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Mean Square Exponential Stability of Semi-Linear Stochastic Perturbed Differential Equation Via Lyapunov Function Approach

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Abstract

In this work, a class of stochastically perturbed differential systems with standard Brownian motion of ordinary unperturbed differential system is considered and studied. The necessary conditions for the existence of a unique solution of the stochastic perturbed semi-linear system of differential equations are suggested and supported by concluding remarks. Some theoretical results concerning the mean square exponential stability of the nominal unperturbed deterministic differential system and its equivalent stochastically perturbed system with the deterministic and stochastic process as a random noise have been stated and proved. The proofs of the obtained results are based on using the stochastic quadratic Lyapunov function method. Form an application point of view of the proposed approach, an illustrative example is considered and implemented.

Keywords: Brownian motion, Mean square Stability, Stochastic differential equation, Lyapunov Function.

استقرارية المعادلات التفاضلية شبه الخطية التصادفية المضطربة باستخدام دوال ليابانوف

فيان قدس يوسف*، راضي علي زبون
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الخلاصة

في هذا البحث ،تم الاخذ بعين الاعتباردراسة صنف نظام تفاضلي تصادفي مضطرب مع حركة براونية لنظام تفاضلي غير مضطرب.لقد اقترحت الشروط الاساسية لضمان وجود الحل ووحدانيته لصنف من نظام شبه خطي تفاضلي تصادفي .لقد صيغت وبرهنت بعض النظريات والنتائج المساعدة لاستقرارية المتوسط الاسي المربع لمعادلات شبه خطية تفاضلية تصادفية لكل من النظام التفاضلي الاعتيادي غير المضطرب والنظام التصادفي المضطرب مع ضوضاء عشوائية وغير عشوائية.اتجاهات البراهين استندت الى نظرية واسلوبية دالة اليابانوف التربيعية التصادفية.مثال توضيحي مع الاستنتاجات كذالك قدمت

1. Introduction

The stochastic differential equation is a rich field of applied sciences and mathematics which has a wide class of real life applications, [1, 2, 3, 4]. The existence of a unique solution

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of stochastic differential equations has been presented in, [5, 6, 7]. Some stochastically perturbed differential equations appear from stochastic differential equation perturbed by some structural uncertainty or differential equations perturbed with deterministic random noise [2, 8]. The existence of a unique solution to some classes of stochastically perturbed differential equations can be found in [2].

The stability analysis for ordinary differential equations has been discussed in [9, 10, 11, 12]. The generalization of stability concepts of the ordinary systems into stochastic differential equations including stability in probability, almost sure stability, and moment stability have been interesting in many research works, [2, 13, 14, 15,16].

The Lyapunov function approach of the first and second types plays a consequential turn in studying the stability of both deterministic as well as stochastic differential equations. The stability of stochastic differential equations by using the Lyapunov function approach may be found in [8,17]. In [10], the quadratic function approach for constructing sufficient condition has been applied to ensure the mean square stability of linear stochastic differential equation with application in a damped pendulum problem. While in [8] the mean square stability of the null solution for a non-linear stochastically perturbed differential equation perturbed with standard Brownian motion is established using some direct Lyapunov function.

In this paper, the generalization approach of [17] and [8] are adapted to ensure the mean square exponential stability of a large class of semi-linear stochastically perturbed differential equations with Brownian motion using the Lyapunov function approach. The sufficient condition for the mean square exponential stability of the null solution is developed. A step by step illustration is proposed with the necessary mathematical requirements.

2. Stochastic Differential Equation (SDE)

Let (Ω, F_t, P) be a complete probability space with $\{F_t\}$, $t \geq 0$ is a filtration which is a family of all increasing sub-algebras of F with conditions (right continuous and increasing while F_0 contains all (P-null sets)) i.e, it satisfies the usual condition. Consider the classical stochastic differential equation, [2]

$$dx(t) = f\big(x(t)\big)dt + g\big(x(t)\big)dB(t) \tag{1}$$

where $B(t) = (B_1(t), B_2(t), \dots, B_d(t))$ is a d-dimensional Brownian motion, such that $E\big(dB(t)\big) = E\big(B_2(t) - B_1(t)\big) = E\big(B_2(t)\big) - E\big(B_1(t)\big) = 0$, where $B_2(t) > B_1(t)$, and

$x \in C_{F_t}([0,T], R^n), T > 0$, $f : R^n \to R^n, f\big(x(t)\big) \in L^1([0,T]; R^n)$, $g: R^n \to R^{n\times d}$
$g(x(t)) \in L^2([0,T]; R^{n\times d})$.
equation (1) can be rewritten as the following integral equation [2]

$$x(t) = x_0 + \int_0^t f\big(x(t)\big)dt + \int_0^t g\big(x(t)\big)dB(t), \ t \in [0,T], T > 0 \tag{2}$$

where the first integral is the deterministic integral and the second integral is the stochastic integral of Itô type with
$E(\int_0^t g(x(t))\, dB(t)) = 0 \ and \ E(\int_0^t g(x(t))\, dB(t))^2 = E(\int_0^t g^2(x(t))dt)$, where the Itô integration cannot be defined and calculated in the ordinary way **[2]**,

with $x_o \in C_{F_{t_0}}([0,T], R^n), where \ C_{F_{t_0}}([0,T], R^n)$ is the family of F_{t_0}- measurable valued random variable with $E\int_0^T \|x_0\|^2 dt < \infty$. Let $f \ and \ g$ be assumed such that the local Lipchitz condition [18].

For every x and y belong to R^n with $\|x\| \vee \|y\| \leq n$ for every n ≥1, where $\|x\| \vee \|y\|$ represent the maximum value between $\|x\| and \|y\|$,there exists a constant (positive) V_n such that for every t belongs to the interval [0,T],

$$\left\|f\big(x(t)\big) - f(y(t))\right\|^2 \vee \left\|g\big(x(t)\big) - g(y(t))\right\|^2 \leq V_n\|x-y\|^2 \quad (3)$$

and the monotone **condition, [19].**

For every $x \in R^n$ and x belongs to the interval [0,T] , we have

$$x^T f\big(x(t)\big) + \frac{1}{2}\|g(x(t))\|^2 \leq \alpha\,(1+\|x\|^2) \quad , \ \alpha > 0 \quad (4)$$

From (3) and (4), the system of the stochastic differential equations (1) or the integral equation (2) has a unique global solution, **[2]**.

Remarks (2.1)

1- The Lyapunov direct method for deterministic stability without solving the differential equation is used to find a Lyapunov function $V(x,t)$.

2- The function $V(x,t)$ is called the Lyapunov function for the deterministic system if $V(x,t)$ is a positive definite and $\dot{V}(x,t) \leq 0$ i.e (the zero solution is stable), and if $V(x,t)$ is positive definite decrescent function and $\dot{V}(x,t)$ is negative definite i.e (the zero solution is asymptotically stable). where $V(x,t)$ is said to be a positive definite if $V(0,t) = 0$ and $V(x,t) \geq \rho\,(\|x\|)$,where $\rho \in \mu$, μ is the family of every nondecreasing continues function and $\|x\|$ denotes the Euclidean vector norm**,** $V(x,t)$ is decrescent if $V(x,t) \leq \rho(\|x\|)$,and $V(x,t)$ is negative definite if $-V(x,t)$ is positive definite, [2, 17].

3. The Mean Square Exponential Stability in the Sense of the Lyapunov Function

In the topic of stochastic differential equations, the stability concept of stochastic differential equations can be generalized by using a Lyapunov function approach as follows: Consider the stochastic differential equations (1) and $V(x,t)$ is the Lyapunov function, by using the chain rule of stochastic function (Itô formulation) with $(dB(t))^2 = dt$, which is satisfied when $B(t)$ is a Brownian motion with $E\big(B(t)\big) = 0$ and $dB(t)$ is linearly increment, (it is also true that $\frac{dB(t)}{dt} = \mathcal{E}(t)$, where $\mathcal{E}(t)$ represents the white noise, for almost every ω the sample path t→ $B(t,\omega)$ is differentiable for no time $t > 0$, thus $\frac{dB(t)}{dt} = \mathcal{E}(t)$ does not really exit. Setting the differential operator

L = $\frac{d}{dt} + \sum_{i=1}^{d} f_i\,\big(x(t)\big)\frac{d}{dx_i} + \frac{1}{2}\sum_{ij=1}^{d}[g(x(t)g^T\,(x(t)]_{ij}\,\frac{d^2}{dx_i dx_j}$, along the solution

then

$$LV(x,t) = V_t(x,t) + V_x(x,t)f(x(t)) + \frac{1}{2}\,trace\big[g^T\big(x(t)\big)V_{xx}(x,t)g\big(x(t)\big)\big]$$

if $(x,t) \leq 0$, $V > 0$, then we get the stochastic stability assertions for equation (1). The zero solution of equation (1) is stochastically stable (stochastically asymptotically stable) if $LV(x,t) \leq 0$ where $V(x,t) \in s_m \times [t_0,\infty)$ and $s_m = \{x \in R^n : \|x\| < m\}$ ($LV(x,t)$ is a negative-definite with $V(x,t)$ is a decrescent function), it is also called stochastically asymptotically stable in the large if $LV(x,t)$ is a negative-definite where $V(x,t)$ decrescent radially unbounded function, [2].

There are three types of stochastic stability, namely stability in probability, moment stability, and Almost sure stability,[20].

In this work, we use the candidate a quadratic Lyapunov function to guarantee the moment exponential stability of some perturbed stochastic process.

The zero solution of equation (1) is called p^{th}moment exponentially stable when there is a positive pair (J, C) where J represents the exponential rate and C represents the constant value, which is obtained from equation (1), such that

$$E\|x(t)\|^p \leq C\|x_0\|^p e^{-J(t-t_0)} \quad \text{For every } x_0 \in R^n, t_0 = 0 \tag{5}$$

In this work, for stability purposes, it is enough to consider the initial condition $x_0 \in R^n$ (constant) instead of the random variable $x_0 \in C_{F_{t_0}}([0,T], R^n)$, [2].
Equation (5) means that the p^{th}moment of the zero solution tends to zero exponentially fast. Letting p =2 in equation (5) then the zero solution is called exponentially stable in the mean square,**[2].**

Problem Formulation (I)

Consider a stochastic differential equation of the form

$$dx(t) = [Ax + f(x(t))]\, dt + g(x(t))dB(t)\,,\ t \in [0,T],\ \forall\, T > 0 \tag{6}$$

With

1. (Ω, F, P) be a complete probability apace with the filtration $\{F_t\}, t \geq o$.
2. $x_0 \in C_{F_{t_0}}([0,T], R^n)$
3. $f: R^n \to R^n$.
4. $g: R^n \to R^{nxd}$.
5. $A \in R^{nxn}$ is a constant matrix.
6. f , g satisfies the local Lipchitz condition (3).
7. $\left(Ax + f(x(t))\right)$ is a drift part and $g(x(t))$ is a stochastic part that satisfies the stochastic property where B is a Brownian motion such that $B(0) = 0$ and$B(t) - B(s)$ is $\aleph(0, t-s)$ which means that B has a normal distribution with mean equal to zero and variance equal to $(t-s)$ for all $t \geq s \geq 0$ and $dB(t)$ is independent increment Brownian motion ,**[13]**.

The following lemma is proposed to ensure the existence of a monotone condition which is standing for solvability requirement of problem formulation (I) as follows:

Lemma (3.1)

Consider the problem formulation (I) and let

$$dx(t) = [Ax + f(x(t))]dt + g(x(t))\, dB(t)\,.$$

If

1- $\langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g(x(t)), g(x(t))\rangle_Q \leq K[1 + \|x\|^2]$, $f(0) = 0$, $g(0) = 0$

2- $\|Ax\| \leq M\|x\|$, and $\|f(x(t))\| \leq M_1\|x\|$ and $\|g(x(t))\| \leq M_2\|x\|$

where $A \in R^{n\times n}$ and $x_0 \in C_{F_{t_0}}([0,T], R^n)$. The constants M ,M_1, and M_2 are upper positive bounds for A, f and g , respectively with $V(x) = x^T Qx, Q = Q^T > 0$, then
$\langle x, Ax\rangle_Q + \langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g(x(t)), g(x(t))\rangle_Q \leq \alpha\, [1 + \|x\|^2]$ with
$\alpha = M + K$ and $K = (M_1 + \frac{1}{2}M_2^2)\sqrt{\lambda_{max}(Q^T Q)}$

Proof

Set a quadratic Itô function $V(x) = x^T Qx, Q = Q^T > 0,$ $where$ $\|Q\| = \sqrt{\lambda_{max}(Q^T Q)}$,where λ_{max} is the maximum eigenvalue for matrix $(Q^T Q)$, with Itô formulation
$dV(x) = LV(x)dt + V_x(x)g(x(t))dB(t)$ which is equivalent to
$dV(x) = V(x + dx) - V(x)$, where x along the solution of (1)

$$dV(x) = x^TQx + x^TQAxdt + x^TQf(x(t))dt + x^TQg(x(t))dB(t) + (Ax)^TQx\,dt + (Ax)^TQf(x(t))(dt)^2 + (Ax)^TQg(x(t))dt\,dB(t) + f(x(t))^TQx\,dt + f(x(t))^TQAx(dt)^2 + f(x(t))^TQf(x(t))(dt)^2 + f(x(t))^TQg(x(t))dtdB(t) + g(x(t))^TQxdB(t) + g(x(t))^TQAx\,dtdB(t) + g(x(t))^TQf(x(t))dt\,dB(t) + g(x(t))^TQg(x(t))(dB(t))^2 - x^TQx \quad (7)$$

Where $dB(t)$ has a normal distribution with mean equal to zero with respect the filtration F_t.
So by taking the expectation to (7) with $E(dB(t)) = 0$ and using condition (2) with multiplication Itô formulation $(dB(t))^2 = dt$, $(dt)^2 = dt \times dB(t) = dB(t) \times dt = 0$ we have that

$$E(dV(x)) = 2\langle x, Ax\rangle_Q dt + 2\langle x, f(x(t))\rangle_Q + \langle g(x(t)), g(x(t))\rangle_Q \quad (8)$$

$E(dV(x)) \le 2\langle x, Ax\rangle_Q dt + [(2M_1\|x\|^2 + M_2^2\|x\|^2)\sqrt{\lambda_{max}(Q^TQ)}]dt$

$E(dV(x)) \le 2\langle x, Ax\rangle_Q dt + 2[(M_1 + \frac{1}{2}M_2^2)\sqrt{\lambda_{max}(Q^TQ)}]\|x\|^2 dt$

$E(dV(x)) \le 2\langle x, Ax\rangle_Q dt + 2[K\|x\|^2]\,dt$,where $K = (M_1 + \frac{1}{2}M_2^2)\sqrt{\lambda_{max}(Q^TQ)}$

$E(dV(x)) \le 2\langle x, Ax\rangle_Q dt + 2[K[1 + \|x\|^2]\,dt$ (9)

from (8) and (9) we obtained

$\langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g(x(t)), g(x(t))\rangle_Q \le K[1 + \|x\|^2]$

From condition (1) and (2) of lemma (3.1) yields that

$\langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g(x(t)), g(x(t))\rangle_Q \le K[1 + \|x\|^2]$ and $\langle x, Ax\rangle_Q \le M\|x\|^2$

$E[dV(x)] \le 2\,(\mathrm{M}\,[1+\|x\|^2] + 2K[1 + \|x\|^2])\,dt$

$E[dV(x)] \le 2\,(\alpha\,[1+\|x\|^2])\,dt$, where $\alpha = M + K$, So

$\langle x, Ax\rangle_Q + \langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g(x(t)), g(x(t))\rangle_Q \le \alpha\,[1+\|x\|^2]$

Then the monotone condition is satisfied.

Theorem (3.2)

Consider the problem formulation (I) with

$dx(t) = [Ax + f(x(t))]dt + g(x(t))dB(t)$, and $f(0) = 0$, $g(0) = 0$

That satisfying the following conditions

1- $x_0 \in C_{F_{t_0}}([0, T], R^n)$

2- $\lambda_i + \lambda_j \ne 0, \forall i \ne j, \lambda_i \in \delta(A) = \{\lambda \in \mathbb{C} : |\lambda I - A| = 0\}$

3- $V = x^TQx$, where Q = Q^T is the unique positive solution of

$A^TQ + QA = -p\ , p > 0$ with $\lambda_{min}(p) > (2M_1 + M_2^2)\sqrt{\lambda_{max}(Q^TQ)}$, where $\lambda_{min}(p)\ and\ \lambda_{min}(Q)$ are the smallest eigenvalues of $p\ and\ Q$, respectively and $\lambda_{max}(p)$, $\lambda_{max}(Q)$ are the largest eigenvalues of $p\ and\ Q$, respectively and Set the positive constant γ such that

$\gamma = \left(\frac{\lambda_{min}(p) + (-2M_1 - M_2^2)\sqrt{\lambda_{max}(Q^TQ)}}{\lambda_{min}(Q)}\right)$ where M_1 and M_2 are positive upper bounds of $f\ and\ g$ where $\|f(x(t))\|_1 \le M_1\|x\|$ and $\|g(x(t))\|_2 \le M_2\|x\|$

and $\|Q\| = \sqrt{\lambda_{max}(Q^TQ)}$

Then the zero solution of system (6) is exponentially stable in the mean square.

Proof

By Lemma (3.1) and since the system (6) satisfies the local Lipchitz condition (1) then it is solvable and has a unique global solution**.**

Set nominate Lyapunov function
$V(x) = x^T Qx \ \ with \ \ Q = Q^T > 0$, with $V(0) = 0$, then
$dV(x) = V(x + dx) - V(x)$, along the solution of (6)

$$dV(x) = \left(x^T + ([Ax + f(x(t))])^T dt + g(x(t))^T dB(t)\right) Q\left(x + [Ax + \ f(x(t)) +] dt + g(x(t)) dB(t)\right) - x^T Qx$$

condition (3) of Theorem (3.2) leads to
$dV(X) \leq$
$-x^T px\, dt + \ 2\|x\|\|Q\|\|f(x(t))\|dt + 2\|x\|\|Q\|\|g(x(t))\|dB(t) +$
$\|g(x(t))\|\|Q\|\|g(x(t))\|dt$
Since $\lambda_{min}(p)\|x\|^2 \leq \ x^T px \leq \lambda_{max}(p)\|x\|^2$, we have that
$dV(x) \leq -\lambda_{min}(p)\|x\|^2 \ dt + 2M_1\sqrt{\lambda_{max}(Q^T Q)}\ \|x\|^2 dt + \ M_2^2\sqrt{\lambda_{max}(Q^T Q)}\ \|x\|^2 dt \ +$
$2\langle x, g(x(t))\rangle_Q \ dB(t)$

$$dV(x) \ \leq \left(\frac{-\lambda_{min}(p) + \sqrt{\lambda_{max}(Q^T Q)}\,(2M_1 + M_2^2)}{\lambda_{min}(Q)}\right) V(x)dt + 2\langle x, g(x(t))\rangle_Q \ dB(t) \qquad (10)$$

Set $\gamma = \left(\frac{\lambda_{min}(p) + (-2M_1 - M_2^2)\sqrt{\lambda_{max}(Q^T Q)}}{\lambda_{min}(Q)}\right)$, since we are in the setting space $x \in \ C_{F_t}([0, T], R^n)$ and by taking the expectation operator to (10) with Brownian motion property and since $E(dB(t)) = 0$ one gets the following differential inequality
$E(dV(x)) \leq -\gamma \ E(V(x))dt$
By integration with $E(dV(x)) = E(V(x + dx) - E(V(x))) = d(E(V(x))$, we obtain that
$EV(x(t)) \leq E(V(x(0))\ e^{-\gamma t}$, if x_0 is constant (for stability purpose) ,then
$EV(x(t)) \leq V(x(0))e^{-\gamma t} \ \leq \lambda_{max}(Q)\|x_0\|^2 e^{-\gamma t}$
So the zero solution of system (6) is exponentially stable in the mean square.

The following Lemma shows that the nominal part of the stochastic differential equation (6) (without perturbation) is exponentially stable (in the ordinary sense) about zero under suitable conditions. This result is needed later on studying the exponential stability of the stochastic differential systems.

Lemma (3.3)

Consider $\frac{dx(t)}{dt} = [Ax + f(x(t))]$ (11)

where the system is solvable **[9]** , $f: R^n \to R^n, f(0) = 0$,with
1- $\lambda_i + \lambda_j \ \neq 0 \ , \forall i \neq j, \ \lambda i \ \in (A) = \{\lambda \in \mathbb{C} \ : |\lambda I - A| = 0\ \}$
2- $V = x^T Qx$, where Q is the unique positive solution of $A^T Q + QA = \ -p \ \ , p > 0$, s.t $\lambda_{min}(p) > 2M_1\sqrt{\lambda_{max} Q^T Q}$ and set the positive constant
$\gamma \ = \ \frac{\lambda_{min}(p) - 2M_1\sqrt{\lambda_{max}(Q^T Q)}}{\lambda_{min}(Q)}$ where M_1 is positive upper bound of f such that $\|f(x(t))\| \leq M_1\|x\|$
Then the zero solution of system (11) is exponentially stable about zero.

Proof
Set $(x) = x^T Qx \ \ with \ \ Q = Q^T > 0$,$V(0) = 0$
$\dot{V}(x) = x^T[AQ + A^T Q]x + 2\langle x, f(x(t))\rangle_Q$, From condition (2), one gets
$\dot{V}(x) \leq -x^T px\, dt + \|f(x(t))\|\ \|Q\|\|x\|\ dt + \|x\|\|Q\|\|f(x(t))\|\ dt$
$\dot{V}(x) \leq \ -\lambda_{min}(p)\|x\|^2 dt + 2M_1\|x\|^2\sqrt{\lambda_{max}(Q^T Q)}\ dt$

$$\leq \left(\frac{-\lambda_{min}(p)+2M_1\sqrt{\lambda_{max}(Q^TQ)}}{\lambda_{min}(Q)}\right)x^TQx\ dt$$

$dV(x) \leq -\gamma\, V(x)\, dt\ , \quad \gamma = \frac{\lambda_{min}(p)-2M_1\sqrt{\lambda_{max}(Q^TQ)}}{\lambda_{min}(Q)}$

$\frac{dV(x)}{V(x)} \leq -\gamma\, dt$, by integration $V(x(t)) \leq V\big(x(0)\big)\, e^{-\gamma t} \leq \lambda_{max}(Q)\|x_0\|^2 e^{-\gamma t}$

Then the zero solution of system (11) is exponentially stable about zero.

Theorem (3.4)

Consider the perturbed stochastic differential equation

$$dx(t)= \big[Ax + f\big(x(t)\big) + F\big(x(t)\big)\big]dt + g\big(x(t)\big)dB(t) \tag{12}$$

which satisfies the following conditions

1- $\lambda_i + \lambda_j \neq 0 , \forall i \neq j,\ \lambda i \in (A) = \{\lambda \in \mathbb{C} : |\lambda I - A| = 0\}$

2- $V = x^TQx$, where Q is the unique positive solution of $A^TQ + QA = -p\ , p > 0$, s.t $\lambda_{min}(p) > 2K + 2M_1\sqrt{\lambda_{max}Q^TQ}$, set the positive constant $\gamma = \frac{\lambda_{min}(p)-(2K+2M_1\sqrt{\lambda_{max}Q^TQ})}{\lambda_{min}(Q)}$ where K is positive constant and M_1 is the positive upper bound of F such that $\|F(x(t))\| \leq M_1\|x\|$

3- $\langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g\big(x(t)\big), g(x(t))\rangle_Q \leq K[1 + \|x\|^2]$

Then the zero solution of system (12) is exponentially stable in the mean square.

Proof:

$dV(x) = V(x + dx) - V(x)$, where x along the solution

$$= \big(x^T + ([Ax + f\big(x(t)\big) + F\big(x(t)\big)])^T dt + g(x(t))^T\big)Q\big(x[Ax + f\big(x(t)\big) + F\big(x(t)\big)]dt + g\big(x(t)\big)dB(t)\big) - x^TQx$$

$dV(x) \leq$

$$x^T[A^TQ + QA]x\ dt + 2\langle x, f\big(x(t)\big)\rangle_Q dt + \langle g\big(x(t)\big), g\big(x(t)\big)\rangle_Q dt + 2\|x\|\|Q\|\big\|F\big(x(t)\big)\big\|dt + 2\langle x, g\big(x(t)\big)\rangle_Q dB(t)$$

by conditions (2) and (3) and some multiplication which is discussed earlier, one gets the following:

$$dV(x) \leq -\gamma(V(x))dt + 2Kdt + 2\langle x, g\big(x(t)\big)\rangle_Q dB(t), \gamma = \left(\frac{\lambda_{min}(p)-2K-2M_1\sqrt{\lambda_{max}(Q^TQ)}}{\lambda_{min}(Q)}\right)$$

By taking the expectation with $E\big(dB(t)\big) = 0$, this yields

$E\big(dV(x)\big) \leq -\gamma E(V(x))\ dt + 2Kdt$ which is equivalent to

$E(V\big(x(t)\big) \leq E\big(V\big(x(0)\big) + 2K\big)e^{-\gamma t} + \frac{2K}{\gamma}$

$\mathrm{E}(V\big(x(t)\big) \leq (V(x(0)) + 2K)e^{-\gamma t} + \frac{2K}{\gamma}$, where $\frac{2K}{\gamma}$ is the rate of convergence

So the zero solution of the stochastic perturbed differential equation is exponentially stable in the mean square.

Remark (3.5)

From Lemma (3.3) and Theorem (3.4), one can conclude that if the nominal deterministic system $\frac{dx(t)}{dt} = \big[Ax + f\big(x(t)\big)\big]$ is exponential stable about zero, and $F(0) = 0\ , \|F(x(t))\| \leq M_1\|x\|$ and $gdB(t)$ satisfy the Brownian motion probability with $\big(dB(t)\big) = 0$, then the system (12) remains exponential stable for small variation dt and random noise.

Example (3.6)

Consider the system

$$dx(t)= \big[Ax + f\big(x(t)\big) + F\big(x(t)\big)\big]dt + g\big(x(t)\big)dB(t) \tag{13}$$

Where f, F, g and $x(0)$ are given in Table.1

Table 1: Summary of drift f, perturbation F, stochastic part g and the initial condition

f	F	g	$x(0)$
$f_1 = x_1 \sin x_1$	$F_1 = \beta_1 x_1 \sin x_1$	$g_1 = x_1 \sin x_1$	$x_1(0) = 2$
$f_2 = x_2 \cos x_2$	$F_2 = \beta_2 x_2 \cos x_2$	$g_2 = x_2 \cos x_2$	$x_2(0) = 3$

With $A = \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}$
Such that , $A = R^{2\times 2}$,$\delta(A) = \{\lambda_{min}(A), \lambda_{max}(A)\} = \{-8, -4\} \Leftrightarrow \lambda_i + \lambda_j \neq 0 , i = j = 1,2$
$\|F(x(t))\| = \sqrt{(\beta_1 x_1 \sin x_1)^2 + (\beta_2 x_2 \sin x_2)^2}$, Since $cos^2\theta \leq 1$ *and* $sin^2\theta \leq 1$
$\|F(x(t))\| \leq (\beta_1^2 x_1^2 + \beta_2^2 x_2^2)^{\frac{1}{2}} \leq 2 \max(\beta_1, \beta_2)\|x\|$
Where $M_1 = 2 \max(\beta_1, \beta_2)$ *and* β_1, β_2 *is constant* , yields $\|F(x(t))\| \leq M_1\|x\|$.
For example $\beta_1 = 0.2$ and $\beta_2 = 0.1$,So $M_1 = 0.2$ then $\|F(x(t))\| \leq 0.2 \ \|x\|$
Now
$A = \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}$ and $p = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$,$p = p^T > 0$ is a symmetric positive definite matrix and
$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}$ with $\lambda_{min}(Q) + \lambda_{max}(Q) \neq 0$
Since $A^T Q + QA = -P$ and Let $p_{12} = p_{21} = 0$, we obtain that
$\begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} + \begin{bmatrix} -4 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} = -\begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix}$. Therefore,
$q_{11} = \frac{p_{11}}{8}$ and $q_{12} = q_{12} = 0$ and $p_{22} \Rightarrow p_{22} = \frac{p_{22}}{16}$
Where $p_{11} = p_{22} = 16$ so $= \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}$, It's clear $p_{11}p_{22} > p_{12}$ and $\lambda_{min}(p) = 16$, $\lambda_{max}(p) = 16$.
so $q_{11} = 2 , q_{22} = 1 , q_{12} = 0$ and $Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite symmetric matrix with $\lambda_{min}(Q) = 1 , \lambda_{max}(Q) = 2$ with $\|Q\| = \sqrt{\lambda_{max}(QQ^T)} = 2$.
from the monotone condition
$\langle x, f(x(t))\rangle_Q + \frac{1}{2}\langle g(x(t)), g(x(t))\rangle_Q \leq K[1 + \|x\|^2]$, we have
$2\,(x_1\ x_2)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 sinx_1 \\ x_2 cosx_2 \end{pmatrix} + (x_1 sinx_1 \quad x_2 cosx_2)\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 sinx_1 \\ x_2 cosx_2 \end{pmatrix}$
$\leq 2(2x_1^2 sinx_1 + x_2^2 cosx_2) + (2x_1^2 sin^2 x_1 + x_2^2 cos^2 x_2)$
since ≤ 1 *and* $cos\theta \leq 1$, $sin^2\theta \leq 1$ *and* $con^2\theta \leq 1$, we obtain that
$2(2x_1^2 + x_2^2) + (2x_1^2 + x_2^2) \leq 2(3\|x\|^2) + 3\|x\|^2 \leq 9\|x\|^2 \leq 9(1 + \|x\|^2)$
So $2K = 9$, with $V(x(o)) = \|x(0)\|_Q^2 = 17$
then $\gamma = \frac{\lambda_{min}(p) - 2K - 2M_1\sqrt{\lambda_{max}(Q^T Q)}}{\lambda_{min}(Q)} = \frac{16 - 9 - 2\times 0.2 \times 2}{1} = 6.2 > 0$
$E(V(x(t)) \leq 26\, e^{-6.2\,t} + 1.5$ so the zero solution of the system (13) is exponentially stable in the mean square.

4. Conclusions

A class of stochastic perturbed differential systems with Brownian motion uncertainty has been considered. The necessary mathematical result for the existence, uniqueness and stability in some sense have been discussed. The mean square exponential stability of stochastically perturbed (unperturbed) differential equation is developed under suitable conditions on the linear nominal parts (deterministic part) of a system with illustrated example.

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