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The Generalized Homogeneous q-Shift Operator ${}_r\Phi_s(D_{xy})$ for q-Identities and q-Integrals

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Abstract

In this paper, we illustrate how to use the generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ in generalizing various well-known q-identities, such as Hiene's transformation, the q-Gauss sum, and Jackson's transfor- mation. For the polynomials $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$, we provide another formula for the generating function, the Rogers formula, and the bilinear generating function of the Srivastava-Agarwal type. In addition, we also generalize the extension of both the Askey-Wilson integral and the Andrews-Askey integral.

Keywords: homogeneous q-shift operator, Hiene's transformation, Jackson's transformation, q-hypergeometric polynomials, generating function, Rogers formula, Srivastava-Agarwal generating function, Askey-Wilson integral, Andrews-Askey q-integral.

مؤثر تحول-q المتجانس العام ${}_r\Phi_s(D_{xy})$ للمتطابقات-q والتكاملات-q

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الخلاصة

في هذا البحث، نوضح كيفية استخدام مؤثر تحول-q المتجانس العام ${}_r\Phi_s(D_{xy})$ لتعميم العديد من متطابقات-q المعروفة، مثل تحويل Hiene، وتحويل جاكسون. بالنسبة إلى متعددات الحدود $\phi_n^{(a,b)}(x,y,c|q)$ ، نقدم صيغة أخرى للدالة المولدة، صيغة روجرز ، والدالة المولدة ثنائية الخطية من نوع Srivastava-Agarwal. بالإضافة إلى ذلك، نقوم أيضًا بتعميم توسيع كل من تكامل-Askey Wilson وتكامل Andrews-Askey .

1. Introduction

The study of basic hypergeometric series began in 1748, when Euler [1,2] investigated the infinite product $1/(q;q)_\infty$ as a generating function for $p(n)$, the number of partitions of a positive integer n into positive integers. Later, mathematicians such as Gauss, Heine, Rogers, Ramannjan, Waston, Slater, and many others made significant contributions to this topic. Because of Andrew and Askey's excellent work, basic hypergeometric series have recently

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been an active research subject again. Because of its various applications to combinatorics, quantum theory, number theory, statistical mechanics, and other fields, the field has grown rapidly [2].

In this paper, the notations used in [1] are followed, and we assume that $|q| < 1$.

For $a \in \mathbb{C}$, the q-shifted factorial is defined by [1]

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The multiple q-shifted factorial is defined by:

$$(a_1, a_2, \dots, a_r; q)_m = (a_1;q)_m (a_2;q)_m \cdots (a_r;q)_m,$$

where $m \in Z$ or ∞.

The basic hypergeometric series ${}_r\phi_s$ is described as follows [1]:

$${}_r\phi_s\left(\begin{matrix}\rho_1, \dots, \rho_r \\ \rho_1, \dots, \rho_s; q, x\end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(\rho_1, \dots, \rho_r; q)_n}{(q, \rho_1, \dots, \rho_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} x^n,$$

where $q \neq 0$ when $r > s + 1$. Note that

$${}_{r+1}\phi_r\left(\begin{matrix}\alpha_1, \dots, \alpha_{r+1} \\ \beta_1, \dots, \beta_r \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1, \dots, \alpha_{r+1}; q)_n}{(q, \beta_1, \dots, \beta_r; q)_n} x^n, \quad |x| < 1.$$

The q-binomial coefficient is given by [1]:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k \, (q;q)_{n-k}} \quad for \quad 0 \leq k \leq n,$$

where $n, k \in \mathbb{N}$.

The following identity will be used in this paper:

$$(q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk}. \tag{1.1}$$

The Cauchy identity is given by:

$$\sum_{m=0}^{\infty} \frac{(\alpha;q)_m}{(q;q)_m} x^m = \frac{(\alpha x;q)_\infty}{(x;q)_\infty}, \quad |x| < 1. \tag{1.2}$$

Letting $\alpha = 0$ in equation (1.2) gives Euler's identity:

$$\sum_{m=0}^{\infty} \frac{x^m}{(q;q)_m} = \frac{1}{(x;q)_\infty}, \quad |x| < 1. \tag{1.3}$$

which has the following inverse:

$$\sum_{m=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}} x^m}{(q;q)_m} = (x;q)_\infty. \tag{1.4}$$

The q-Chu-Vandermonde's sum is given by [1]:

$${}_2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{1.5}$$

The q-analog of the Chu-Vandermonde summation is [3]:

$$\begin{bmatrix} n+m \\ k \end{bmatrix} = \sum_{i=0}^{k} \begin{bmatrix} n \\ i \end{bmatrix} \begin{bmatrix} m \\ k-i \end{bmatrix} q^{(n-i)(k-i)}. \tag{1.6}$$

The q-Gauss sum is given by [1]:

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ d \end{matrix}; q, d/ab\right) = \frac{(d/b, d/a; q)_\infty}{(d, d/ab; q)_\infty}. \tag{1.7}$$

Heine's transformation of ${}_2\phi_1$ series [1, Appendix III, equation (III.1)] is:

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(b, az; q)_\infty}{(d, z; q)_\infty}\ {}_2\phi_1\left(\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right). \tag{1.8}$$

Jackson's transformation of ${}_2\phi_1$ series [1, Appendix III, equation (III.4)] is:

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ d \end{matrix}; q, x\right) = \frac{(ax; q)_\infty}{(x; q)_\infty}\ {}_2\phi_2\left(\begin{matrix} a, d/b \\ d, ax \end{matrix}; q, bx\right). \tag{1.9}$$

The transformations of ${}_3\phi_2$ series [1, Appendix III, equations (III.9) and (III.10)] are:

$$ {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right) = \frac{(e/a, de/bc; q)_\infty}{(e, de/abc; q)_\infty}\ {}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; q, e/a\right). \tag{1.10}$$

$$ = \frac{(b, de/ab, de/bc; q)_\infty}{(d, e, de/abc; q)_\infty}\ {}_3\phi_2\left(\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; q, b\right). \tag{1.11}$$

Hahn polynomials are defined as follows [4, 5]:

$$ \phi_n^{(\alpha)}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha; q)_k\, x^k. \tag{1.12}$$

The Cauchy polynomials are defined as follows [6 -9]:

$$ P_n(x, y) = (x - y)(x - qy) \cdots (x - q^{n-1}y) = (y/x; q)_n\ x^n,$$

which has the following generating function:

$$ \sum_{n=0}^{\infty} P_n(x, y) \frac{t^n}{(q; q)_n} = \frac{(yt; q)_\infty}{(xt; q)_\infty}, \quad |xt| < 1.$$

In 1965, Al-Salam and Carlitz [10] defined the following polynomials:

$$ u_n^{(a)}(x; q) = (-a)^n q^{\binom{n}{2}}\ {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}; q, {}^{qx}/_{a}\right).$$

In 2003, Chen et. al [11] presented the homogeneous q-difference operator D_{xy}, which performs on functions in two variables as follows:

$$ D_{xy}\{f(x, y)\} = \frac{f(x, q^{-1}y) - f(qx, y)}{x - q^{-1}y}.$$

In 2010, Chen et. al [12] extended the definition of Al-Salam-Carlitz polynomials as follows:

$$ U_n(x, y, a; q) = (-1)^n q^{\binom{n}{2}} a^n\ {}_2\phi_1\left(\begin{matrix} q^{-n}, {}^{y}/_{x} \\ 0 \end{matrix}; q, {}^{qx}/_{a}\right). \tag{1.13}$$

The Rogers formula for $U_n(x, y, a; q)$ is [12, 13]:

$$ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} U_{n+m}(x, y, a; q) \frac{t^n}{(q; q)_n} \frac{\ell^m}{(q; q)_m}$$

$$ = \frac{(a\ell, y\ell; q)_\infty}{(x\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (x\ell; q)_k (at)^k}{(q; q)_k (a\ell, y\ell; q)_k}\ {}_2\phi_1\left(\begin{matrix} y/x, 0 \\ y\ell q^k \end{matrix}; q, xt\right), \tag{1.14}$$

provided that $\max\{|x\ell|, |xt|\} < 1$.

In 2014, Abdlhusein [14] provided a transformation of ${}_1\phi_1$ series:

$$ {}_1\phi_1\left(\begin{matrix} xt \\ yt \end{matrix}; q, ys\right) = \frac{(xt, ys; q)_\infty}{(yt; q)_\infty} \ {}_2\phi_1\left(\begin{matrix} y/x, 0 \\ ys \end{matrix}; q, xt\right), \quad |yt| < 1. \tag{1.15} $$

In 2020 Cao et al. [15] defined the generalized Verma-Jain polynomials.

$$ \omega_n^{\binom{d,e,f}{g,h}}(u, v, z|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(d, e, f; q)_k}{(g, h; q)_k} z^k P_{n-k}(u, v). \tag{1.16} $$

In 2021 Cao et al. [16] explained the Srivastava-Agarwal type bilinear generating function for the q-polynomials $\omega_n^{\binom{d,e,f}{g,h}}(u, v, z|q)$.

$$ \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)\, \omega_n^{\binom{d,e,f}{g,h}}(u, v, z|q) \frac{t^n}{(q;q)_n} = \frac{(vt, \alpha x; q)_\infty}{(ut, x; q)_\infty} $$

$$ \times \sum_{n=0}^{\infty} \frac{(\alpha, ut; q)_n}{(q, vt, q/x; q)_n} q^n \ {}_3\phi_2\left(\begin{matrix} d, e, f \\ g, h \end{matrix}; q, ztq^n\right), \quad \max\{|ut|, |zt|, |x|\} < 1. \tag{1.17} $$

In 2022, the generalised q-hypergeometric polynomials were defined by Reshem and Saad [17] as follows*:*

$$ \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \frac{(a_1, \cdots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} c^k P_{n-k}(x, y). \tag{1.18} $$

In 2022, Abdul-Ghani and Saad [18] defined the generalised homogeneous q-shift operator ${}_r\Phi_s$ as follows:

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right) = \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(q, b_1, \ldots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} \left(cD_{xy}\right)^k. $$

Abdul-Ghani and Saad [18] found the following identities for the operator ${}_r\Phi_s$:

Theorem 1.1 [18]. *Let the operator* ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)$ *be defined as in abov, then*

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ \mathrm{b}_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\{P_n(x, y)\} = \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q). \tag{1.19} $$

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\left\{\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} = \frac{(yt;q)_\infty}{(xt;q)_\infty} \ {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ct\right), \tag{1.20} $$

provided that $|xt| < 1$.

$$ {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\left\{\frac{P_k(x, y)}{(yt;q)_k}\frac{(yt;q)_\infty}{(xt;q)_\infty}\right\} $$

$$ = \frac{(yt;q)_\infty}{(xt;q)_\infty} t^{-k} \sum_{j=0}^{k} \frac{(q^{-k}, xt; q)_j}{(q, yt; q)_j} q^j \ {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ctq^j\right), \quad |xt| < 1. \tag{1.21} $$

Abdul-Ghani and Saad [18] gave the following results:

Theorem 1.2 [18]. *Let* $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$ *be defined as* (1.18)*, then*

The generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$ is*:*

$$ \sum_{n=0}^{\infty} \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) \frac{t^n}{(q;q)_n} = \frac{(yt;q)_\infty}{(xt;q)_\infty} \ {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ct\right), |xt| < 1. \tag{1.22} $$

The Rogers formula for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$ is*:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\phi_{n+k}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^n}{(q;q)_n}\frac{\ell^k}{(q;q)_k}=\frac{(y\ell;q)_\infty}{(t/\ell,x\ell;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(x\ell;q)_k}{(q,y\ell,q\ell/t;q)_k}q^k\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right),\max\left\{\left|{}^t/_\ell\right|,|\ell x|\right\}<1.\quad(1.23)$$

The Srivastava-Agarwal type generating function for $\phi_n^{(a,b)}(x,y,c|q)$ is:

$$\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)P_n(u,v)\frac{t^n}{(q;q)_n}=\frac{(v/u,yut;q)_\infty}{(xut;q)_\infty}$$
$$\times\sum_{j=0}^{\infty}\frac{(xut;q)_j(v/u)^j}{(q,yut;q)_j}\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,cutq^j\right),\qquad |xut|<1.\quad(1.24)$$

The Srivastava-Agarwal type bilinear generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$ is:

$$\sum_{n=0}^{\infty}\phi_n^{(\alpha)}(x|q)\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(u,v,c|q)\frac{t^n}{(q;q)_n}$$
$$=\frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,ut;q)_n}{(q,vt,q/x;q)_n}q^n\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,ctq^n\right),\quad(1.25)$$

where $\max\{|ut|,|x|\}<1$.

The q-integral was introduced by F.H. Jackson [19] as follows:

$$\int_0^b f(t)d_qt=b(1-q)\sum_{k=0}^{\infty}f(bq^k)q^k.$$

and

$$\int_a^b f(t)d_qt=\int_0^b f(t)d_qt-\int_0^a f(t)d_qt\,.$$

The Askey-Wilson integral is given by [20, 21]:

$$\int_o^\pi\frac{h(\cos2\theta;1)}{h(\cos\theta;a,b,\ell,d)}d\theta=\frac{2\pi(ab\ell d;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d;q)_\infty},\quad(1.26)$$

where $\max\{|a|,|b|,|\ell|,|d|\}<1$and

$$h(\cos\theta;a)=(ae^{i\theta},ae^{-i\theta};q)_\infty.$$
$$h(\cos\theta;a_1,a_2,\ldots,a_n)=h(\cos\theta;a_1)h(\cos\theta;a_2)\ldots h(\cos\theta;a_n).$$

In 2008, Chen and Gu [22] introduced an extension of the Askey–Wilson integral:

$$\int_o^\pi\frac{(e^{2i\theta},e^{-2i\theta},fge^{i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},\ell e^{i\theta},\ell e^{-i\theta},de^{i\theta},de^{-i\theta},ge^{i\theta};q)_\infty}$$
$$\times\ {}_3\phi_2\left(\begin{matrix}f,ae^{i\theta},ae^{-i\theta}\\ fge^{i\theta},ab\end{matrix};q,ge^{-i\theta}\right)d\theta$$
$$=\frac{2\pi(ab\ell d,\ell fg;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d,\ell g;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}f,a\ell,b\ell\\ fg\ell,ab\ell d\end{matrix};q,dg\right),\quad(1.27)$$

where $\max\{|a|,|b|,|\ell|,|d|,|g|\}<1$.

The Andrews-Askey integral is presented as follows [20, 23]:

$$\int_e^f\frac{(qt/e,qt/f;q)_\infty}{(at,bt;q)_\infty}d_qt=\frac{(1-q)f(q,e/f,qf/e,abef;q)_\infty}{(ae,be,af,bf;q)_\infty}.\quad(1.28)$$

In 2018, Liu [24] introduced the q-integral:

$$\int_e^f \frac{(qt/e, qt/f, abut; q)_\infty}{(at, bt, vt; q)_\infty} d_q t = \frac{(1-q)f(q, e/f, qf/e, av\,ef, bvef, abu/v; q)_\infty}{(ae, be, ve, af, bf, vf; q)_\infty} \times {}_3\phi_2\left(\begin{matrix} v\,e, vf, vef/u \\ av\,ef, bvef; q, abu/v \end{matrix}\right). \tag{1.29}$$

We use the generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ in this paper to do the following: We constructed a generalization of numerous well-known q-identities in Section 2, including Hiene's transformation and Jackson's transformation. In Section 3, we present another generating function formula, Rogers Formula, as well as the bilinear generating function of the Srivastava-Agarwal type for the polynomials $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$. Finally, in Section 4, we generalized the Askey-Wilson and Andrews-Askey integrals' extensions.

2. Generalization of q-identities

In this section, we established a generalization of various well-known q-identities, including Hiene's transformation, q-Gauss sum, and Jackson's transformation using the generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$.

Theorem 2.1 (Generalization of Hiene's transformation of ${}_2\phi_1$ series). *For* $\max\{|d|, |x|\} < 1$, *we have*

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, d; q)_n} x^n \; {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, bq^n \\ b_1, \cdots, b_s, dq^n; q, ct \end{matrix}\right) = \frac{(b, ax; q)_\infty}{(d, x; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b, x; q)_n}{(q, ax; q)_n} b^n \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s; q, ctq^n \end{matrix}\right). \tag{2.1}$$

Proof. Setting $a = y/x$ and $x \to xt$ in Heine's transformation (1.8), we get

$$\sum_{n=0}^{\infty} \frac{(b; q)_n}{(q, d; q)_n} t^n P_n(x, y) = \frac{(b; q)_\infty}{(d; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b; q)_n}{(q; q)_n} b^n \frac{(ytq^n; q)_\infty}{(xtq^n; q)_\infty}. \tag{2.2}$$

Applying the operator ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s; q, cD_{xy} \end{matrix}\right)$ on both sides of (2.2), we get

$$\sum_{n=0}^{\infty} \frac{(b; q)_n}{(q, d; q)_n} t^n \; {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s; q, cD_{xy} \end{matrix}\right)\{P_n(x, y)\} = \frac{(b; q)_\infty}{(d; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b; q)_n}{(q; q)_n} b^n \; {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s; q, cD_{xy} \end{matrix}\right)\left\{\frac{(ytq^n; q)_\infty}{(xtq^n; q)_\infty}\right\}.$$

$$\sum_{n=0}^{\infty} \frac{(b; q)_n}{(q, d; q)_n} t^n \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) = \frac{(b; q)_\infty}{(d; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b; q)_n}{(q; q)_n} b^n \frac{(ytq^n; q)_\infty}{(xtq^n; q)_\infty} \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s; q, ctq^n \end{matrix}\right).$$

(By using (1.19) *and* (1.20))

$$\sum_{n=0}^{\infty} \frac{(b; q)_n}{(q, d; q)_n} t^n \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) = \frac{(b, yt; q)_\infty}{(d, xt; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b, xt; q)_n}{(q, yt; q)_n} b^n \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s; q, ctq^n \end{matrix}\right). \tag{2.3}$$

By using (1.18), we obtain
LHS of (2.3)

$$= \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} c^k \sum_{n=k}^{\infty} \frac{(b; q)_n}{(d; q)_n (q; q)_{n-k}} t^n P_{n-k}(x, y)$$

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} c^k \frac{(b; q)_{n+k}}{(d; q)_{n+k} (q; q)_n} t^{n+k} P_n(x, y)$$

$$= \sum_{n=0}^{\infty} \frac{(b; q)_n}{(d; q)_n (q; q)_n} t^n P_n(x, y) \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r, bq^n; q)_k}{(q, b_1, \dots, b_s, dq^n; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k$$

$$= \sum_{n=0}^{\infty} \frac{(y/x, b; q)_n}{(q, d; q)_n} (xt)^n \ {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, bq^n \\ b_1, \cdots, b_s, dq^n \end{matrix}; q, ct\right).$$

Hence equation (2.3) can be written as

$$\sum_{n=0}^{\infty} \frac{(y/x, b; q)_n}{(q, d; q)_n} (xt)^n \ {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, bq^n \\ b_1, \cdots, b_s, dq^n \end{matrix}; q, ct\right) = \frac{(b, yt; q)_\infty}{(d, xt; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/b, xt; q)_n}{(q, yt; q)_n} b^n \ {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ctq^n\right). \quad (2.4)$$

Setting $y/x = a$ and $xt \to x$ in equation (2.4), we get the required result.

Setting $c = 0$ in equation (2.1), we obtain Heine's transformation (1.8).
If $x = d/ab$ in equation (2.1), we get a generalization of q-Gauss sum (1.7).

Corollary 2.1.1 (Generalization of q-Gauss sum). *For* $\max\{|d|, |d/ab|\} < 1$, *we have*

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, d; q)_n} (d/ab)^n \ {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, bq^n \\ b_1, \cdots, b_s, dq^n \end{matrix}; q, ct\right) = \frac{(d/b, d/a; q)_\infty}{(d, d/ab; q)_\infty} \ {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, b \\ b_1, \cdots, b_s, \frac{d}{a} \end{matrix}; q, ct\right). \quad (2.5)$$

Proof. Putting $x = d/ab$ in equation (2.1), we obtain

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, d; q)_n} (d/ab)^n \ {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, bq^n \\ b_1, \cdots, b_s, dq^n \end{matrix}; q, ct\right)$$

$$= \frac{(b, d/b; q)_\infty}{(d, d/ab; q)_\infty} \sum_{n=0}^{\infty} \frac{(d/ab; q)_n}{(q; q)_n} b^n \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ctq^n)^k$$

$$= \frac{(b, d/b; q)_\infty}{(d, d/ab; q)_\infty} \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_k}{(q, b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k \sum_{n=0}^{\infty} \frac{(d/ab; q)_n}{(q; q)_n} (bq^k)^n$$

$$= \frac{(d/b, d/a; q)_\infty}{(d, d/ab; q)_\infty} \sum_{k=0}^{\infty} \frac{(a_1, \cdots, a_r, b; q)_k}{(q, b_1, \dots, b_s, d/a; q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} (ct)^k \quad \text{(by using (1.2))}$$

$$= \frac{(d/b, d/a; q)_\infty}{(d, d/ab; q)_\infty} \ {}_{r+1}\phi_{s+1}\left(\begin{matrix} a_1, \cdots, a_r, b \\ b_1, \cdots, b_s, d/a \end{matrix}; q, ct\right).$$

- For $c = 0$ in equation (2.5), we obtain q-Gauss sum (1.7).

- Setting $b = 0$ and $d = 0$ in equation (2.3), we obtain the generating function for $\phi_n^{(a,b)}(x, y, c|q)$ (equation (1.22)).

• Letting $d = 0$, $b = v/u$ and $t \to tu$ in equation (2.3), we obtain the Srivastava- Agarwal type generating function for $\phi_n^{(a,b)}(x, y, c|q)$ (equation (1.24)).

Theorem 2.2 (Generalization of Jackson's transformation of ${}_2\phi_1$ series). *For* $|x| < 1$, *we have*

$$\sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, d; q)_n} x^n \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cq^n\right)$$

$$= \frac{(ax; q)_\infty}{(x; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a; q)_n}{(q, ax; q)_n} x^n \sum_{k=0}^{n} \frac{(q^{-n}, b; q)_k}{(q, d; q)_k} q^k \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cq^k\right), \quad (2.6)$$

Proof. Setting $b = x$, $d = y$ and $x = f$ in equation (1.9), we get

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} f^n \frac{(x; q)_n}{(y; q)_n} = \frac{(af; q)_\infty}{(f; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a; q)_n}{(q, af; q)_n} f^n \frac{P_n(x, y)}{(y; q)_n}$$

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} f^n \frac{(yq^n; q)_\infty}{(xq^n; q)_\infty} = \frac{(af; q)_\infty}{(f; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a; q)_n}{(q, af; q)_n} f^n \frac{P_n(x, y)}{(y; q)_n} \frac{(y; q)_\infty}{(x; q)_\infty}. \quad (2.7)$$

Applying the operator ${}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)$ on both sides of equation (2.7), we get

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} f^n \; {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\left\{\frac{(x; q)_n}{(y; q)_n}\right\}$$

$$= \frac{(af; q)_\infty}{(f; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a; q)_n}{(q, af; q)_n} f^n \; {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cD_{xy}\right)\left\{\frac{P_n(x, y)}{(y; q)_n} \frac{(y; q)_\infty}{(x; q)_\infty}\right\}$$

$$\sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} f^n \frac{(yq^n; q)_\infty}{(xq^n; q)_\infty} \; {}_r\Phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cq^n\right)$$

$$= \frac{(af; q)_\infty}{(f; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (a; q)_n}{(q, af; q)_n} f^n$$

$$\times \frac{(y; q)_\infty}{(x; q)_\infty} \sum_{k=0}^{n} \frac{(q^{-n}, x; q)_k}{(q, y; q)_k} q^k \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, cq^k\right). \quad (2.8)$$

(By using (1.20) and (1.21))

Setting $x = b$, $y = d$ and $f = x$ in equation (2.8), we get the desired result.

• For $c = 0$ in equation (2.6) and by using equation (1.5), we obtain equation (1.9).

3. Another formula to the identities for $\phi_n^{(a,b)}(x, y, c|q)$

In this section, we use the generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$ to give another formula for: the generating function, the Rogers formula and the Srivastava-Agarwal type bilinear generating functions for polynomials $\phi_n^{(a,b)}(x, y, c|q)$. By providing some specific value, we recover the generating function, Rogers Formula and Srivastava-Agarwal type bilinear generating functions for the polynomials $\phi_n^{(a,b)}(x, y, c|q)$.

Theorem 3.1 (Another generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$). *Let* $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$ *be defined as* in (1.18), *then*

$$\sum_{n=0}^{\infty}\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^n}{(q,\lambda;q)_n} = \frac{(yt;q)_\infty}{(xt,\lambda;q)_\infty}\ {}_r\phi_s\begin{pmatrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{pmatrix};q,ct\Bigg)\ {}_1\phi_1\begin{pmatrix}xt\\ yt\end{pmatrix};q,\lambda\Bigg),\quad |xt|<1. \tag{3.1}$$

Proof. Setting $s\to\lambda/y$ in equation (1.15), we get

$$\frac{1}{(\lambda;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}\lambda^n}{(q;q)_n}\frac{(ytq^n;q)_\infty}{(xtq^n;q)_\infty}=\sum_{n=0}^{\infty}P_n(x,y)\frac{t^n}{(q,\lambda;q)_n}. \tag{3.2}$$

Applying the operator ${}_r\Phi_s\begin{pmatrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s;q,cD_{xy}\end{pmatrix}$ to both sides of (3.2) and then using (1.19) and (1.20), we obtain (3.1).

Setting $\lambda=0$ in equation (3.1), we obtain the generating function for the polynomials $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$ (equation (1.22)).

Theorem 3.2 (Another Rogers formula for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$). *Let* $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$ *be defined as in* (1.18), *then*

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\phi_{i+j}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^i}{(q;q)_i}\frac{\ell^j}{(q;q)_j}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(atq^j)^n}{(q,a\ell;q)_n}=\frac{(y\ell;q)_\infty}{(x\ell,t/\ell;q)_\infty}$$
$$\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(x\ell;q)_k\ (at)^k}{(q,y\ell,a\ell;q)_k}\sum_{m=0}^{\infty}\frac{(-1)^m q^{-\binom{m}{2}}(x\ell q^k;q)_m\ (q^{-k}t/\ell)^m}{(q,y\ell q^k;q)_m(t/\ell q^{-(k+m)};q)_{m+k}}$$
$$\times\ {}_r\phi_s\begin{pmatrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s;q,c\ell q^{k+m}\end{pmatrix},\quad \max\{|x\ell|,|t/\ell|\}<1. \tag{3.3}$$

Proof. We will prove this theorem by using equation (1.14).

$$\text{LHS of (1.14)} =\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}U_{n+m}(x,y,a;q)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{\binom{n+m}{2}}a^{n+m}\ {}_2\phi_1\begin{pmatrix}q^{-(n+m)},y/x\\ 0\end{pmatrix};q,qx/a\Bigg)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}.$$
(By using (1.13))
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n+m}\begin{bmatrix}n+m\\ k\end{bmatrix}(-1)^{n+m+k}q^{\binom{n+m}{2}+\binom{k}{2}-(n+m)k+k}a^{n+m-k}P_k(x,y)$$
$$\times\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}.\quad \text{(By using (1.1))}$$

$$\text{RHS of (1.14)}=\frac{(a\ell,y\ell;q)_\infty}{(x\ell;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(x\ell;q)_k(at)^k}{(q;q)_k(a\ell,y\ell;q)_k}\sum_{n=0}^{\infty}\frac{(y/x;q)_n(xt)^n}{(q,y\ell q^k;q)_n}$$
$$=(a\ell;q)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(at)^k}{(q;q)_k(a\ell;q)_k}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{P_n(x,y)}{(y\ell q^k;q)_n}\frac{(y\ell q^k;q)_\infty}{(x\ell q^k)_\infty}.$$

As a result, equation (1.14) can be rewritten as follows:

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n+m}\begin{bmatrix}n+m\\k\end{bmatrix}(-1)^{n+m+k}q^{\binom{n+m}{2}+\binom{k}{2}-(n+m)k+k}a^{n+m-k}P_k(x,y)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$
$$=(a\ell;q)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(at)^k}{(q;q)_k(a\ell;q)_k}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}\frac{P_n(x,y)}{(y\ell q^k;q)_n}\frac{(y\ell q^k;q)_\infty}{(x\ell q^k)_\infty}. \tag{3.4}$$

Applying the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)$ for both sides of equation (3.4), we obtain

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n+m}\begin{bmatrix}n+m\\k\end{bmatrix}(-1)^{n+m+k}q^{\binom{n+m}{2}+\binom{k}{2}-(n+m)k+k}a^{n+m-k}\phi_k^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c)$$
$$\times\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$
$$=(a\ell;q)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(at)^k}{(q;q)_k(a\ell;q)_k}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}(\ell q^k)^{-n}\frac{(y\ell q^k;q)_\infty}{(x\ell q^k)_\infty}\sum_{m=0}^{n}\frac{(q^{-n},x\ell q^k;q)_m}{(q,y\ell q^k;q)_m}\,q^m$$
$$\times\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\b_1,\cdots,b_s\end{matrix};q,c\ell q^{k+m}\right).\quad\big(\text{By using (1.19) and (1.21)}\big) \tag{3.5}$$

LHS of (3.5)

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n+m}\begin{bmatrix}n+m\\k\end{bmatrix}(-1)^{n+m+k}q^{\binom{n+m}{2}+\binom{k}{2}-(n+m)k+k}a^{n+m-k}\phi_k^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$$
$$\times\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{n+m}\sum_{i=0}^{k}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}m\\k-i\end{bmatrix}q^{(n-i)(k-i)}(-1)^{n+m+k}q^{\binom{n+m}{2}+\binom{k}{2}-(n+m)k+k}a^{n+m-k}$$
$$\times\phi_k^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}.\quad(\text{By using (1.6)})$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{m}\begin{bmatrix}n\\i\end{bmatrix}\begin{bmatrix}m\\j\end{bmatrix}q^{(n-i)j}(-1)^{n+m+i+j}q^{\binom{n+m}{2}+\binom{i+j}{2}-(n+m)(i+j)+(i+j)}$$
$$\times a^{n+m-(i+j)}\phi_{i+j}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^n}{(q;q)_n}\frac{\ell^m}{(q;q)_m}$$
$$=\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}(-1)^n q^{\binom{n}{2}}(aq^j)^n\phi_{i+j}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^{n+i}}{(q;q)_i(q;q)_n}\frac{\ell^j}{(q;q)_j}$$
$$\times\sum_{m=0}^{\infty}\frac{(-1)^m q^{\binom{m}{2}}(a\ell q^n)^m}{(q;q)_m}$$
$$=(a\ell;q)_\infty\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\phi_{i+j}^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)\frac{t^i}{(q;q)_i}\frac{\ell^j}{(q;q)_j}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\binom{n}{2}}(atq^j)^n}{(q,a\ell;q)_n}.$$

(By using (1.4))

RHS of (3.5)

$$=(a\ell;q)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}(at)^k}{(q;q)_k(a\ell;q)_k}\sum_{n=0}^{\infty}\frac{t^n}{(q;q)_n}(\ell q^k)^{-n}\frac{(y\ell q^k;q)_\infty}{(x\ell q^k)_\infty}$$

$$\times \sum_{m=0}^{n} \frac{(q^{-n}, x\ell q^k; q)_m}{(q, y\ell q^k; q)_m} \; q^m \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, c\ell q^{k+m}\right)$$

$$= \frac{(a\ell, y\ell; q)_\infty}{(x\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (x\ell; q)_k (at)^k}{(q; q)_k (, y\ell, a\ell; q)_k} \sum_{n=0}^{\infty} \frac{(t/\ell q^{-k})^n}{(q; q)_n}$$

$$\times \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix} q^{\binom{m}{2} - nm + m} \frac{(x\ell q^k; q)_m}{(y\ell q^k; q)_m} \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, c\ell q^{k+m}\right)$$

(By using (1.1))

$$= \frac{(a\ell, y\ell; q)_\infty}{(x\ell; \mathrm{q})_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (x\ell; q)_k (at)^k}{(q; q)_k (, y\ell, a\ell; q)_k} \sum_{m=0}^{\infty} (-1)^m q^{\binom{m}{2} + m} \frac{(x\ell q^k; q)_m}{(q, y\ell q^k; q)_m}$$

$$\times \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, c\ell q^{k+m}\right) \sum_{n=m}^{\infty} \frac{(t/\ell q^{-k})^n q^{-nm}}{(q; q)_{n-m}}$$

$$= \frac{(a\ell, y\ell; q)_\infty}{(x\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (x\ell; q)_k (at)^k}{(q; q)_k (, y\ell, a\ell; q)_k} \sum_{m=0}^{\infty} (-1)^m q^{-\binom{m}{2}} \frac{(x\ell q^k; q)_m}{(q, y\ell q^k; q)_m}$$

$$\times (t/\ell q^{-k})^m \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, c\ell q^{k+m}\right) \frac{1}{(q^{-(m+k)} t/\ell; q)_\infty} \qquad \left(\text{By using (1.3)}\right)$$

$$= \frac{(a\ell, y\ell; q)_\infty}{(x\ell, t/\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (x\ell; q)_k (at)^k}{(q; q)_k (, y\ell, a\ell; q)_k} \sum_{m=0}^{\infty} \frac{(-1)^m q^{-\binom{m}{2}} (t/\ell q^{-k})^m (x\ell q^k; q)_m}{(q, y\ell q^k; q)_m (t/\ell q^{-(k+m)}; q)_{m+k}}$$

$$\times \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, c\ell q^{k+m}\right).$$

Consequently, equation (3.5) might be rewritten as follows:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi_{i+j}^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q) \frac{t^i}{(q; q)_i} \frac{\ell^j}{(q; q)_j} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} (atq^j)^n}{(q, a\ell; q)_n}$$

$$= \frac{(y\ell; q)_\infty}{(\mathrm{x}\ell, t/\ell; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} (x\ell; q)_k (at)^k}{(q; q)_k (, y\ell, a\ell; q)_k} \sum_{m=0}^{\infty} \frac{(-1)^m q^{-\binom{m}{2}} (q^{-k} t/\ell)^m (x\ell q^k; q)_m}{(q, y\ell q^k; q)_m (t/\ell q^{-(k+m)}; q)_{m+k}}$$

$$\times \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, c\ell q^{k+m}\right).$$

- Setting $a = 0$ in equation (3.3), we obtain Rogers formula for the polynomials $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$ (equation (1.23)).

Theorem 3.3 (Another Srivastava-Agarwal type bilinear generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$). *Let* $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x, y, c|q)$ *be defined as in* (1.18), *then*

$$\sum_{k=0}^{\infty} \frac{(d, e, f; q)_k}{(q, g, h; q)_k} (zt)^k \sum_{n=0}^{\infty} \phi_{n+k}^{(\alpha)}(x|q) \phi_n^{(\boldsymbol{a},\boldsymbol{b})}(u, v, c|q) \frac{t^n}{(q; q)_n} = \frac{(vt, \alpha x; q)_\infty}{(ut, x; q)_\infty}$$

$$\times \sum_{n=0}^{\infty} \frac{(\alpha, ut; q)_n}{(q, vt, q/x; q)_n} q^n \; {}_3\phi_2\left(\begin{matrix} d, e, f \\ g, h \end{matrix}; q, ztq^n\right) \; {}_r\phi_s\left(\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}; q, ctq^n\right), \qquad (3.6)$$

provided that $\max\{|ut|, |ct|, |x|\} < 1$.

Proof. Equation (1.17) will be used to prove this theorem.

$$\text{LHS of (1.17)} = \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|q)\omega_n^{\binom{d,e,f}{g,h}}(u,v,z|q)\frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty} \phi_n^{(\alpha)}(x|\mathrm{q})\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\frac{(d,e,f;q)_k}{(g,h;q)_k}z^k P_{n-k}(u,v) \quad \text{(By using (1.16))}$$

$$= \sum_{k=0}^{\infty}\frac{(d,e,f;q)_k}{(g,h;q)_k}(zt)^k\sum_{n=0}^{\infty}\phi_{n+k}^{(\alpha)}(x|q)\frac{t^n}{(q;q)_n}P_n(u,v).$$

$$\text{RHS of (1.17)} = \frac{(vt,\alpha x;q)_\infty}{(ut,x;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha,ut;q)_n}{(q,vt,q/x;q)_n}q^n\ {}_3\phi_2\left(\begin{matrix} d,e,f \\ g,h \end{matrix};q,ztq^n\right)$$

$$= \frac{(\alpha x;q)_\infty}{(x;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q,q/x;q)_n}q^n\ {}_3\phi_2\left(\begin{matrix} d,e,f \\ g,h \end{matrix};q,ztq^n\right)\frac{(vtq^n;q)_\infty}{(utq^n;q)_\infty}.$$

Equation (1.17) can be rewritten as

$$\sum_{k=0}^{\infty}\frac{(d,e,f;q)_k}{(q,g,h;q)_k}(zt)^k\sum_{n=0}^{\infty}\phi_{n+k}^{(\alpha)}(x|q)P_n(u,v)\frac{t^n}{(q;q)_n}$$
$$= \frac{(\alpha x;q)_\infty}{(x;q)_\infty}\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q,q/x;q)_n}q^n\ {}_3\phi_2\left(\begin{matrix} d,e,f \\ g,h \end{matrix};q,ztq^n\right)\frac{(vtq^n;q)_\infty}{(utq^n;q)_\infty}. \tag{3.7}$$

Applying the operator ${}_r\Phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,cD_{uv}\right)$ for both sides of (3.7) and by using (1.19) and (1.20), we get the required result.

Setting $r=3$, $s=2$, $z=0$, $c=z$, $a=(d,e,f)$ and $b=(g,h)$ in equation (3.6), we obtain Srivastava-Agarwal type bilinear generating functions for $\omega_n^{(d,e,f,g,h)}(u,v,z|q)$ (equation (1.17)).

Setting $z=0$ in equation (3.6), we obtain Srivastava-Agarwal type bilinear generating function for $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$ (equation (1.25)).

4. Generalization of q-integral

By utilizing the generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$, we generalized an extesion of the Askey-Wilson integral and an extesion the Andrews-Askey integral in this section.

Theorem 4.1 (Generalization an extension of the Askey-Wilson integral). *For* $\max\{|a|,|b|,|\ell|,|d|,|g|\}<1$, *we have*

$$\int_0^{\pi}\frac{(e^{2i\theta},e^{-2i\theta},fge^{i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},\ell e^{i\theta},\ell e^{-i\theta},de^{i\theta},de^{-i\theta},ge^{i\theta};q)_\infty}\sum_{n=0}^{\infty}\frac{(ae^{i\theta},be^{i\theta};q)_n}{(q,ab;q)_n}$$
$$\times e^{-2in\theta}\sum_{k=0}^{n}\frac{(q^{-n},ge^{i\theta};q)_k}{(q,fge^{i\theta};q)_k}q^k\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,ce^{i\theta}q^k\right)d\theta$$
$$= \frac{2\pi(ab\ell d,\ell fg;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d,\ell g;q)_\infty}\sum_{n=0}^{\infty}\frac{(a\ell,b\ell;q)_n}{(q,ab\ell d;q)_n}(d/\ell)^n\sum_{k=0}^{n}\frac{(q^{-n},g\ell;q)_k}{(q,fg\ell;q)_k}q^k$$
$$\times\ {}_r\phi_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix};q,c\ell q^k\right), \tag{4.1}$$

Proof. Equation (1.27) can be written as

$$\int_{o}^{\pi}\frac{(e^{2i\theta},e^{-2i\theta},fge^{i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},\ell e^{i\theta},\ell e^{-i\theta},de^{i\theta},de^{-i\theta},ge^{i\theta};q)_\infty}\times\sum_{n=0}^{\infty}\frac{(f,ae^{i\theta},be^{i\theta};q)_n}{(q,fge^{i\theta},ab;q)_n}(ge^{-i\theta})^n d\theta$$
$$=\frac{2\pi(\ell gf,ab\ell d;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d,\ell g;q)_\infty}\sum_{n=0}^{\infty}\frac{(f,a\ell,b\ell;q)_n}{(q,\ell fg,abd\ell;q)_n}(gd)^n. \quad (4.2)$$

Let $f=y/x$, $g=x$ in (4.2), we obtain

$$\int_{o}^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},\ell e^{i\theta},\ell e^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}\sum_{n=0}^{\infty}\frac{(ae^{i\theta},be^{i\theta};q)_n}{(q,ab;q)_n}\times(e^{-i\theta})^n\frac{P_n(x,y)}{(ye^{i\theta};q)_n}\frac{(ye^{i\theta};q)_\infty}{(xe^{i\theta};q)_\infty}d\theta$$
$$=\frac{2\pi(ab\ell d;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d;q)_\infty}\sum_{n=0}^{\infty}\frac{(a\ell,b\ell;q)_n}{(q,abd\ell;q)_n}d^n\frac{P_n(x,y)}{(y\ell;q)_n}\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}. \quad (4.3)$$

Applying the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)$ on both sides of equation (4.3), we have

$$\int_{o}^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},\ell e^{i\theta},\ell e^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}\sum_{n=0}^{\infty}\frac{(ae^{i\theta},be^{i\theta};q)_n}{(q,ab;q)_n}\times(e^{-i\theta})^{\mathrm{n}}\ {}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\left\{\frac{P_n(x,y)}{(ye^{i\theta};q)_n}\frac{(ye^{i\theta};q)_\infty}{(xe^{i\theta};q)_\infty}\right\}d\theta$$
$$=\frac{2\pi(ab\ell d;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d;q)_\infty}\sum_{n=0}^{\infty}\frac{(a\ell,b\ell;q)_n}{(q,abd\ell;q)_n}d^n\times\ {}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,cD_{xy}\right)\left\{\frac{P_n(x,y)}{(y\ell;q)_n}\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}\right\}. \quad (4.4)$$

$$\int_{o}^{\pi}\frac{(e^{2i\theta},e^{-2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta},be^{i\theta},be^{-i\theta},\ell e^{i\theta},\ell e^{-i\theta},de^{i\theta},de^{-i\theta};q)_\infty}\sum_{n=0}^{\infty}\frac{(ae^{i\theta},be^{i\theta};q)_n}{(q,ab;q)_n}(e^{-i\theta})^n\times\frac{(ye^{i\theta};\mathrm{q})_\infty}{(xe^{i\theta};q)_\infty}(e^{-i\theta})^n\sum_{k=0}^{n}\frac{(q^{-n},xe^{i\theta};q)_k}{(q,ye^{i\theta};q)_k}q^k\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,ce^{i\theta}q^k\right)d\theta$$
$$=\frac{2\pi(ab\ell d;q)_\infty}{(q,ab,a\ell,ad,b\ell,bd,\ell d;q)_\infty}\sum_{n=0}^{\infty}\frac{(a\ell,b\ell;q)_n}{(q,abd\ell;q)_n}d^n\frac{(y\ell;q)_\infty}{(x\ell;q)_\infty}(\ell)^{-n}\sum_{k=0}^{n}\frac{(q^{-n},x\ell;q)_k}{(q,y\ell;q)_k}q^k\times\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,c\ell q^k\right).\quad\text{(By using (1.21))} \quad (4.5)$$

Replacing $(y/x,x)$ by (f,g) in (4.5), we obtain equation (4.1).

For $c=0$ and then $g=0$ in equation (4.1), we obtain the Askey-Wilson integral (1.26).
For $c=0$ in equation (4.1), we obtain the extension of the Askey-Wilson integral (1.27).

Theorem 4.2 (Generalization an extension of the Andrews-Askey integral). *For* $\max\{|at|,|bt|,|vt|,|ae|,|be|,|af|,|bf|,|ve|,|f/e|\}<1$, *we have*

$$\int_e^f \frac{(qt/e,qt/f,abut;q)_\infty}{(at,bt,vt;q)_\infty}\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,ct\right)d_qt$$
$$=\frac{(1-q)f(q,e/f,qf/e,abef,abue;q)_\infty}{(ae,be,af,bf,ve;q)_\infty}\sum_{n=0}^{\infty}\frac{(ae,be;q)_n}{(q,abef;q)_n}(f/e)^n$$
$$\times\sum_{k=0}^{n}\frac{(q^{-n},ve;q)_k}{(q,abue;q)_k}q^k\ {}_r\phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,ceq^k\right). \quad (4.6)$$

Proof. Using equation (1.11), equation (1.29) can be written as

$$\int_e^f \frac{(qt/e,qt/f,abut;q)_\infty}{(at,bt,vt;q)_\infty}d_qt$$
$$=\frac{(1-q)f(q,e/f,qf/e,abef,abeu;q)_\infty}{(ae,be,ve,af,bf;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}ae,be,{}^{abu}/_v\\ abef,abeu\end{matrix};q,vf\right). \quad (7.4)$$

Let $abu=v$ and $v=u$ in (4.6), we have

$$\int_e^f \frac{(qt/e,qt/f;q)_\infty}{(at,bt;q)_\infty}\frac{(vt;q)_\infty}{(ut;q)_\infty}d_qt$$
$$=\frac{(1-q)f(q,e/f,qf/e,abef;q)_\infty}{(ae,be,af,bf;q)_\infty}\sum_{n=0}^{\infty}\frac{(ae,be;q)_n}{(q,abef;q)_n}f^n\frac{P_n(u,v)}{(ve;q)_n}\frac{(ve;q)_\infty}{(ue;q)_\infty}. \quad (4.8)$$

Applying the operator ${}_r\Phi_s\left(\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,cD_{uv}\right)$ on both sides of equation (4.8) and by using equations (1.19) and (1.21), then replacing (v,u) by (abu,v) in (4.8), we obtain equation (4.6).

For $c=0$ and $abu=v$ in equation (4.6), we obtain Andrews-Askey integral (1.28).

For $c=0$ in (4.6) and by using equations (1.5) and (1.10), we obtain the following corollary:

Corollary 4.2.2 *For* $\max\{|at|,|bt|,|vt|,|ae|,|be|,|ve|,|af|,|\mathrm{b}f|,|vf|,|bu|\}<1$, *we have*

$$\int_e^f \frac{(qt/e,qt/f,abut;q)_\infty}{(at,bt,vt;q)_\infty}d_qt$$
$$=\frac{(1-q)f(q,e/f,qf/e,abef,aefv,bu;q)_\infty}{(ae,be,ve,af,\mathrm{b}f,vf;q)_\infty}\ {}_3\phi_2\left(\begin{matrix}ae,af,{}^{evf}/_u\\ abef,aefv\end{matrix};q,bu\right). \quad (4.9)$$

Conclusions

We perform the following using the generalized homogeneous q-shift operator ${}_r\Phi_s(D_{xy})$:

1. We generalized some well-known q-identities.
2. We offer an additional formulas for the polynomials $\phi_n^{(\boldsymbol{a},\boldsymbol{b})}(x,y,c|q)$.
3. We generalized an extension of some well-known integrals.

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