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Applications of q-Difference Equation and q-Operator $_r\Phi_s(\theta)$ in q-Polynomials

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Abstract

We provide a q-operator form solution to a generalized q-difference equation involving $(r+s+2)$-variables. We introduce a q-polynomials $\Psi_n^{(A,B)}(x,y,c|q)$. The generating function, two Rogers formulas, and two types of Srivastava-Agarwal generating functions for the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ are established using the q-difference equation technique.

Keywords: generalized q-difference equation, generalized q-polynomials, generating function, Rogers formula, Srivastava-Agarwal type generating function.

تطبيقات معادلة الفروقات-q العامة والمؤثر-q العام $_r\Phi_s(\theta)$ في متعددات الحدود-q

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الخلاصة

نقدم المؤثر-q كحل لمعادلة الفروقات-q العامة التي تتضمن $(r+s+2)$ من المتغيرات. نعرف متعددات الحدود-q $\Psi_n^{(A,B)}(x,y,c|q)$. برهنا الدالة المولدة، صيغتين لروجرز ، ونوعين من الدالة المولدة ل Srivastava–Agarwal لمتعددات الحدود $\Psi_n^{(A,B)}(x,y,c|q)$ باستخدام اسلوب معادلة الفروقات-q.

1. Introduction

The q-series notations and definitions used in this paper are the same as those in [1]. Since $0<|q|<1$ is assume.

Let $a\in\mathbb{C}$. The q-shifted factorial is defined by [1]:

$$(a;q)_0=1,\qquad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k),\qquad (a,q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)$$

The multiple q-shifted factorial is defined as:

$$(a_1,a_2,\cdots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n$$

The basic hypergeometric series $_r\phi_s$ is given by [1]:

$$_r\phi_s\left(\begin{matrix}a_0,a_1,\cdots,a_{r-1}\\ b_1,b_2,\cdots,b_s\end{matrix};q,x\right)=\sum_{n=0}^{\infty}\frac{(a_0;q)_n(a_1;q)_n\cdots(a_{r-1};q)_n}{(q;q)_n(b_1;q)_n(b_2;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}x^n,$$

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where $r, s \in \mathbb{N}$; $a_0, \dots, a_{r-1} \in \mathbb{C}$; $b_1, \dots, b_s \in \mathbb{C}\backslash\{q^{-k}, k \in \mathbb{N}\}$ are assumed to be none of the denominator factors is evaluated to zero. This series converges absolutely for all x if $r \leq s$ and for $|x| < 1$ if $r = s + 1$. Note that

$$_{s+1}\phi_s\left({a_0, a_2, \cdots, a_s \atop b_1, b_2, \cdots, b_s}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_0, a_2, \dots, a_s; q)_n}{(q, b_1, b_2, \dots, b_s; q)_n} \; x^n, \quad |x| < 1.$$

The q-binomial coefficients are given by [1]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad 0 \leq k \leq n. \tag{1.1}$$

In this paper, the following identities will be used [1]:

$$(a;q)_n = (q^{1-n}/a;q)_n (-a)^n q^{\binom{n}{2}}. \tag{1.2}$$

$$(a;q)_{n-k} = \frac{(a;q)_n}{(q^{1-n}/a;q)_k} (-1)^k \left(\frac{q}{a}\right)^k q^{\binom{k}{2}-nk}. \tag{1.3}$$

$$(aq^{-n};q)_k = \frac{(a;q)_k (q/a;q)_n q^{-nk}}{(q^{1-k}/a;q)_n}. \tag{1.4}$$

$$(aq^n;q)_k = \frac{(a;q)_k (aq^k;q)_n}{(a;q)_n}. \tag{1.5}$$

$$(aq^k;q)_{n-k} = \frac{(a;q)_n}{(a;q)_k}. \tag{1.6}$$

Cauchy identity is described by:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1.7}$$

The following identity was discovered by Euler as a special case of the Cauchy identity (1.7) [1]:

$$\sum_{n=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}, \quad |x| < 1.$$

The q-Chu-Vandermonde sum [1, Appendix II, equation (II.7)] is:

$$_2\phi_1\left({q^{-n}, a \atop c}; q, cq^n/a\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{1.8}$$

Jackson's transformation of $_2\phi_1$ [1, Appendix III, equation (III.4)] is:

$$_2\phi_1\left({a, b \atop c}; q, z\right) = \frac{(az;q)_\infty}{(z;q)_\infty} \; _2\phi_2\left({a, c/b \atop c, az}; q, bz\right). \tag{1.9}$$

The transformations of $_3\phi_2$ series [1, Appendix III, equation (III.13)] is:

$$_3\phi_2\left({q^{-n}, b, c \atop d, e}; q, \frac{deq^n}{bc}\right) = \frac{(e/c;q)_n}{(e;q)_n} \; _3\phi_2\left({q^{-n}, c, d/b \atop d, cq^{1-n}/e}; q, q\right). \tag{1.10}$$

The Cauchy polynomials are provided by [2],[3],[4]:

$$P_n(x;y) = (x-y)(x-qy)\dots(x-yq^{n-1}) = (y/x;q)_n x^n,$$

has the generating function

$$\sum_{k=0}^{\infty} P_k(x,y) \frac{t^k}{(q,q)_k} = \frac{(yt;q)_\infty}{(xt;q)_\infty}, \quad |xt| < 1. \tag{1.11}$$

In 2013, Saad and Sukhi [5] presented the Rogers formula for $P_n(x,y)$ as:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}$$
$$=\frac{(yt;q)_\infty}{(xv,xt;q)_\infty}\ {}_1\phi_1\left(\begin{matrix}xt\\ yt\end{matrix};q,ys\right),\quad max\{|xv|,|xt|\}<1. \tag{1.12}$$

The operator θ is defined by [6]-[9]:

$$\theta\{f(a)\}=\frac{f(aq^{-1})-f(a)}{aq^{-1}}.$$

The Leibniz formula for θ is [8]

$$\theta^n\{f(a)g(a)\}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\theta^k\{f(a)\}\theta^{n-k}\{g(aq^{-k})\}. \tag{1.13}$$

We will use θ_x for the operator θ acting on the variable x.

Let k be a nonnegative integer. The operator θ has the following properties [10]:

$$\theta_a^k\left\{\frac{(at;q)_\infty}{(av;q)_\infty}\right\}=v^k q^{-\binom{k}{2}}(t/v;q)_k\frac{(at;q)_\infty}{(av/q^k;q)_\infty}. \tag{1.14}$$

In 2015, Reshem [11] introduced the following identity:

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(-1)^n q^{\binom{n}{2}}x^n=(x;q)_\infty\sum_{k=0}^{\infty}\frac{q^{k^2-k}(ax)^k}{(q,x;q)_k}. \tag{1.15}$$

In 2015, Fang [12] defined the generalized q-operator $F(a_0,\dots,a_s,b_1,\dots,b_s,-c\ \theta_b)$:

$$F(a_0,\dots,a_s,b_1,\dots,b_s,-c\ \theta_b)=\ {}_{s+1}\phi_s\left(\begin{matrix}a_0,a_1,\dots,a_s\\ b_1,b_2,\dots,b_s\end{matrix};q,-c\ \theta_b\right). \tag{1.16}$$

Fang [12] presented the following generalized q-difference equation:

Theorem 1.1. [12]. *Let $f(a_0,\dots,a_s,b_1,\dots,b_s,b,c)$ be a $2s+3$-variable analytic function in a neighborhood of $(a_0,\dots,a_s,b_1,\dots,b_s,b,c)=(0,0,\dots,0)\in\mathbb{C}^{2s+3}, s\in\mathbb{N}$, satisfying the q-difference equation*

$$bq^{-1}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}f\left(a_0,\dots,a_s,b_1,\dots,b_s,b,cq^j\right)$$
$$+c\sum_{j=0}^{s+1}(-1)^j A_j[f\left(a_0,\dots,a_s,b_1,\dots,b_s,bq^{-1},cq^j\right)$$
$$-f(a_0,\dots,a_s,b_1,\dots,b_s,b,cq^j)]=0, \tag{1.17}$$

where

$$b=q,\quad B_0=A_0=1,\quad B_1=\sum_{j=0}^{s}b_i,\quad B_2=\sum_{0\le i<j\le s}b_ib_j$$
$$B_3=\sum_{0\le i<j<k\le s}b_ib_jb_k,\dots,\quad B_{s+1}=b_0b_1\dots b_s,\quad A_1=\sum_{i=0}^{s}a_i$$
$$A_2=\sum_{0\le i<j\le s}a_ia_j,\quad A_3=\sum_{0\le i<j<k\le s}a_ia_ja_k,\dots,\quad A_{s+1}=a_0a_1\dots a_s.$$

Then

$$f(a_0,\dots,a_s,b_1,\dots,b_s,b,c)=F(a_0,\dots,a_s,b_1,\dots,b_s,-c\ \theta_b)\{f(a_0,\dots,a_s,b_1,\dots.,b_s,b,0)\}.$$

Using the q-difference equation (1.17), Fang [12] proved the following operator identities:

Theorem 1.2. [12]. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, u, v, a_i, b_i \in \mathbb{C}, i = 1,2, \dots, s$, *then*

$$F(a_0, \dots, a_s, b_1, \dots, b_s, -c\,\theta_b)\left\{\frac{(bu, bv; q)_\infty}{(bw; q)_\infty}\right\}$$

$$= \frac{(bu, bv; q)_\infty}{(bw; q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, \dots, a_s; q)_n}{(q, b_1 \dots, b_s; q)_n} \left(\frac{qc}{b}\right)^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \frac{(bu/q^k, bv/q^k; q)_k}{(bw/q^k; q)_k}. \quad (1.18)$$

Corollary 1.2.1. [12]. *If* $max\{|bu|, |cu|\} < 1, u, a_i, b_i \in \mathbb{C}, i = 1,2, \dots, s$, *then*

$$F(a_0, \dots, a_s, b_1, \dots, b_s, -c\,\theta_b)\{(bu; q)_\infty\} = (bu; q)_\infty \;{}_{s+1}\phi_s\left(\begin{matrix} a_0, a_1, \dots, a_s \\ b_1, b_2, \dots, b_s \end{matrix}; q, cu\right). \quad (1.19)$$

Also, Fang [12] defined the q-polynomials $\widetilde{H}_n$:

$$\widetilde{H}_n(a_0, \dots, a_s; b_1, \dots, b_s; b, c) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_0, a_1, \dots, a_s; q)_k}{(b_1, \dots, b_s; q)_k} (-1)^k q^{\binom{k+1}{2} - nk} c^k b^{n-k}, \quad (1.20)$$

which have the generating function

$$\sum_{n=0}^{\infty} \widetilde{H}_n \frac{(-1)^n q^{\binom{n}{2}} u^n}{(q; q)_n} = (bu; q)_\infty \;{}_{s+1}\phi_s\left(\begin{matrix} a_0, a_1, \dots, a_s \\ b_1, b_2, \dots, b_s \end{matrix}; q, cu\right). \quad (1.21)$$

In 2020, Cao et al. [13] constructed the new generalized Al-Salam-Carlitz polynomials as:

$$\phi_n^{\binom{a,b,c}{d,e}}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a, b, c; q)_k}{(d, e; q)_k} x^{n-k} y^k.$$

$$\psi_n^{\binom{a,b,c}{d,e}}(x, y|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(-1)^k q^{k(n-k)} (a, b, c; q)_k}{(d, e; q)_k} x^{n-k} y^k. \quad (1.22)$$

They [13] gave the following results:

$$\sum_{n=0}^{\infty} \phi_n^{\binom{a,b,c}{d,e}}(x, y|q) \frac{P_n(s, t)}{(q; q)_n} = \frac{(xs; q)_\infty}{(xt; q)_\infty} \;{}_4\phi_3\left(\begin{matrix} a, b, c, \frac{s}{t} \\ d, e, xs \end{matrix}; q, yt\right), \max\{|xt|, |yt|\} < 1. \quad (1.23)$$

$$\sum_{n=0}^{\infty} \psi_n^{\binom{a,b,c}{d,e}}(x, y|q) \frac{t^n}{(q; q)_n} = (xt; q)_\infty \;{}_3\phi_3\left(\begin{matrix} a, b, c \\ 0, d, e \end{matrix}; q, -yt\right), \quad \max\{|xt|, |yt|\} < 1. \quad (1.24)$$

Also, they offered the following q-difference equation:

Theorem 1.3. [13]. *Let* $f(a, b, c, d, e, x, y)$ *be a seven-variable analytic function in a neighborhood of* $(a, b, c, d, e, x, y) = (0,0,0,0,0,0,0) \in \mathbb{C}^7$. *Then* $f(a, b, c, d, e, x, y)$ *can be expanded in terms of* $\psi_n^{\binom{a,b,c}{d,e}}(x, y|q)$ *if and only if satisfies the difference equation:*

$$\begin{aligned} &x\{f(a, b, c, d, e, xq, y) - f(a, b, c, d, e, xq, yq) \\ &\quad -(d + e)q^{-1}[f(a, b, c, d, e, xq, yq) - f(a, b, c, d, e, xq, yq^2)] \\ &\quad +deq^{-2}[f(a, b, c, d, e, xq, yq^2) - f(a, b, c, d, e, xq, yq^3)]\} \\ &= y\{f(a, b, c, d, e, xq, yq) - f(a, b, c, d, e, x, yq) \\ &\quad -(a + b + c)[f(a, b, c, d, e, xq, yq^2) - f(a, b, c, d, e, x, yq^2)] \\ &\quad +(ab + ac + bc)[f(a, b, c, d, e, xq, yq^3) - f(a, b, c, d, e, x, yq^3)] \\ &\quad -abc[f(a, b, c, d, e, xq, yq^4) - f(a, b, c, d, e, x, yq^4)]\}. \end{aligned} \quad (1.25)$$

In 2021, Saad and Khalaf [14] defined the generalized q-operator $F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_b)$ as follows:

$$F(a_0,a_1,\ldots,a_{r-1},b_1,\ldots,b_s,-c\,\theta_b)=\sum_{n=0}^{\infty}\frac{(a_0,a_1,\ldots,a_{r-1})_n}{(q,b_1,\ldots,b_s)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(-c\,\theta_b)^n$$
$$= {}_r\phi_s\left(\begin{matrix}a_0,a_1,\ldots,a_{r-1}\\ b_1,\ldots,b_s\end{matrix};q,-c\,\theta_b\right). \quad (1.26)$$

They [14] generalized numerous well-known q-identities, such as the q-Chu-Vandermonde summation formula, the q-Pfaf-Saalschütz summation formula, and Hein's q-Gauss summation formula, by employing the operator $F(a_0,a_1,\ldots,a_{r-1},b_1,\ldots,b_s,-c\,\theta_b)$.

In 2022, Saad and Reshem [15] used q-Gospers algorithm [16,17] to verify that the function $f(a,b,c)$ satisfies the q-difference equation.

The paper is structured as follows: In Section 2, a solution to a generalized q-difference equation is presented in a generalized q-operator form. In section 3, several identities for the operator $F(a_0,\ldots,a_{r-1},b_1,\ldots,b_s,-c\,\theta_b)$ are provided. In section 4, we introduce a generalized q-polynomials $\Psi_n^{(A,B)}(x,y,c|q)$. The generating function for the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ is found using the q-difference equation technique. In section 5, two Rogers formulas for the q-polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ and $\widetilde{H}_n$ are demonstrated using the method of the q-difference equation. In Section 6, two types of Srivastava-Agarwal generating functions for the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ are provided.

2. The Generalized q-Difference Equation

This section describes a solution to a generalized q-difference equation in a generalized q-operator form, which is a generalization to Fang's work [12].

Lemma 2.1. [18]. *If $f(x_1,x_2,\ldots,x_k)$ is analytic at the origin $(0,0,\ldots,0)\in\mathbb{C}^k$, then f can be expanded in an absolutely convergent power series*

$$f(x_1,x_2,\ldots,x_k)=\sum_{n_1,n_2,\ldots,n_k}^{\infty}\alpha_{n_1,n_2,\ldots,n_k}x_1^{n_1}x_2^{n_2}\ldots x_k^{n_k}.$$

Theorem 2.2. *Let $f(a_0,a_1\ldots,a_{r-1},b_1,\ldots,b_s,b,c)$ be an $(r+s+2)$-variable analytic function in a neighborhood of $(a_0,a_1,\ldots,a_{r-1},b_1,\ldots,b_s,b,c)=(0,0,\ldots,0)\in\mathbb{C}^{r+s+2}$ satisfying the q-difference equation*

$$(-q)^{1+s-r}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}f\left(a_0,a_1,\ldots,a_{r-1},b_1,\ldots,b_s,b,cq^{j+r-s-1}\right)$$
$$=-c\sum_{j=0}^{r}(-1)^j A_j\theta_b\left\{f\left(a_0,a_1,\ldots,a_{r-1},b_1,\ldots,b_s,b,cq^j\right)\right\}, \quad (2.1)$$

where

$$b_0=q,\quad B_0=A_0=1,\quad B_1=\sum_{j=0}^{s}b_i,\quad B_2=\sum_{0\le i<j\le s}b_ib_j$$
$$B_3=\sum_{0\le i<j<k\le s}b_ib_jb_k\quad,\ldots,\quad B_{s+1}=b_0b_1\ldots b_s,\quad A_1=\sum_{i=0}^{r-1}a_i$$
$$A_2=\sum_{0\le i<j\le r-1}a_ia_j,\quad A_3=\sum_{0\le i<j<k\le r-1}a_ia_ja_k\quad,\ldots,\quad A_r=a_0a_1\ldots a_{r-1}.$$

Then

$$f(a_0,\dots,a_{r-1},b_1,\dots,b_s,b,c)$$
$$= F(a_0,\dots,a_{r-1},b_1,\dots,b_s,-c\ \theta_b)\{f(a_0,\dots,a_{r-1},b_1,\dots,b_s,b,0)\}.$$

Proof. By using Lemma 2.1, we suppose that

$$f(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,b,c)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\alpha_{m,n}b^m c^n. \tag{2.2}$$

$$(-q)^{1+s-r}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}\sum_{m,n=0}^{\infty}\alpha_{m,n}b^m c^n q^{n(j+r-s-1)}$$
$$=-c\sum_{j=0}^{r}(-1)^j A_j\theta_b\left\{\sum_{m,n=0}^{\infty}\alpha_{m,n}b^m c^n q^{jn}\right\}.$$

$$\sum_{n=0}^{\infty}\sum_{j=0}^{s+1}(-1)^j B_j(-1)^{1+s-r}q^{j(n-1)+(n-1)(r-s-1)}\sum_{m=0}^{\infty}\alpha_{m,n}b^m c^n$$
$$=\sum_{n=0}^{\infty}\sum_{j=0}^{r}(-1)^j A_j q^{jn}\theta_b\left\{\sum_{m=0}^{\infty}\alpha_{m,n}b^m c^{n+1}\right\}$$
$$=\sum_{n=1}^{\infty}\sum_{j=0}^{r}(-1)^j A_j q^{j(n-1)}\theta_b\left\{\sum_{m=0}^{\infty}\alpha_{m,n-1}b^m\right\}c^n.$$

Equating the coefficients of c^n, we obtain

$$\sum_{j=0}^{s+1}(-1)^j B_j(-1)^{1+s-r}q^{j(n-1)+(n-1)(r-s-1)}\sum_{m=0}^{\infty}\alpha_{m,n}b^m$$
$$=\sum_{j=0}^{r}(-1)^j A_j q^{j(n-1)}\theta_b\left\{\sum_{m=0}^{\infty}\alpha_{m,n-1}b^m\right\}.$$

$$\sum_{m=0}^{\infty}\alpha_{m,n}b^m=\frac{(-1)^{1+s-r}q^{(n-1)(1+s-r)}\sum_{j=0}^{r}(-1)^j A_j q^{j(n-1)}}{\sum_{j=0}^{s+1}(-1)^j B_j q^{j(n-1)}}\theta_b\left\{\sum_{m=0}^{\infty}\alpha_{m,n-1}b^m\right\}$$
$$=\frac{(-1)^{1+s-r}q^{(n-1)(1+s-r)}\prod_{j=0}^{r-1}(1-a_j q^{n-1})}{\prod_{j=0}^{s}\left(1-b_j q^{n-1}\right)}(-1)\theta_b\left\{\sum_{m=0}^{\infty}\alpha_{m,n-1}b^m\right\}.$$

We discover through repetition that

$$\sum_{m=0}^{\infty}\alpha_{m,n}b^m=\frac{(a_0,a_1,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}(-\theta_b)^n\left\{\sum_{m=0}^{\infty}\alpha_{m,0}b^m\right\}. \tag{2.3}$$

Setting $c=0$ in (2.2), we get

$$f(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,b,0)=\sum_{m=0}^{\infty}\alpha_{m,0}b^m. \tag{2.4}$$

From equations (2.2), (2.3) and (2.4), the required result is obtained.

- Letting $r=s+1$ in Theorem 2.2, we get Theorem 1.1 given by Fang [12].
- Setting $(r,s)=(3,3)$ and $(a_0,a_1,a_2,b_1,b_2,b_3,x,y,c)=(a,b,c,d,e,0,x,0,-y)$ in equation (2.1), we get equation (1.25) obtained by Cao et al. [13].

3. Identities for the q-Operator $F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_b)$

A number of identities for the operator $F(a_0, \dots, a_{r-1}, b_1 \dots, b_s, -c\,\theta_b)$ are given in this section. These identities are generalizations of the results obtained by Fang [12].

Lemma 3.1. Let θ_a^n be the Leibniz rule for θ_a and $a \neq 0$.

$$\theta_a^n\left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\right\} = \left(-\frac{q}{a}\right)^n \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n}, q/au, q/at; q)_k}{(q, q/ab, q/ad; q)_k}\left(\frac{u\,tq^n}{b\,d}\right)^k. \quad (3.1)$$

Proof. Using (1.13) and (1.14),

$$\theta_a^n\left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\right\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \theta_a^k\left\{\frac{(au; q)_\infty}{(ab; q)_\infty}\right\} \theta_a^{n-k}\left\{\frac{(atq^{-k}; q)_\infty}{(adq^{-k}; q)_\infty}\right\}$$

$$= \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k}(-1)^k q^{-2\binom{k}{2}+nk} \frac{b^k (u/b; q)_k (au; q)_\infty}{(abq^{-k}; q)_\infty} \frac{q^{-\binom{n-k}{2}}(dq^{-k})^{n-k}(t/d; q)_{n-k}(atq^{-k}; q)_\infty}{(adq^{-n}; q)_\infty}$$

$$= \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\sum_{k=0}^{n} \frac{(q^{-n}, u/b; q)_k}{(q, q/ab, q^{1-n}d/t; q)_k}\frac{(t/d; q)_n}{(q/ad; q)_n} q^k\left(-\frac{q}{a}\right)^n \qquad \text{(by using (1.3))}$$

$$= \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\left(-\frac{q}{a}\right)^n \frac{(t/d; q)_n}{(q/ad; q)_n}\sum_{k=0}^{n} \frac{(q^{-n}, u/b, q/at; q)_k}{(q, q/ab, q^{1-n}d/t; q)_k} q^k$$

$$= \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\left(-\frac{q}{a}\right)^n \sum_{k=0}^{n} \frac{(q^{-n}, q/au, q/at; q)_k}{(q, q/ab, q/ad; q)_k}\left(\frac{utq^n}{bd}\right)^k \qquad \text{(by using (1.10))}$$

$$= \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\left(-\frac{q}{a}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, q/au, q/at \\ q/ab, q/ad \end{matrix}; q; \frac{utq^n}{b\,d}\right).$$

In the following theorem, we shall show how to satisfy the q-difference equation (2.1):

Theorem 3.2. *For* $a_0 = q^{-m}$, $m \in \mathbb{N}, a, b, d, u, t, a_i, b_j \in \mathbb{C}, a \neq 0, i = 0, \dots, r-1, j = 1, \dots, s$, *then*

$$F(a_0, a_1, \dots, a_{r-1}, b_1 \dots, b_s, -c\,\theta_a)\left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\right\} = \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{a}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, q/au, q/at \\ q/ab, q/ad \end{matrix}; q, \frac{utq^n}{b\,d}\right), \quad (3.2)$$

provided $\max\{|ab|, |ad|, |ut/bd|\} < 1$.

Proof. Let $f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, a, c)$ be the right hand-side of (3.2).

$$(-q)^{1+s-r}\sum_{k=0}^{s+1} \frac{(-1)^j B_j}{q^j} f(a_0, a_1, \dots, a_{r-1}, b_1 \dots, b_s, a, cq^{j+r-s-1})$$

$$= (-q)^{1+s-r}\sum_{k=0}^{s+1} \frac{(-1)^j B_j}{q^j}\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\sum_{n=0}^{\infty} \frac{(a_0, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}$$

$$\times\left(\frac{qcq^{j+r-s-1}}{a}\right)^n {}_3\phi_2\left(\begin{matrix} q^{-n}, q/au, q/at \\ q/ab, q/ad \end{matrix}; q, \frac{utq^n}{b\,d}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{s+1} (-1)^j B_j \frac{(a_0, a_1, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-r} (-c)^n$$
$$\times \left(-\frac{q}{a}\right)^n \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty} \; {}_3\phi_2\left(\begin{matrix} q^{-n}, q/au, q/at \\ q/ab, q/ad \end{matrix}; q, \frac{utq^n}{b\,d}\right)$$
$$= \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-r} (-c)^n \theta_a^n \left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\right\}$$
$$\times \prod_{j=0}^{s} (1 - b_j q^{n-1})$$
$$= \sum_{n=1}^{\infty} \frac{(a_0, \dots, a_{r-1}; q)_{n-1}}{(q, b_1, \dots, b_s; q)_{n-1}} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-r} (-c)^n \theta_a^n \left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\right\}$$
$$\times \prod_{j=0}^{r-1} (1 - a_j q^{n-1})$$
$$= \sum_{n=0}^{\infty} \frac{(a_0, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^{n+2} q^{\binom{n+1}{2}-n}\right]^{1+s-r} (-c)^{n+1} \theta_a \theta_a^n \left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}\right\}$$
$$\times \prod_{j=0}^{r-1} (1 - a_j q^n)$$
$$= -c \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} (-c)^n$$
$$\times \theta_a \left\{\left(-\frac{q}{a}\right)^n \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty} \; {}_3\phi_2\left(\begin{matrix} q^{-n}, q/au, q/at \\ q/ab, q/ad \end{matrix}; q, \frac{utq^n}{b\,d}\right)\right\} \sum_{j=0}^{r} (-1)^j A_j q^{jn}$$
$$= -c \sum_{j=0}^{r} (-1)^j A_j \theta_a \left\{\frac{(au, at; q)_\infty}{(ab, ad; q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{r-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} \left(\frac{qcq^j}{a}\right)^n\right.$$
$$\left.\times \; {}_3\phi_2\left(\begin{matrix} q^{-n}, \frac{q}{au}, \frac{q}{at} \\ \frac{q}{ab}, \frac{q}{ad} \end{matrix}; q, \frac{utq^n}{b\,d}\right)\right\}$$
$$= -c \sum_{j=0}^{r} (-1)^j A_j \, \theta_a \{f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, a, cq^j)\}.$$

So, $f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, a, c)$ satisfies the q-difference equation (2.1). Note that $f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, a, 0) = \frac{(au, at; q)_\infty}{(ab, ad; q)_\infty}$.
From Theorem 2.2, the required result is obtained.

- Setting $t = 0$ in equation (3.2), we obtain

Corollary 3.2.2. *If* $a_0 = q^{-m}$, $m \in \mathbb{N}, a, b, d, u, a_i, b_j \in \mathbb{C}, i = 0, \dots, r-1, j = 1, \dots, s, a \neq 0$, *then*

$$F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_a)\left\{\frac{(au; q)_\infty}{(ab, ad; q)_\infty}\right\} = \frac{(au; q)_\infty}{(ab, ad; q)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(a_0,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{a}\right)^n \ {}_2\phi_2\left(\begin{matrix} q^{-n},q/au \\ q/ab,q/ad \end{matrix};q,\frac{uq^{n+1}}{ab\,d}\right), \quad (3.3)$$

where $\max\{|ab|,|ad|\} < 1$.

- For $u = 0$ in equation (3.3), we get

Corollary 3.2.3. *For* $a_0 = q^{-m}$, $m \in \mathbb{N}, a, b, d, a_i, b_j \in \mathbb{C}, i = 0,\dots,s, j = 1,\dots,s, a \neq 0$, *then*

$$F(a_0,a_1,\dots,a_{r-1},b_1\dots,b_s,-c\,\theta_a)\left\{\frac{1}{(ab,ad;q)_\infty}\right\} = \frac{1}{(ab,ad;q)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(a_0,a_1,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{a}\right)^n \ {}_1\phi_2\left(\begin{matrix} q^{-n} \\ q/ab,q/ad \end{matrix};q,\frac{q^{n+2}}{a^2bd}\right). \quad (3.4)$$

- When $d = 0$ in equation (3.2), we have

Corollary 3.2.4. *For* $a_0 = q^{-m}$,$m \in \mathbb{N}, a, b, u, t, a_i, b_j \in \mathbb{C}, i = 0,\dots,s, j = 1,\dots,s, a \neq 0$, *then*

$$F(a_0,a_1,\dots,a_{r-1},b_1\dots,b_s,-c\,\theta_a)\left\{\frac{(au,at;q)_\infty}{(ab;q)_\infty}\right\} = \frac{(au,at;q)_\infty}{(ab;q)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(a_0,\dots,a_{r-1};q)_n}{(q,b_1\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{a}\right)^n \ {}_3\phi_1\left(\begin{matrix} q^{-n},q/au,q/at \\ q/ab \end{matrix};q,\frac{aut}{bq^{1-n}}\right). \quad (3.5)$$

- Letting $r = s + 1$ in equation (3.5), we obtain Theorem 1.2 obtained by Fang [12] (equation (1.18)).
- If $(b,t) = (0,0)$ in equation (3.5), we get

Corollary 3.2.5. *For* $a_0 = q^{-m}$, $m \in \mathbb{N}, a, u, t, a_i, b_j \in \mathbb{C}, i = 0,\dots,s, j = 1,,s$, *then*

$$F(a_0,\dots,a_{r-1},b_1\dots,b_s,-c\,\theta_a)\{(au;q)_\infty\} = (au;q)_\infty \ {}_r\phi_s\left(\begin{matrix} a_0,\dots,a_{r-1} \\ b_1,\dots,b_s \end{matrix};q,cu\right). \quad (3.6)$$

- Letting $r = s + 1$ in equation (3.6), we obtain Corollary 1.2.1. obtained by Fang [12] (equation (1.19)).

4. The Generating Function for the Generalized q-Polynomials $\Psi_n^{(A,B)}(x,y,c|q)$

In this section, a generalized q-polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ is defined. Using the q-difference equation method, the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ are represented by the operator $F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)$ and we find that the generating function for the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$.

Let $x, y, c \in \mathbb{C}$. We define a generalized q-polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ as follows:

$$\Psi_n^{(A,B)}(x,y,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_0,\dots,a_{r-1};q)_k}{(b_1,\dots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (-1)^k q^{\binom{k+1}{2}-nk} c^k P_{n-k}(x,q^k y), \quad (4.1)$$

where $A = (a_0,\dots,a_{r-1})$, $B = (b_1,\dots,b_s)$.

- When $r = s + 1$ and $(y, x) = (0, b)$ in equation (4.1), the generalized q-polynomials $\Psi_n^{(A,B)}(x, y, c|q)$ become the q-polynomials $\widetilde{H}_n = \widetilde{H}_n(a_0, \dots, a_s; b_1, \dots, b_s; b, c)$ established by Fang [12] (equation (1.20)).

- If $(r, s) = (3,3)$ and $(a_0, a_1, a_2, b_1, b_2, b_3, y, c) = (a, b, c, d, e, 0,0, -y)$ in equation (4.1), we get the new generalized Al-Salam-Carlitz q-polynomials $\psi_n^{\binom{a,b,c}{d,e}}(x, y|q)$ described by Cao et al. [13] (equation (1.22)).

Theorem 4.1 Let $\Psi_n^{(A,B)}(x, y, c|q)$ be defined as in (4.1), then

$$F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\{P_n(x, y)\} = \Psi_n^{(A,B)}(x, y, c|q).$$

Proof. Let

$$f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, x, c) = \Psi_n^{(A,B)}(x, y, c|q)$$

$$(-q)^{1+s-r}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j} f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, x, cq^{j+r-s-1})$$

$$= (-q)^{1+s-r}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(a_0, \dots, a_{r-1}; q)_k}{(b_1, \dots, b_s; q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(cq^{j+r-s-1}\right)^k$$

$$\times (-1)^k q^{\binom{k+1}{2}-nk} P_{n-k}(x; q^k y)$$

$$= \sum_{k=0}^{\infty}\frac{(a_0, a_1, \dots, a_{r-1}; q)_k}{(q, b_1, \dots, b_s; q)_k}\left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-r}(qc)^k (q^{-n}; q)_k (yq^k/x; q)_{n-k} x^{n-k}$$

$$\times \sum_{j=0}^{s+1}(-1)^j B_j q^{j(k-1)} \quad \text{(by using (1.1))}$$

$$= \sum_{k=0}^{\infty}\frac{(a_0, a_1, \dots, a_{r-1}; q)_k}{(q, b_1, \dots, b_s; q)_k}\left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k (q^{-n}; q)_k \frac{(y/x; q)_n}{(y/x; q)_k} x^n$$

$$\times \sum_{j=0}^{s+1}(-1)^j B_j q^{j(k-1)} \quad \text{(by using (1.6))}$$

$$= \sum_{k=0}^{\infty}\frac{(a_0, a_1, \dots, a_{r-1}; q)_k}{(q, b_1, \dots, b_s; q)_k}\left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k \frac{(-y)^n q^{\binom{n}{2}}(q^{1-n}x/y; q)_n}{(y/x; q)_k}(q^{-n}; q)_k$$

$$\times \sum_{j=0}^{s+1}(-1)^j B_j q^{j(k-1)} \quad \text{(by using (1.2))}$$

$$= \sum_{k=0}^{\infty}\frac{(a_0, a_1, \dots, a_{r-1}; q)_k}{(q, b_1, \dots, b_s; q)_k}\left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-r}(-c)^k (-y)^n q^{\binom{n}{2}}$$

$$\times \left(\frac{-q}{x}\right)^k \frac{(q^{1-n}x/y; q)_\infty}{(qx/y; q)_k}\ {}_2\phi_1\left(\begin{matrix} q^{-k}, yq^n/x \\ y/x \end{matrix}; q, q^{k-n}\right)\prod_{j=0}^{s}(1 - b_j q^{k-1})$$

$$= \sum_{k=1}^{\infty}\frac{(a_0, a_1, \dots, a_{r-1}; q)_{k-1}}{(q, b_1, \dots, b_s; q)_{k-1}}\left[(-1)^{k+1} q^{\binom{k}{2}-(k-1)}\right]^{1+s-r}(-c)^k (-y)^n q^{\binom{n}{2}}$$

$$\times\, \theta_x^k \left\{\frac{(q^{1-n}x/y;q)_\infty}{(qx/y;q)_\infty}\right\} \prod_{j=0}^{r-1} (1-a_j q^{k-1})$$

$$= \sum_{k=0}^{\infty} \frac{(a_0,a_1,\dots,a_{r-1};q)_k}{(q,b_1,\dots,b_s;q)_k} \left[(-1)^{k+2} q^{\binom{k+1}{2}-k}\right]^{1+s-r} (-c)^{k+1}(-y)^n q^{\binom{n}{2}} \theta_{q,x}$$

$$\times\, \theta_x^k \left\{\frac{(q^{1-n}x/y;q)_\infty}{(qx/y;q)_\infty}\right\} \prod_{j=0}^{r-1} (1-a_j q^{k})$$

$$= -c \sum_{j=0}^{r} (-1)^j A_j \theta_x \sum_{k=0}^{\infty} \frac{(a_0,a_1,\dots,a_{r-1};q)_k}{(q,b_1,\dots,b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (q^j c)^k (-y)^n q^{\binom{n}{2}}$$

$$\times \left\{\left(\frac{q}{x}\right)^k \frac{(q^{1-n}x/y;q)_\infty}{(qx/y;q)_k}\ {}_2\phi_1\left(\begin{matrix} q^{-k}, yq^n/x \\ y/x \end{matrix}; q, q^{k-n}\right)\right\} \quad \text{(by using (3.1))}$$

$$= -c \sum_{j=0}^{r} (-1)^j A_j \theta_x \left\{\sum_{k=0}^{\infty} \frac{(a_0,\dots,a_{r-1};q)_k}{(b_1,\dots,b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \left(cq^j\right)^k \right.$$

$$\left. \times \frac{(-y)^n q^{\binom{n}{2}} (q^{1-n}x/y;q)_n}{(y/x;q)_k} (q^{-n};q)_k \right\} \quad \text{(by using (1.8))}$$

$$= -c \sum_{j=0}^{r} (-1)^j A_j \theta_x \{\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_0,\dots,a_{r-1};q)_k}{(b_1,\dots,b_s;q)_k} \left[(-1)^k q^{\binom{n}{2}}\right]^{1+s-r} \left(cq^j\right)^k$$

$$\times\, (-1)^k q^{\binom{k+1}{2}-nk} P_{n-k}(x;yq^k)\}$$

$$= -c \sum_{j=0}^{r} (-1)^j A_j \theta_x \{f(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,b,cq^j)\}.$$

So, $f(a_0,a_1,\dots,a_{t-1},b_1,\dots,b_s,x,c)$ satisfies the q-difference equation (2.1). Note that

$$f(a_0,a_1,\dots,a_{t-1},b_1,\dots,b_s,x,0) = P_n(x;y).$$

From Theorem 2.2, the desired result is obtained.

Theorem 4.2 (Generating function for $\Psi_n^{(A,B)}(x,y,c|q)$). *If* $a_0 = q^{-G}$, $G \in \mathbb{N}$, *and* $|yt| < 1$, *then*

$$\sum_{n=0}^{\infty} \Psi_n^{(A,B)}(x,y,c|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}$$

$$= (xt;q)_\infty \sum_{k=0}^{\infty} \frac{(a_0,a_1,\dots,a_{r-1};q)_k}{(q,b_1,\dots,b_s;q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k\ {}_0\phi_1\left(\begin{matrix} 0 \\ xt \end{matrix}; q, ytq^k\right). \tag{4.2}$$

Proof. Put

$$f_L = f_L(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s;x;c) = \sum_{n=0}^{\infty} \Psi_n^{(A,B)}(x,y,c|q) \frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}.$$

We can check f_L satisfies the q-difference equation (2.1) in the same technique used in Theorem 4.1. So

$f_L = F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\{f(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,x,0)\}$

$$= F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\left\{\sum_{n=0}^{\infty} P_n(x,y)\frac{(-1)^n q^{\binom{n}{2}} t^n}{(q;q)_n}\right\}$$
$$= F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\left\{\sum_{n=0}^{\infty} \frac{(y/x;q)_n}{(q;q)_n}(-1)^n q^{\binom{n}{2}}(xt)^n\right\}$$
$$= F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\left\{(xt;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-n}(yt)^n}{(q,xt;q)_n}\right\} \quad \text{(by using (1.15))}$$
$$= \sum_{n=0}^{\infty} \frac{q^{n^2-n}(yt)^n}{(q;q)_n} F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\{(xtq^n;q)_\infty\}$$
$$= (xt;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2-n}(yt)^n}{(q,xt;q)_n}\; {}_r\phi_s\left(\begin{matrix} a_0, a_1, \dots, a_{r-1} \\ b_1, \dots, b_s \end{matrix}; q, ctq^n\right) \quad \text{(by using (3.6))}$$
$$= (xt;q)_\infty \sum_{k=0}^{\infty} \frac{(a_0, a_1, \dots, a_{r-1};q)_k}{(q, b_1, \dots, b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k\; {}_0\phi_1\left(\begin{matrix} 0 \\ xt \end{matrix}; q, ytq^k\right).$$

- When $r = s+1$ and $(x,y,t) = (b,0,u)$ in (4.2), we obtain the generating function for the q-polynomials $\widetilde{H}_n = \widetilde{H}_n(a_0, \dots, a_s; b_1, \dots, b_s; b, c)$ achieved by Fang [12] (equation (1.21)).
- For $(r,s) = (3,3)$ and $(a_0, a_1, a_2, b_1, b_2, b_3, y, c) = (a, b, c, d, e, 0, 0, -y)$ in equation (4.2), we get the generating function for Al-Salam-Carlitz q-polynomials $\psi_n^{\binom{a,b,c}{d,e}}(x, y|q)$ described by Cao et al. [13] (equation (1.24)).

5. Two Types of Rogers Formula for the Polynomials $\Psi_n^{(A,B)}(x, y, c|q)$

In this section, two types of Rogers formula for the q-polynomials $\Psi_n^{(A,B)}(x, y, c|q)$ and $\widetilde{H}_n$ are shown.

Theorem 5.1 *If* $a_0 = q^{-G}$, $G \in \mathbb{N}$, *and* $max\{|xt|, |xv|\} < 1$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Psi_{n+m}^{(A,B)}(x, y, c|q)\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}$$
$$= \frac{(yt;q)_\infty}{(xv, xt;q)_\infty}\sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{r-1};q)_n}{(q, b_1, \dots, b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^n$$
$$\times \sum_{m=0}^{\infty} \frac{(q^{-n};q)_m q^{2\binom{m}{2}}}{(q, q/xt, q/xv;q)_m}\left(\frac{q^{n+2}}{x^2vt}\right)^m\; {}_1\phi_1\left(\begin{matrix} xtq^{-m} \\ yt \end{matrix}; q, yv\right). \tag{5.1}$$

Proof. Put

$$f_L = f_L(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s; x; c) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \Psi_{n+m}^{(A,B)}(x, y, c|q)\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}.$$

We can check f_L satisfies the q-difference equation (2.1) in the same technique used in Theorem 4.1. So

$$f_L = F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\theta_x)\{f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, x, 0)\}$$

$$= F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\ \theta_x)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_{n+m}(x,y)\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}\right\}$$

$$F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\theta_x)\left\{\frac{(yt;q)_\infty}{(xv,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(xt;q)_k(-yv)^k q^{\binom{k}{2}}}{(q,yt;q)_k}\right\} \text{ (by } using \text{ (1.12))}$$

$$= (yt;q)_\infty \sum_{k=0}^{\infty}\frac{(-yv)^k q^{\binom{k}{2}}}{(q,yt;q)_k} F(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, -c\theta_x)\left\{\frac{1}{(xv, xtq^k;q)_\infty}\right\}$$

$$= \frac{(yt;q)_\infty}{(xv,xt;q)_\infty}\sum_{k=0}^{\infty}\frac{(xt;q)_k(-yv)^k q^{\binom{k}{2}}}{(q,yt;q)_k}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^n$$

$$\times\ {}_1\phi_2\left(\begin{matrix} q^{-n} \\ q/xv, q/xtq^k\end{matrix}; q, \frac{q^{n-k+2}}{x^2vt}\right) \quad \text{(by using (3.4))}$$

$$= \frac{(yt;q)_\infty}{(xv,xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0,a_1,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^n$$

$$\times\sum_{m=0}^{\infty}\frac{(q^{-n};q)_m q^{2\binom{m}{2}}}{(q,q/xt,q/xv;q)_m}\left(\frac{q^{n+2}}{x^2vt}\right)^m\ {}_1\phi_1\left(\begin{matrix} xtq^{-m} \\ yt\end{matrix}; q, yv\right). \quad \text{(by using (1.4))}$$

Theorem 5.2 *If* $a_0 = q^{-G}$, $G \in \mathbb{N}$, *and* $max\{|yt|, |yv|\} < 1$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Psi_{n+m}^{(A,B)}(x,y,c|q)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}$$

$$= (xv;q)_\infty\sum_{k=0}^{\infty}\frac{(a_0,\dots,a_{r-1};q)_k}{(q,b_1\dots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k\sum_{n=0}^{\infty}\frac{(y/x;q)_n(-xt)^n q^{\binom{n}{2}}}{(q,xv;q)_n}$$

$$\times\sum_{m=0}^{\infty}\frac{q^{m^2-m}(yvq^{2n})^m}{(q,xvq^n;q)_m}\ {}_3\phi_1\left(\begin{matrix} q^{-k}, yq^n/x, q/xvq^{n+m} \\ y/x\end{matrix}; q, \frac{xvq^m}{q^{1-k}}\right). \tag{5.2}$$

Proof. Put

$$f_L = f_L(a_0, \dots, a_{r-1}, b_1, \dots, b_s; x; c)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Psi_{n+m}^{(A,B)}(x,y,c|q)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}.$$

We can check f_L satisfies the q-difference equation (2.1) in the same technique used in Theorem 4.1. So

$$f_L = F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\ \theta_x)\{f(a_0, a_1, \dots, a_{r-1}, b_1, \dots, b_s, x, 0)\}$$

$$= F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\ \theta_x)\left\{\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}P_{n+m}(x,y)(-1)^{n+m}q^{\binom{n+m}{2}}\frac{t^n}{(q;q)_n}\frac{v^m}{(q;q)_m}\right\}$$

$$= F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\ \theta_x)\left\{\sum_{n=0}^{\infty}\frac{P_n(x,y)(-t)^n q^{\binom{n}{2}}}{(q;q)_n}\sum_{m=0}^{\infty}\frac{P_m(x,yq^n)(-vq^n)^m q^{\binom{m}{2}}}{(q;q)_m}\right\}$$

$$= F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\left\{\sum_{n=0}^{\infty} \frac{(y/x;q)_n(-xt)^n q^{\binom{n}{2}}}{(q;q)_n} \times \sum_{m=0}^{\infty} \frac{(yq^n/x;q)_m(-xvq^n)^m q^{\binom{m}{2}}}{(q;q)_m}\right\}$$

$$= F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\left\{\sum_{n=0}^{\infty} \frac{(q^{1-n}x/y;q)_n(yt)^n q^{2\binom{n}{2}}}{(q;q)_n} \times (xvq^n;q)_\infty \sum_{m=0}^{\infty} \frac{q^{m^2-m}(yvq^{2n})^m}{(q,xvq^n;q)_m}\right\} \quad \text{(by using (1.2), (1.15))}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2\binom{n}{2}}(yt)^n}{(q;q)_n} \sum_{m=0}^{\infty} \frac{q^{m^2-m}(yvq^{2n})^m}{(q;q)_m} \times F(a_0, \dots, a_{r-1}, b_1, \dots, b_s, -c\,\theta_x)\left\{\frac{(xq^{1-n}/y, xvq^{n+m};q)_\infty}{(xq/y;q)_\infty}\right\}$$

$$= (xv;q)_\infty \sum_{k=0}^{\infty} \frac{(a_0, \dots, a_{r-1};q)_k}{(q, b_1 \dots, b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k \sum_{n=0}^{\infty} \frac{(q^{1-n}x/y;q)_n(yt)^n q^{2\binom{n}{2}}}{(q,xv;q)_n} \times \sum_{m=0}^{\infty} \frac{q^{m^2-m}(yvq^{2n})^m}{(q,xvq^n;q)_m}\ {}_3\phi_1\left(\begin{matrix} q^{-k}, yq^n/x, q/xvq^{n+m} \\ y/x \end{matrix}; q, \frac{xvq^m}{q^{1-k}}\right) \quad (by\ using\ (3.5))$$

$$= (xv;q)_\infty \sum_{k=0}^{\infty} \frac{(a_0, \dots, a_{r-1};q)_k}{(q, b_1 \dots, b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k \sum_{n=0}^{\infty} \frac{(y/x;q)_n(-xt)^n q^{\binom{n}{2}}}{(q,xv;q)_n} \times \sum_{m=0}^{\infty} \frac{q^{m^2-m}(yvq^{2n})^m}{(q,xvq^n;q)_m}\ {}_3\phi_1\left(\begin{matrix} q^{-k}, yq^n/x, q/xvq^{n+m} \\ y/x \end{matrix}; q, \frac{xvq^m}{q^{1-k}}\right).$$

- Setting $r = s + 1$ and $(x, y, t) = (b, 0, u)$ in equations (5.1) and (5.2), we get the following two types of Rogers formula for the polynomials $\widetilde{H}_n$.

Corollary 5.2.6. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$, *we have*

- If $\max\{|bu|, |bv|\} < 1$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widetilde{H}_{n+m} \frac{u^n}{(q;q)_n}\frac{v^m}{(q;q)_m} = \frac{1}{(bu, bv;q)_\infty}\sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_s;q)_n}{(q, b_1, \dots, b_s;q)_n}\left(\frac{qc}{b}\right)^n\ {}_1\phi_2\left(\begin{matrix} q^{-n} \\ q/bv, q/bu \end{matrix}; q, \frac{q^{n+2}}{b^2vu}\right).$$

- If $b \neq 0$ and $|qc/b| < 1$, then

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \widetilde{H}_{n+m}(-1)^{n+m} q^{\binom{n+m}{2}} \frac{u^n}{(q;q)_n}\frac{v^m}{(q;q)_m} = (bv;q)_\infty\ {}_{s+1}\phi_s\left(\begin{matrix} a_0, a_1, \dots, a_s \\ b_1, \dots, b_s \end{matrix}; q, \frac{qc}{b}\right)\ {}_1\phi_1\left(\begin{matrix} 0 \\ bv \end{matrix}; q, bu\right).$$

6. Two Types of Srivastava-Agarwal Generating Functions

In this section, two types of Srivastava-Agarwal generating functions for the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ and $\widetilde{H}_n$ are introduced.

Theorem 6.1 *For* $a_0=q^{-G}$, $G\in\mathbb{N}$ *and* $|xt|<1$, *then*

$$\sum_{n=0}^{\infty}\Psi_n^{(A,B)}(x,y,c|q)\frac{(\lambda;q)_n}{(\mu;q)_n}\frac{t^n}{(q;q)_n}$$
$$=\frac{(xt\lambda;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(\lambda,q^{1-n}y/x\mu;q)_n(x\mu t)^n q^{2\binom{n}{2}}}{(q,\mu,xt\lambda;q)_n}\sum_{k=0}^{\infty}\frac{(a_0,\dots,a_{r-1};q)_k}{(q,b_1\dots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k$$
$$\times\ {}_3\phi_2\left(\begin{matrix}q^{-k},yq/x\mu,q^{1-n}/xt\lambda\\ q/xt,yq^{1-n}/x\mu\end{matrix};q,q^k\lambda\right). \tag{6.1}$$

Proof. Let $f_L=f_L(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s;a;c)$ =LHS of equation (6.1). By using the same technique used in Theorem 4.1 to check f_L satisfies q-difference equation (2.1). We have

$$f_L=F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\{f(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,b,0)\}$$
$$=F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\sum_{n=0}^{\infty}P_n(x,y)\frac{(\lambda;q)_n}{(q,\mu;q)_n}t^n\right\}$$
$$=F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\sum_{n=0}^{\infty}\frac{(y/x,\lambda;q)_n}{(q,\mu;q)_n}(xt)^n\right\}$$
$$=F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\frac{(xt\lambda;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(\lambda,x\mu/y;q)_n(-yt)^n q^{\binom{n}{2}}}{(q,\mu,xt\lambda;q)_n}\right\}$$
(by using (1.9))
$$=\sum_{n=0}^{\infty}\frac{(\lambda;q)_n(-yt)^n q^{\binom{n}{2}}}{(q,\mu;q)_n}F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\frac{(xt\lambda q^n,x\mu/y;q)_\infty}{(xt,x\mu q^n/y;q)_\infty}\right\}$$
$$=\frac{(xt\lambda;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(\lambda,x\mu/y;q)_n(-yt)^n q^{\binom{n}{2}}}{(q,\mu,xt\lambda;q)_n}\sum_{k=0}^{\infty}\frac{(a_0,\dots,a_{r-1};q)_k}{(q,b_1\dots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k$$
$$\times\ {}_3\phi_2\left(\begin{matrix}q^{-k},yq/x\mu,q^{1-n}/xt\lambda\\ q/xt,yq^{1-n}/x\mu\end{matrix};q,q^k\lambda\right)\quad\text{(by using (3.2))}$$
$$=\frac{(xt\lambda;q)_\infty}{(xt;q)_\infty}\sum_{n=0}^{\infty}\frac{(\lambda,q^{1-n}y/x\mu;q)_n(x\mu t)^n q^{2\binom{n}{2}}}{(q,\mu,xt\lambda;q)_n}\sum_{k=0}^{\infty}\frac{(a_0,\dots,a_{r-1};q)_k}{(q,b_1\dots,b_s;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^k$$
$$\times\ {}_3\phi_2\left(\begin{matrix}q^{-k},yq/x\mu,q^{1-n}/xt\lambda\\ q/xt,yq^{1-n}/x\mu\end{matrix};q,q^k\lambda\right).$$

This completes the proof of Theorem 6.1.

Theorem 6.2 *If* $a_0=q^{-G}$, $G\in\mathbb{N}$ *and* $|xu|<1$, *then*

$$\sum_{n=0}^{\infty}\Psi_n^{(A,B)}(x,y,c|q)\frac{\phi_n^{\binom{t,w,z}{d,e}}(u,v|q)}{(q;q)_n}$$

$$= \frac{(uy;q)_\infty}{(ux;q)_\infty} \sum_{n=0}^{\infty} \frac{(t,w,z,y/x;q)_n}{(d,e,uy;q)_n} (vx)^n \sum_{n=0}^{\infty} \frac{(a_0,\dots,a_{r-1};q)_n}{(q,b_1,\dots,b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} \left(\frac{qc}{x}\right)^n$$
$$\times \; {}_2\phi_2 \left(\begin{matrix} q^{-n}, yq^n/x \\ q/ux, y/x \end{matrix}; q, \frac{q}{xu}\right). \tag{6.2}$$

Proof. Put

$$f_L = f_L(a_0,\dots,a_{r-1},b_1,\dots,b_s;b;c) = \sum_{n=0}^{\infty} \Psi_n^{(A,B)}(x,y,c|q) \frac{\phi_n^{\binom{t,w,z}{d,e}}(u,v|q)}{(q;q)_n}.$$

We can check f_L satisfy the q-difference equation (2.1) in the same way used in Theorem 4.1. Hence

$$f_L = F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\{f(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,b,0)\}$$
$$= F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\sum_{n=0}^{\infty} \phi_n^{\binom{t,w,z}{d,e}}(u,v|q) \frac{P_n(x,y)}{(q;q)_n}\right\}$$
$$= F(a_0,a_1,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\frac{(yu;q)_\infty}{(xu;q)_\infty} \sum_{n=0}^{\infty} \frac{(t,w,z,y/x;q)_n}{(q,d,e,uy;q)_n}(xv)^n\right\} \text{ (by using (1.23))}$$
$$= (yu;q)_\infty \sum_{n=0}^{\infty} \frac{(t,w,z;q)_n}{(q,d,e,yu;q)_n} (-yv)^n q^{\binom{n}{2}} F(a_0,\dots,a_{r-1},b_1,\dots,b_s,-c\,\theta_x)\left\{\frac{(q^{1-n}x/y;q)_\infty}{(xu,xq/y;q)_\infty}\right\}$$
$$= \frac{(yu;q)_\infty}{(xu;q)_\infty} \sum_{n=0}^{\infty} \frac{(t,w,z,y/x;q)_n}{(q,d,e,yu;q)_n}(xv)^n \sum_{m=0}^{\infty} \frac{(a_0,\dots,a_{r-1};q)_m}{(q,b_1,\dots,b_s;q)_m}\left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-r}\left(\frac{qc}{x}\right)^m$$
$$\times \; {}_2\phi_2\left(\begin{matrix} q^{-n}, yq^n/x \\ q/ux, y/x \end{matrix} q, \frac{q^{1-n+m}}{xu}\right). \quad \text{(by using (3.3))}$$

• Letting $r = s + 1$ and $(x, y) = (b, 0)$ in equations (6.1) and (6.2), the following two types of Srivastava-Agarwal formula for the polynomials $\widetilde{H}_n$ are obtained.

Corollary 6.2.7. *For $a_0 = q^{-G}$ and $G \in \mathbb{N}$, then*

- If $|qc/b| < 1$, then

$$\sum_{n=0}^{\infty} \widetilde{H}_n \frac{(\lambda;q)_n}{(\mu;q)_n} \frac{t^n}{(q;q)_n} = \sum_{n=0}^{\infty} \frac{(\lambda;q)_n}{(\mu;q)_n} \frac{(bt)^n}{(q;q)_n} \; {}_{s+2}\phi_{s+1}\left(\begin{matrix} a_0,a_1,\dots,a_s,q^{-n} \\ b_1 \dots, b_s \end{matrix}; q, \frac{qc}{b}\right).$$

- If $b \neq 0$ and $|bu| < 1$, then

$$\sum_{n=0}^{\infty} \widetilde{H}_n \frac{\phi_n^{\binom{t,w,z}{d,e}}(u,v|q)}{(q;q)_n} = \frac{1}{(bu;q)_\infty} \sum_{n=0}^{\infty} \frac{(t,w,z;q)_n}{(q,d,e;q)_n}(bv)^n \sum_{m=0}^{\infty} \frac{(a_0,\dots,a_s;q)_m}{(q,b_1,\dots,b_s;q)_m}\left(\frac{qc}{b}\right)^m$$
$$\times \; {}_1\phi_1\left(\begin{matrix} q^{-m} \\ q/bu \end{matrix}; q, \frac{q^{1-n+m}}{bu}\right).$$

Conclusions

1. We provide a generalization of Fang's work [12] by providing a solution to a generalized q-difference equation in the form of a generalized q-operator.
2. Many q-difference equations can be produced by assigning some specific values to the generalized q-difference equation described in Theorem 2.2.
3. By assigning some special values to the generalized q-polynomial $\Psi_n^{(A,B)}(x,y,c|q)$,

several q-polynomials can be obtained.

4. Get numerous identities for the polynomials $\Psi_n^{(A,B)}(x,y,c|q)$ using the q-difference equation approach.
5. Give applications of generalized q-difference equations for other q-integrals and q-polynomials.

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