)\right)^{\alpha_3}ds\right)^{\frac{1}{\alpha_3}},$$

Integrating the last inequality from t to $\delta(t)$, we get:

$$\omega_3'\big(\delta(t)\big)-\omega_3'(t)\geq \omega_1\left(\tau_1\big(\sigma_3(t)\big)\right)\int_t^{\delta(t)}\left(\frac{1}{\zeta_3(s)}\int_s^{\delta(s)} q_3(\xi)(1-p_1\big(\sigma_3(\xi)\big))^{\alpha_3}\,d\xi\right)^{\frac{1}{\alpha_2}}ds.$$

Integrating the last inequality from t_2 to t we get:

$$\omega_3(t)-\omega_3(t_2) \geq \omega_1\left(\tau_1\big(\sigma_3(t_2)\big)\right)\int_{t_2}^{t}\int_s^{\delta(s)}\left(\frac{1}{\zeta_3(\xi)}\int_\xi^{\delta(\xi)} q_3(\theta)(1 - p_1\big(\sigma_1(\theta)\big))^{\alpha_3}\,d\theta\right)^{\frac{1}{\alpha_3}}d\xi\,ds,$$

As $t\to\infty,\ \omega_3(t)\to\infty$. This implies $y_3(t)\to\infty$. This leads to $\lim_{t\to\infty} y_1(t)=\infty=\lim_{t\to\infty} y_2(t)=\infty=\lim_{t\to\infty} y_3(t)=\infty=0$. The proof of the cases when $Y(t)\in \boldsymbol{K}_7,\boldsymbol{K}_8$, are similar to the proof of **case 4**; hence, the proof is completed.

Theorem 5.3 Assume that $0<p_{1,2,3}(t)<1$. Let $y_{1,2,3}(t)$ be a NOS of (E), With $\lambda=-1$ and suppose the corresponding $\omega_{1,2,3}(t)$ belongs to L_1. If

$$\begin{cases}\int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{\zeta_1(u)}\int_u^{\infty} q_1(s)\,ds\right]^{\frac{1}{\alpha_1}}du\,dv=\infty,\\ \int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{\zeta_2(u)}\int_u^{\infty} q_2(s)\,ds\right]^{\frac{1}{\alpha_2}}du\,dv=\infty,\\ \int_{t_0}^{\infty}\int_v^{\infty}\left[\frac{1}{\zeta_3(u)}\int_u^{\infty} q_3(s)\,ds\right]^{\frac{1}{\alpha_3}}du\,dv=\infty\end{cases} \tag{33}$$

Then either $\lim_{t\to\infty} y_{1,2,3}(t)=\lim_{t\to\infty}\omega_{1,2,3}(t)=\infty$, or $\lim_{t\to\infty} y_{1,2,3}(t)=\lim_{t\to\infty}\omega_{1,2,3}(t)=0$.

Proof. Suppose that y_1,y_2,y_3 are the positive solution of (E), with $\lambda=-1$. Where the corresponding functions $\omega_{1,2,3}(t)$ belong to L_1. Then there are two cases to consider:

Case I. If $\omega_i'(t)>0$ and $\omega_i''(t)>0$. From Lemma 4.1, we have

$$\lim_{t\to\infty} y_{1,2,3}(t)=\lim_{t\to\infty}\omega_{1,2,3}(t)=\infty.$$

Case II. If $\omega_{1,2,3}'(t)<0$ and $\omega_{1,2,3}''(t)>0$. since $\omega_2(t)>0$ and $\omega_2'(t)<0$, then there exists a finite $\hbar_2$ such that

$$\lim_{t\to\infty}\omega_2(t)=\hbar_2.$$

We shall prove that $\hbar_2=0$. Assume that $\hbar_2>0$. Then for any $\varepsilon>0$, we have $\hbar_2<\omega_2(t)<\hbar_2+\varepsilon$, eventually. Choose $0<\varepsilon<\frac{\hbar_2(1-p_2)}{p_2}$. It is easy to verify that

$$y_2(t)=\omega_2(t)-p_2y_2\big(\tau_2(t)\big)>\hbar_2-p_2(\hbar_2+\varepsilon)=k_2(\hbar_2+\varepsilon)>k_2\omega_2(t),$$

Where $k_2=\frac{\hbar_2-p_2(\hbar_2+\varepsilon)}{\hbar_2+\varepsilon}>0$. Using the above inequality, we obtain from (E),

$$\begin{aligned}&\left(\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1}\right)'\leq -k_2^{\alpha_1}q_1(t)\omega_2^{\alpha_1}\big(\sigma_1(t)\big)\quad (a)\\ &\left(\zeta_2(t)\big(\omega_2''(t)\big)^{\alpha_2}\right)'\leq -k_3^{\alpha_2}q_2(t)\omega_3^{\alpha_2}\big(\sigma_2(t)\big)\quad (b)'\\ &\left(\zeta_3(t)\big(\omega_3''(t)\big)^{\alpha_3}\right)'\leq -k_1^{\alpha_3}q_3(t)\omega_1^{\alpha_3}\big(\sigma_3(t)\big)\quad (c)\end{aligned} \tag{34}$$

Integrating the first inequality of (34a) from t to ∞, we get

$$\zeta_1(t)\big(\omega_1''(t)\big)^{\alpha_1} \geq k_2^{\alpha_1}\int_t^\infty q_1(t)\omega_2^{\alpha_1}\big(\sigma_1(t)\big)\,ds$$

Using $\omega_2\big(\sigma_1(t)\big) \geq \hbar_2$, we see that

$$\omega_1''(t) \geq k_2\hbar_2\left[\frac{1}{\zeta_1(t)}\int_t^\infty q_1(s)ds\right]^{\frac{1}{\alpha_1}}, \tag{35}$$

Integrating (35) from t to ∞, we obtain

$$-\omega_1'(t) \geq k_2\hbar_2\int_t^\infty\left[\frac{1}{\zeta_1(u)}\int_u^\infty q_1(s)ds\right]^{\frac{1}{\alpha_1}}du, \tag{36}$$

Integrating (36) from t_1 to t, we obtain

$$-\omega_1(t)+\omega_1(t_1) \geq k_2\hbar_2\int_{t_1}^t\int_v^\infty\left[\frac{1}{\zeta_1(u)}\int_u^\infty q_1(s)ds\right]^{\frac{1}{\alpha_1}}du\,dv$$

As $t\longrightarrow\infty$, $\lim_{t\longrightarrow\infty}\omega_1(t)=-\infty$, this is in contradiction with the positivity of $\omega_1(t)$. Therefore, $\hbar_2=0$. That leads to $\lim_{t\longrightarrow\infty}\omega_2(t)=0$. Inequality $0\leq y_2(t)\leq\omega_2(t)$ implies $\lim_{t\longrightarrow\infty}y_2(t)=0$. In the same way, we can prove that $\lim_{t\longrightarrow\infty}y_3(t)=0$ and $\lim_{t\longrightarrow\infty}y_1(t)=0$. Hence, the proof is completed.

Corollary 5.2 Suppose that $\lambda=-1$ and $(9),(33)$ holds. Then every unbounded solution to the system (E) is either oscillatory or $\lim_{t\to\infty}y_i(t)=\infty$.

Proof. Suppose that system (E) has a non-oscillatory solution $Y(t)=(y_1(t),y_2(t),y_3(t))^{\mathrm{T}}$ so by Lemma 4.1 and Table 3, there are only the possible cases $\boldsymbol{L}_1-\boldsymbol{L}_8$ to consider for $t\geq t_1\geq t_0$. If $Y(t)$ is bounded, then by Theorem 5.2, it follows that $Y(t)$ is either oscillatory or $Y(t)\to 0$ as $t\to\infty$. If $Y(t)$ is unbounded, then from Table 3, there are the following cases to consider:

Case 1. Suppose that $Y(t)\in\boldsymbol{L}_1,\boldsymbol{L}_2$. By Lemma 4.1, it follows $\lim_{t\to\infty}y_i(t)=\infty,\ \ i=1,2,3.$

Case 2. Suppose that $Y(t)\in\boldsymbol{L}_3$. Then there are two subcases to consider:

Subcase I. If $\omega_i'(t)>0$, then from Lemma 4.1, we have

$$\lim_{t\to\infty}y_{1,2,3}(t)=\lim_{t\to\infty}\omega_{1,2,3}(t)=\infty.$$

Subcase II. If $\omega_{1,2}'(t)>0$ and $\omega_3'(t)<0$. By Lemma 4.1, it follows

$$\lim_{t\to\infty}y_{1,2}(t)=\lim_{t\to\infty}\omega_{1,2}(t)=\infty.$$

since $\omega_3(t)>0$ and $\omega_3'(t)<0$, then there exists a finite $\hbar_3$ such that

$$\lim_{t\longrightarrow\infty}\omega_3(t)=\hbar_3.$$

We shall prove that $\hbar_3=0$. Assume that $\hbar_3>0$. Then for any $\varepsilon>0$, we have $\hbar_3<\omega_3(t)<\hbar_3+\varepsilon$, eventually. Choose $0<\varepsilon<\frac{\hbar_3(1-p_3)}{p_3}$. It is easy to verify that

$$y_3(t)=\omega_3(t)-p_3y_3\big(\tau_2(t)\big)>\hbar_3-p_3(\hbar_3+\varepsilon)=k_3(\hbar_3+\varepsilon)>k_3\omega_3(t),$$

Where $k_3=\frac{\hbar_3-p_3(\hbar_3+\varepsilon)}{\hbar_3+\varepsilon}>0$. Using the above inequality, we obtain from (E)

Integrating the second inequality of (34) from t to ∞, we get

$$\zeta_2(t)\big(\omega_2''(t)\big)^{\alpha_1}\leq k_3^{\alpha_1}\int_t^\infty q_2(t)\omega_3^{\alpha_2}\big(\sigma_2(t)\big)\,ds$$

Using $\omega_3\big(\sigma_1(t)\big)\geq\hbar_3$, we see that

$$\omega_2''(t)\leq k_3\hbar_3\left[\frac{1}{\zeta_2(t)}\int_t^\infty q_2(s)ds\right]^{\frac{1}{\alpha_2}}, \tag{37}$$

Integrating (37) from t to ∞, we obtain

$$-\omega_2'(t) \le k_2 \hbar_2 \int_t^\infty \left[\frac{1}{\zeta_2(u)} \int_u^\infty q_2(s) ds\right]^{\frac{1}{\alpha_2}} du, \qquad (38)$$

Integrating (38) from t_1 to t, we obtain

$$-\omega_2(t) + \omega_2(t_1) \le k_3 \hbar_3 \int_{t_1}^t \int_v^\infty \left[\frac{1}{\zeta_2(u)} \int_u^\infty q_2(s) ds\right]^{\frac{1}{\alpha_2}} du\, dv$$

As $t \longrightarrow \infty$, $\lim_{t\longrightarrow\infty} \omega_2(t) = -\infty$, this is in contradiction with the positivity of $\omega_2(t)$. Therefore, $\hbar_3 = 0$. That leads to $\lim_{t\longrightarrow\infty} \omega_3(t) = 0.$ inequality $0 \le y_3(t) \le \omega_3(t)$ implies $\lim_{t\longrightarrow\infty} y_3(t) = 0.$
The proof of the cases when $Y(t) \in \boldsymbol{K}_4 - \boldsymbol{K}_8$ is similar to the proof of **case 2**.
We will give some examples of the result of the three-dimensional half-linear systems (TDHLS) of third-order of neutral type.

Example 5.1 Consider a TDHLS of third-order of neutral type:

$$\begin{cases} \left(y_1(t) + \frac{1}{2} y_1(t-2)\right)''' = -\left(\frac{1}{2e^2} + \frac{1}{4}\right) e^t y_2(t-1) \\ \left(y_2(t) + \frac{3}{4} y_2(t-1)\right)''' = -\left(\frac{16}{e^2} + 12\right) e^{-t} y_3(t-2), \\ \left(y_3(t) + \frac{1}{3} y_3(t-1)\right)''' = -\left(\frac{1}{e^3} + \frac{1}{4e^2}\right) y_1(t-3) \end{cases} \qquad (39)$$

$\lambda = -1, \alpha_1 = \alpha_2 = \alpha_1 = 1,$
$\zeta_1(t) = \zeta_2(t) = \zeta_3(t) = 1,$
$\tau_1(t) = t-2, \tau_2(t) = t-1, \tau_3(t) = t-1,$
$\sigma_1(t) = t-1, \sigma_2(t) = t-2, \sigma_3(t) = t-3,$
$p_1(t) = \frac{1}{2}, p_2(t) = \frac{3}{4}, p_3(t) = \frac{1}{3},$
$q_1(t) = \left(\frac{1}{2e^2} + \frac{1}{4}\right) e^t, q_2(t) = \left(\frac{16}{e^2} + 12\right) e^{-t}, q_3(t) = \left(\frac{1}{e^3} + \frac{1}{4e^2}\right).$
Since
$\omega_1(t) = y_1(t) + \frac{1}{2} y_1(t-2) > 0, \omega_1'(t) < 0, \omega_1''(t) > 0,$
$\omega_2(t) = y_2(t) + \frac{3}{4} y_2(t-1) > 0, \omega_2'(t) < 0, \omega_2''(t) > 0,$
$\omega_3(t) = y_3(t) + \frac{1}{3} y_3(t-1) > 0, \omega_1'(t) < 0, \omega_1''(t) > 0,$

$$\lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \left(\frac{1}{\zeta_i(\xi)}\right)^{\frac{1}{\alpha_i}} d\xi\, ds = \lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} d\xi\, ds = \lim_{t\to\infty} \int_T^t (\delta(s) - s)\, ds \ge c_1 \lim_{t\to\infty} (t-T) = \infty, i = 1,2,3.$$

And

$$\lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \frac{1}{\zeta_1(\xi)} \int_\xi^{\delta(\xi)} q_1(\theta)\left(1 - p_2(\sigma_1(\theta))\right) d\theta\, d\xi ds = \infty$$

$$\lim_{t\to\infty} \int_T^t \int_s^{\delta(s)} \frac{1}{\zeta_2(\xi)} \int_\xi^{\delta(\xi)} q_2(\theta)\left(1 - p_3(\sigma_2(\theta))\right) d\theta\, d\xi ds = \infty$$

$$\lim_{t\to\infty}\int_T^t\int_s^{\delta(s)}\frac{1}{\zeta_3(\xi)}\int_\xi^{\delta(\xi)} q_3(\theta)\left(1-p_1\big(\sigma_3(\theta)\big)\right)d\theta\, d\xi ds=\infty$$

Then $(e^{-t}, 2e^{-2t}, e^{-t})\in L_1$, from Lemma 4.1 and Theorem 5.2, we get,
$\lim_{t\to\infty} y_1(t)=\lim_{t\to\infty} y_2(t)=\lim_{t\to\infty} y_3(t)=0.$

Example 5.2 Consider a TDHLS of third-order of neutral type:

$$\begin{cases}\left(e^{-t}\left(\left[y_1(t)+\frac{1}{2}y_1(t-1)\right]''\right)^3\right)'=2e^6\left(1+\frac{1}{2e}\right)^3 e^{-4t}y_2^3(t-1)\\ \left(e^{-\frac{t}{2}}\left[y_2(t)+\frac{1}{3}y_2(t-2)\right]''\right)'=2e^6(6e^3+2e^{-1})^3e^{-\frac{t}{2}}y_3(t-3)\\ \left(\left(\left[y_3(t)+\frac{3}{5}y_3(t-3)\right]''\right)^{\frac{3}{5}}\right)'=\frac{3}{5e^3}\left(1+\frac{3}{5}e^{-3}\right)^{\frac{3}{5}}y_1^{\frac{3}{5}}(t-5)\end{cases}\qquad(40)$$

$\lambda=1, \alpha_1=3, \alpha_2=1, \alpha_1=\frac{3}{5},$
$\zeta_1(t)=e^{-t}, \zeta_2(t)=e^{-t}, \zeta_3(t)=1,$
$\tau_1(t)=t-1, \tau_2(t)=t-2, \tau_3(t)=t-3,$
$\sigma_1(t)=t-1, \sigma_2(t)=t-3, \sigma_3(t)=t-5,$
$p_1(t)=\frac{1}{2}, p_2(t)=\frac{1}{3}, p_3(t)=\frac{3}{5},$
$q_1(t)=2e^6\left(1+\frac{1}{2e}\right)^3 e^{-4t}, q_2(t)=2e^6(6e^3+2e^{-1})^3e^{-\frac{t}{2}},$
$q_3(t)=\frac{3}{5e^3}\left(1+\frac{3}{5}e^{-3}\right)^{\frac{3}{5}}.$
$\omega_1(t)=y_1(t)+\frac{1}{2}y_1(t-1)>0, \omega_1'(t)>0, \omega_1''(t)>0.$
$\omega_2(t)=y_2(t)+\frac{1}{3}y_2(t-2)>0, \omega_2'(t)>0, \omega_2''(t)>0.$
$\omega_3(t)=y_3(t)+\frac{3}{5}y_3(t-3)>0, \omega_1'(t)>0, \omega_1''(t)>0.$
Then $(e^{t}, 2e^{2t}, e^{t})\in K_1$, from lemma 3.1 and corollary 5.1, we get,

$$\lim_{t\to\infty} y_1(t)=\lim_{t\to\infty} y_2(t)=\lim_{t\to\infty} y_3(t)=\infty.$$

Example 5.3 Consider a TDHLS of third-order of neutral type:

$$\begin{cases}\left(y_1(t)+\frac{1}{2}y_1(t-\pi)\right)'''=\frac{1}{2}y_2\left(t-\frac{\pi}{2}\right)\\ \left(y_2(t)+\frac{1}{2}y_2(t-2\pi)\right)'''=\frac{1}{2}y_3(t-\pi)\\ \left(y_3(t)+\frac{3}{4}y_3(t-\pi)\right)'''=\frac{1}{4}y_1(t-2\pi)\end{cases}\quad,\qquad(41)$$

$\lambda=1, \alpha_1=\alpha_2=\alpha_3=1,$
$\zeta_1(t)=\zeta_2(t)=\zeta_3(t)=1,$
$\tau_1(t)=t-\pi, \quad \tau_2(t)=t-2\pi, \quad \tau_3(t)=t-\pi,$
$\sigma_1(t)=t-\frac{\pi}{2}, \quad \sigma_2(t)=t-\pi, \quad \sigma_3(t)=t-2\pi,$

$\wp_1(t) = \frac{1}{2}, \wp_2(t) = \frac{1}{2}, \wp_3(t) = \frac{3}{4},$
$q_1(t) = \frac{1}{2}, q_2(t) = \frac{1}{2}, q_3(t) = \frac{1}{4}.$
Since
$\omega_1(t), \omega_2(t)$ and $\omega_3(t)$ are bounded. By Theorem 5.1, every bounded solution to system (41) oscillates. So the solution $(\sin t, \sin t, \cos t)$ oscillates.

6. Conclusions

Knowing and calculating all possible cases of positive solutions to the third-order three-dimensional half-linear system with neutral type.
Oscillation: This trend revolves around studying and obtaining sufficient conditions for obtaining the oscillation of positive solutions for the three-dimensional half-linear system with the neutral type of the third order.
Asymptotic behavior : The required sufficient conditions are drawn in this direction to obtain the convergence to zero or divergence of all non-oscillatory solutions of a half linear system of neutral differential equations of third order when $t \to \infty$. All obtained results are included with illustrative examples.

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