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Some Results of Fixed Point for Single Value Mapping in Fuzzy Normed Space with Applications

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Abstract

Fixed point theory is not only essential in mathematics but also to a vast array of applications. In this approach, we provide some fixed point theorems for fuzzy normed space. The triangle property of a fuzzy norm is introduced first. This property is used to demonstrate some fixed point solutions for self-mappings in fuzzy normed space. To demonstrate the importance of the obtained results, an application for the existence of a solution to the ordinary differential equation and the Fredholm integral equation is constructed.

Keywords: Fixed point, Fuzzy norm, Fuzzy normed space, Triangular property.

بعض نتائج النقطة الصامدة للدوال ذات القيمة الفردية في الفضاء المعياري االضبابي مع التطبيقات

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الخلاصة

نظرية النقطة الصامدة ليست ضرورية فقط في الرياضيات ولكن أيضًا لمجموعة واسعة من التطبيقات. في هذا النهج ، نقدم بعض نظريات النقطة الصامدة للفضاء المعياري الضبابي. أولاً يتم تقديم خاصية المثلث لمعيار ضبابي. تُستخدم هذه الخاصية لتوضيح بعض حلول النقاط الصامدة للدوال الذاتية في فضاء معياري ضبابي. لإثبات أهمية النتائج التي تم الحصول عليها ، تم إنشاء تطبيق لوجود حل للمعادلة التفاضلية الاعتيادية ومعادلة فريدهولم التكاملية.

1. Introduction and Preliminaries

The Banach contraction principle (BCP) states that a self-mapping in a complete metric space meeting a contraction condition possesses a unique fixed point [1]. Due to its relevance, several authors have generalized the BCP theorem in different ways. On other hand, functional analysis is one of the most important areas of contemporary mathematics. It plays an essential

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role In the theory of differential equations, representation theory, and probability, as well as in the study of many different properties of different spaces, such as metric space, Hilbert space, Banach space, and others see[2-8]. Zadeh[9] presented the ingenious notion of a fuzzy set in his renowned paper, many mathematicians became aware of the myriad possibilities for expanding the classical conclusions in the new fuzzy framework and of their countless applications.

Kramosil and Michalek[10] were the pioneers of the notion of fuzzy metric spaces. Following this, the notion of fuzzy metric space is modified by George and Veeramani [11]. There have been several papers pertaining to fuzzy metric spaces see[12-14]. Katsaras [15] was the first to propose the concept of fuzzy norms in linear spaces.

Many other mathematicians, such as Felbin[16], Cheng and Mordeson[17], and others, afterward presented the concept of fuzzy normed linear spaces in various ways. Besides, fuzzy normed linear spaces have been studied in several works, see[18-23]. In this paper, we will utilize Nadaban and Dzitac's notion of the fuzzy norm[24]. Assume that L is a non-empty vector space over the field $\mathbb{F}$ ($\mathbb{C}$ *or* $\mathbb{R}$). A fuzzy normed space is represented by the triplet (L, $\tilde{\mathcal{F}}_{\mathcal{N}}$,⊛), where ⊛ is a continuous t-norm and $\tilde{\mathcal{F}}_{\mathcal{N}}$ represent a fuzzy set on $L \times \mathbb{R}$ fulfilling the following for all x, y ∈L:

(1) $\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}, 0) = 0$,
(2) $\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}, \tau) = 1, \forall \tau > 0$ if and only if $\mathrm{x} = 0$,
(3) $\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{rx}, \tau) = \tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}, \tau/|\mathrm{r}|)$, for each $0 \neq \mathrm{r} \in \mathbb{R}, \tau \geq 0$
(4) $\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}, \tau) \circledast \tilde{\mathcal{F}}_{\mathcal{N}}(y, s) \leq \tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} + y, \tau + s), \forall \tau, s \geq 0$
(5) $\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}, .)$is left continuous for all $\mathrm{x} \in L$, and $\lim_{\tau \to \infty} \tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}, \tau) = 1$.

In this paper, we use the triangular property of fuzzy normed space to prove some fixed point theorems. The fixed point theorems are used to study the existence of a unique solution of the ordinary differential equation and the Fredholm integral equation. The rest of this study consists of the following subsections: In Section 2, we offer the triangle property of a fuzzy norm and show the fixed point theorem for a fuzzy normed space. Our main results are applied in the last portion.

2. MAIN RESULTS

In this section, the triangular property of the fuzzy norm is introduced. Using this property, some fixed point theorems for contractive self-mappings in a fuzzy normed space are established.

Definition 2.1: Let $(L, \tilde{\mathcal{F}}_{\mathcal{N}}, \circledast)$ be a fuzzy normed space. The fuzzy norm $\tilde{\mathcal{F}}_{\mathcal{N}}$ termed as triangular if the condition described below are met:

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} - y, \tau)} - 1 \leq \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} - z, \tau)} - 1\right) + \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(y - z, \tau)} - 1\right)$$

for each $\mathrm{x}, y, z \in L$ and $\tau > 0$.

Theorem 2.2: Suppose that (L,$\tilde{\mathcal{F}}_{\mathcal{N}}$, ⊛) is a fuzzy Banach space, $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular. Let $\Gamma: \mathrm{L} \to \mathrm{L}$ be a mapping with

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\Gamma\mathrm{x} - \Gamma y, \tau)} - 1 \leq \lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} - y, \tau)} - 1\right) + \lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} - \Gamma y, \tau), \tilde{\mathcal{F}}_{\mathcal{N}}(y - \Gamma\mathrm{x}, \tau)\}} - 1\right)$$
$$+\lambda_3\left(\frac{1}{\max\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} - y, \tau), \tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x} - \Gamma y, \tau), \tilde{\mathcal{F}}_{\mathcal{N}}(y - \Gamma\mathrm{x}, \tau)\}} - 1\right)$$

$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}-\Gamma\mathrm{x},\tau)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(y-\Gamma y,\tau)}-1\right) \tag{1}$$

for all $\mathrm{x}, y \in \mathrm{L}$, $\tau > 0$, λ_1, $\lambda_2, \lambda_3, \lambda_4 \in [0,1)$ with $\lambda_1 + 2\lambda_2 + 2\lambda_4 < 1$. Then Γ possesses a fixed point in L**.**

Proof: Let $\mathrm{x}_\circ \in L$ and construct a sequence of points in L such that

$$\Gamma\mathrm{x}_j = \mathrm{x}_{j+1}$$

$j \geq 0$. Then,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1=\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\Gamma\mathrm{x}_{j-1}-\Gamma\mathrm{x}_j,\tau)}-1$$

$$\leq \lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right)+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\Gamma\mathrm{x}_j,\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\Gamma\mathrm{x}_{j-1},\tau)\}}-1\right)$$

$$+\lambda_3\left(\frac{1}{\max\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\Gamma\mathrm{x}_j,\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\Gamma\mathrm{x}_{j-1},\tau)\}}-1\right)$$

$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\Gamma\mathrm{x}_{j-1},\tau)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\Gamma\mathrm{x}_j,\tau)}-1\right)$$

$$=\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right)+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_{j+1},\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_j,\tau)\}}-1\right)$$

$$+\lambda_3\left(\frac{1}{\max\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_{j+1},\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_j,\tau)\}}-1\right)$$

$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\right)$$

$$=\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right)+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_{j+1},\tau),1\}}-1\right)$$

$$+\lambda_3\left(\frac{1}{\max\{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau),\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_{j+1},\tau),1\}}-1\right)+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right.$$

$$\left.+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\right)$$

$$=\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right)+\lambda_2\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_{j+1},\tau)}-1\right)\quad+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1+\right.$$

$$\left.\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\right).$$

Further by the triangular property of $\tilde{\mathcal{F}}_{\mathcal{N}}$ we have,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\leq\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right)$$

$$+\lambda_2\left(\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right)+\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\right)\right)$$

$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\right).$$

Following simplification, we arrive at:

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_j-\mathrm{x}_{j+1},\tau)}-1\leq\vartheta\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\mathrm{x}_{j-1}-\mathrm{x}_j,\tau)}-1\right) \tag{2}$$

where $\vartheta = \frac{\lambda_1+\lambda_2+\lambda_4}{1- \lambda_2- \lambda_4} < 1$, since $\lambda_2, \lambda_3, \lambda_4 \geq 0$ with $\lambda_1 + 2\lambda_2 + 2\lambda_4 < 1$

Now, by induction and using (2), we obtain
$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(x_j - x_{j+1}, \tau)} - 1 \leq \vartheta^{j}\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(x_{j-1} - x_j, \tau)} - 1\right) \to 0$ as $j \to \infty$.
Accordingly,

$$\lim_{j\to\infty} \tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - x_{j+1}, \tau\right) = 1, \text{ for } \tau > 0. \quad (3)$$

Since $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular, we have

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - x_k, \tau\right)} - 1 \leq \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - x_{j+1}, \tau\right)} - 1\right) + \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j+1} - x_{j+2}, \tau\right)} - 1\right)$$
$$+ \cdots \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{k-1} - x_k, \tau\right)} - 1\right)$$
$$\leq \left(\vartheta^{j} + \vartheta^{j+1} + \cdots + \vartheta^{k-1}\right)\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_\circ - x_1, \tau\right)} - 1\right)$$
$$\leq \frac{\vartheta^{j}}{1-\vartheta^{j}}\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_\circ - x_1, \tau\right)} - 1\right) \to 0 \text{ as } j \to \infty.$$

Consequently, $\{x_j\}$ is Cauchy sequence in L. By assumption L is complete. Thus there is $y_1 \in$ L such that

$$\lim_{j\to\infty} \tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - y_1, \tau\right) = 1, \text{ for } \tau > 0. \quad (4)$$

Now, we prove that $\Gamma y_1 = y_1$. Since $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - \Gamma y_1, \tau\right)} - 1 \leq \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - x_j, \tau\right)} - 1\right) + \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - \Gamma y_1, \tau\right)} - 1\right) \quad (5)$$

for $\tau > 0$. By (1), (3), and (4) we have,

$$\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - \Gamma y_1, \tau\right)} - 1\right) = \frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\Gamma x_{j-1} - \Gamma y_1, \tau\right)}$$
$$\leq \lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - y_1, \tau\right)} - 1\right)$$
$$+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - \Gamma y_1, \tau\right), \tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - \Gamma x_{j-1}, \tau\right)\}} - 1\right)$$
$$+\lambda_3\left(\frac{1}{\max\{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - y_1, \tau\right), \tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - \Gamma y_1, \tau\right), \tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - \Gamma x_{j-1}, \tau\right)\}} - 1\right)$$
$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - \Gamma x_{j-1}, \tau\right)} - 1 + \frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - \Gamma y_1, \tau\right)} - 1\right)$$
$$= \lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - y_1, \tau\right)} - 1\right) + \lambda_2\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - \Gamma y_1, \tau\right)} - 1\right)$$
$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - \Gamma x_{j-1}, \tau\right)} - 1 + \frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - \Gamma y_1, \tau\right)} - 1\right).$$

Since $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular, then

$$\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_j - \Gamma y_1, \tau\right)} - 1\right) \leq \lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - y_1, \tau\right)} - 1\right)$$
$$+\lambda_2\left(\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(x_{j-1} - y_1, \tau\right)} - 1\right) + \left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1 - \Gamma y_1, \tau\right)} - 1\right)\right)$$

$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_{j-1}-\Gamma\mathrm{x}_{j-1},\tau\right)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)}-1\right).$$

Hence $\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_{j}-\Gamma y_1,\tau\right)}-1\right)\leq(\lambda_2+\lambda_4)\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)}-1\right).$

Since this is true for each j then,

$$\limsup_{j\to\infty}\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_{j}-\Gamma y_1,\tau\right)}-1\right)\leq(\lambda_2+\lambda_4)\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)}-1\right)$$

By (4) and (5) we obtain

$$\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)}-1\right)\leq(\lambda_2+\lambda_4)\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)}-1\right)$$

Note that $\lambda_2+\lambda_4<1$ because $\lambda_2,\lambda_3,\lambda_4\geq 0$ with $\lambda_1+2\lambda_2+2\lambda_4<1$. Then,

$$\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)=1$$

Therefore $\Gamma y_1=y_1$. Consequently, y_1 is a fixed point of Γ.

Corollary 2.3: Suppose that $(\mathrm{L},\tilde{\mathcal{F}}_{\mathcal{N}}\ \circledast)$ is a fuzzy Banach space where $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular. Let $\Gamma:\mathrm{L}\to\mathrm{L}$ be a mapping such that

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\Gamma\mathrm{x}-\Gamma y,\tau\right)}-1\leq\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}-y,\tau\right)}-1\right)+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}-\Gamma y,\tau\right),\tilde{\mathcal{F}}_{\mathcal{N}}\left(y-\Gamma\mathrm{x},\tau\right)\}}-1\right)$$
$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}-\Gamma\mathrm{x},\tau\right)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y-\Gamma y,\tau\right)}-1\right)\qquad(6)$$

for all $\mathrm{x},y\in\mathrm{L}$, $\tau>0$, $\lambda_1,\lambda_2,\lambda_4\in[0,1)$ with $\lambda_1+2\lambda_2+2\lambda_4<1$. Then Γ has a unique fixed point in L**.**

Proof: According to the proof of Theorem 2.2, y_1 is a fixed point of Γ in L such that $\Gamma y_1=y_1$. For the sake of uniqueness, consider x_1 be another fixed point of Γ in L such that $\Gamma\mathrm{x}_1=\mathrm{x}_1$. Then,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1=\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\Gamma\mathrm{x}_1-\Gamma y_1,\tau\right)}-1$$
$$\leq\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\right)+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-\Gamma y_1,\tau\right),\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma\mathrm{x}_1,\tau\right)\}}-1\right)$$
$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-\Gamma\mathrm{x}_1,\tau\right)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\Gamma y_1,\tau\right)}-1\right)$$

Consequently,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\leq\lambda_1\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\right)+\lambda_2\left(\frac{1}{\min\{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right),\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\mathrm{x}_1,\tau\right)\}}-1\right)$$
$$+\lambda_4\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-\mathrm{x}_1,\tau\right)}-1+\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-y_1,\tau\right)}-1\right)$$

Now we have two following cases:

(i)If $\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)$ is a minimum term in $\{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right),\tilde{\mathcal{F}}_{\mathcal{N}}\left(y_1-\mathrm{x}_1,\tau\right)\}$then after simplification we get,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\leq(\lambda_1+\lambda_2)\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\right)$$
$$\leq(\lambda_1+\lambda_2)^2\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\right)$$
$$\vdots$$
$$\leq(\lambda_1+\lambda_2)^{j}\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}\left(\mathrm{x}_1-y_1,\tau\right)}-1\right)$$

$\rightarrow 0$ as $j \rightarrow \infty$

since $\lambda_1 + \lambda_2 < 1$, therefore $\tilde{\mathcal{F}}_{\mathcal{N}}(x_1 - y_1, \tau) = 1$. Hence $x_1 = y_1$ for $\tau > 0$.

(ii) Similarly to (i), by condition (3) $\tilde{\mathcal{F}}_{\mathcal{N}}(y_1 - x_1, \tau) = \tilde{\mathcal{F}}_{\mathcal{N}}\left(x_1 - y_1, \frac{\tau}{|-1|}\right)$

$= \tilde{\mathcal{F}}_{\mathcal{N}}(x_1 - y_1, \tau)$. Therefore $x_1 = y_1$.

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3. APPLICATIONS

This section applies Theorem 2.2 to the study of the existence and uniqueness of solutions to the second-order differential equation and the Fredholm integral equation. The boundary value problem for the second-order differential equation is as follows:

$$x''(t) = f(t, x(t)),\ t \in [0,1] \text{ with } x(0) = x(1) = 0 \tag{7}$$

where $f: [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function. Problem (7) is equivalent to the integral equation:

$$x(t) = \int_0^1 \mathcal{G}(t, \vartheta)\ f\big(\vartheta, x(\vartheta)\big) d\vartheta \quad \text{for } t \in [0,1] \tag{8}$$

where $\mathcal{G}(t, \vartheta)$ is the Green's function defined by,

$$\mathcal{G}(t, \vartheta) = \begin{cases} t(1-\vartheta) & if \quad 0 \le t \le \vartheta \le 1 \\ \vartheta(1-t) & if \quad 0 \le \vartheta \le t \le 1 \end{cases}$$

Let $\Upsilon = C([0,1])$ be the space of all continuous functions defined on [0,1] and let $\tilde{\mathcal{F}}_{\mathcal{N}}: \Upsilon \times \mathbb{R} \rightarrow [0,1]$ be a fuzzy norm specified as:

$\tilde{\mathcal{F}}_{\mathcal{N}}(x, \tau) = \frac{\tau}{\tau + \|x\|}$ for each $x \in \Upsilon$, $\tau > 0$ where $\|x\| = sup|x(t)|$.

Then $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular and $(\Upsilon, \tilde{\mathcal{F}}_{\mathcal{N}}\ \circledast)$ be a fuzzy Banach space.

Now to prove the existing result for the boundary value problem (7) by applying Theorem 2.2.

Theorem3.1. Assume that

(i) There is function $\zeta: [0,1] \rightarrow \mathbb{R}$ which is continuous with

$$|f(\vartheta, x) - f(\vartheta, y)| \le 8\zeta(\vartheta)|x - y|$$

for each $x, y \in \Upsilon$, and $\vartheta \in [0,1]$.

(ii) $\max_{\vartheta \in [0,1]} \zeta(\vartheta) = \beta$ where $0 < \beta < 1$.

Then there is only one solution to eq. (7) in Υ.

Proof: Specify the mapping $\Gamma: \Upsilon \rightarrow \Upsilon$ by

$$\Gamma x(t) = \int_0^1 \mathcal{G}(t, \vartheta)\ f\big(\vartheta, x(\vartheta)\big) d\vartheta$$

for each $x \in \Upsilon$ and $t \in [0,1]$. Then problem (7) is the same as finding a fixed point of Γ in Υ.

Let $x, y \in \Upsilon$, we have

$$\begin{aligned}
\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\Gamma x(t) - \Gamma y(t), \tau)} - 1 &= \frac{|\Gamma x(t) - \Gamma y(t)|}{\tau} \\
&= \frac{\left|\int_0^1 \mathcal{G}(t,\vartheta) f\big(\vartheta, x(\vartheta)\big) d\vartheta - \int_0^1 \mathcal{G}(t,\vartheta) f\big(\vartheta, y(\vartheta)\big) d\vartheta\right|}{\tau} \\
&= \frac{\int_0^1 \mathcal{G}(t,\vartheta) \left|f\big(\vartheta, x(\vartheta)\big) - f\big(\vartheta, y(\vartheta)\big)\right| d\vartheta}{\tau} \\
&\le \frac{\int_0^1 \mathcal{G}(t,\vartheta) 8\zeta(\vartheta) |x(t) - y(t)| d\vartheta}{\tau} \\
&\le 8\beta \frac{|x(t) - y(t)|}{\tau} \int_0^1 \mathcal{G}(t,\vartheta)\, d\vartheta.
\end{aligned}$$

Since $\int_0^1 \mathcal{G}(t, \vartheta)\, dr = \frac{t}{2} - \frac{t^2}{2}$ for $t \in [0,1]$,we have

$\sup_{t \in [0,1]} \int_0^1 \mathcal{G}(t, \vartheta) = \frac{1}{8}$.

Therefore,

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\Gamma x(t)-\Gamma y(t),\tau)}-1 \le \beta\frac{|x(t)-y(t)|}{\tau}$$
$$=\beta\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(x(t)-y(t),\tau)}-1\right)$$

As a result, the operator Γ meets the requirements of Theorem 2.2 with $\beta=\lambda_1$ and $\lambda_2=\lambda_3=\lambda_4=0$ in (1). Consequently, Γ possesses a unique fixed point $x^* \in \Upsilon$, meaning that x^* is a solution to the boundary value problem (7).

Next, we study the existence and unique solution to Fredholm integral equations as an application of Theorem 2.2. Let $\Upsilon=C([0,a])$ represent the space of all real-valued continuous functions on the interval $[0,a]$, where $0<a\in\mathbb{R}$. We now present a special instance of a Fredholm integral equation of the second type, as shown below:

$$x(t)=\int_0^a f\left(t,x(\vartheta),x'(\vartheta)\right)d\vartheta \tag{9}$$

where $t\in[0,a]$ and $f:[0,a]\times[0,a]\times\mathbb{R}\longrightarrow\mathbb{R}$. The binary operation $\circledast$ is defined by $\mu\circledast\sigma=\sigma\circledast\mu, \forall\mu,\sigma\in[0,a]$. The standard fuzzy norm $\tilde{\mathcal{F}}_{\mathcal{N}}:\Upsilon\times(0,\infty)\to[0,1]$ is expressed as follows:

$\tilde{\mathcal{F}}_{\mathcal{N}}(x,\tau)=\frac{\tau}{\tau+\aleph(x)}$ for all $x\in\Upsilon$ and $\tau>0$ where $\aleph(x)=\|x\|=|x(t)|$.

Then it is simple to verify that $\tilde{\mathcal{F}}_{\mathcal{N}}$ is triangular and $(\Upsilon,\tilde{\mathcal{F}}_{\mathcal{N}},\circledast)$ is a fuzzy normed space.

Theorem 3.2. Suppose that

(i) There is a continuous function $h:[0,a]\times[0,a]\longrightarrow[0,\infty)$ and $0<\alpha\in\mathbb{R}$ such that for all $x,y\in\Upsilon$ we have

$$\left|f\left(t,x(\vartheta),x'(\vartheta)\right)-f\left(t,y(\vartheta),y'(\vartheta)\right)\right|\le\alpha h(t,\vartheta)|x(\vartheta)-y(\vartheta)|$$

(ii) $\int_0^a h(t,\vartheta)d\vartheta\le 1$.

Then there is only one solution to equation (9) in ϒ.

Proof: Specify the mapping $\Gamma:\Upsilon\to\Upsilon$ by

$$\Gamma x(t)=\int_0^a f\left(t,x(\vartheta),x'(\vartheta)\right)d\vartheta$$

Let $x,y\in\Upsilon$, we have

$$\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(\Gamma x(t)-\Gamma y(t),\tau)}-1=\frac{|\Gamma x(t)-\Gamma y(t)|}{\tau}$$
$$=\frac{\left|\int_0^a f\left(t,x(\vartheta),x'(\vartheta)\right)d\vartheta-\int_0^a f\left(t,y(\vartheta),y'(\vartheta)\right)d\vartheta\right|}{\tau}$$
$$\le\frac{\int_0^a\left|f\left(t,x(\vartheta),x'(\vartheta)\right)-f\left(t,y(\vartheta),y'(\vartheta)\right)\right|d\vartheta}{\tau}$$
$$\le\frac{\int_0^a\alpha h(t,\vartheta)|x(\vartheta)-y(\vartheta)|d\vartheta}{\tau}$$
$$\le\alpha\frac{|x(\vartheta)-y(\vartheta)|}{\tau}\int_0^a h(t,\vartheta)\,d\vartheta$$
$$\le\alpha\frac{|x(\vartheta)-y(\vartheta)|}{\tau}$$
$$=\alpha\left(\frac{1}{\tilde{\mathcal{F}}_{\mathcal{N}}(x(t)-y(t),\tau)}-1\right).$$

Consequently, the operator Γ meets the requirements of Theorem 2.2 with $\alpha=\lambda_1$ and $\lambda_2=\lambda_3=\lambda_4=0$ in (1). Then Γ possesses a unique fixed point $x^*\in\Upsilon$, i.e x^* is a solution to equation (9).

4. Conclusions

In this work, the triangle property on the fuzzy norm is proposed, and fixed point theorems on fuzzy normed spaces are proven using this property. As an application of the fixed point theorems, the existence and uniqueness of the solution to the ordinary differential equation and Fredholm integral equation were investigated to demonstrate the significance of our results. This work offers a good jumping-off point for discussions and future investigations into the application of fixed point theorems in fuzzy normed space to the study of the existence and uniqueness of the solution to different kinds of equations.

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