DOI: 10.24996/ijs.2025.66.12.28

ISSN: 0067-2904

Variational Approach for Solving Oxygen Diffusion Problem with Time-Fractional Derivative

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Received: 3/11/2023 Accepted: 24/11/2024 Published: 30/12/2025

Abstract

One significant problem is the simultaneous absorption and diffusion of oxygen into the cells. When it comes to medical applications, it is crucial. There are two stages to the mathematical formulation of the problem. The stable case in which there is no oxygen transition in the isolated cell is examined in the first stage, and the moving boundary problem pertaining to the oxygen absorbed by the cell's tissues is examined in the second stage. A shifting border is a crucial component of the oxygen diffusion problem. In this paper, we present mathematical model of oxygen diffusion problem with time fractional derivative in Caputo sense. This problem is difficult to solve analytically, so we will use the variational approach to find the corresponding formula for the problem. The well-known Magri's approach formula cannot be used to find a variational formula corresponding to the presented problem, so a modified formula of Magri's approach will be found, and then the corresponding model is solved numerically. So the resulting solution is an approximate analytical solution.

Keywords: Variational approach, Moving boundary value problem, Caputo fractional derivative, Stefan problem, Magri’s approach.

نهج تغايري لحل مسألة انتشار الأوكسجين ذات المشتقة الكسورية بالنسبة للزمن

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الخلاصة

احدى المشاكل المهمة هي مشكلة امتصاص الأوكسجين وانتشاره في الخلايا في وقت واحد. عندما يتعلق الأمر بالتطبيقات الطبية فهو أمر بالغ الأهمية. هنالك مرحلتان لأيجاد الصياغة الرياضية للمشكلة. يتم في المرحلة الأولى فحص الحالة المسترة التي لا يحدث فيها انتقال في الخلية المعزولة، كما يتم في المرحلة الثانية فحص مشكلة الحدود المتحركة الخاصة بالأوكسجين الذي تمتصه أنسجة الخلية. يعد تغيير الحدود عنصرا حاسما في مشكلة انتشار الأوكسجين. في هذا البحث نعرض نموذج رياضي لمسألة انتشار

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الأوكسجين ذات مشتقة كسورية بالنسبة للزمن وحسب تعريف كابتو. من الصعب حل هذه المسألة تحليليا، لذلك سنستخدم نهج ماكري التغايري لأيجاد الصيغة التغايرية المقابلة للمسألة. لا يمكن استخدام صيغة ماكري المعروفة لأيجاد الصيغة المقابلة للمسألة المعروضة، لذلك سيتم ايجاد صيغة معدلة لمنهج ماكري، ومن ثم حل النموذج المقابل عدديا. لذا فأن الحل الناتج هو حلا تحليليا تقريبي.

1. Introduction

Crank and Gupta provide the oxygen diffusion problem model [1]. Two steps of mathematics are involved in the oxygen diffusion problem. When the oxygen is present in the initial stage, the stable state. The cell surface is isolated after being injected into the cell from either the inside or the outside. In the second step, tissues begin to absorb the oxygen that was injected. This level results to the problem of moving boundary. Finding a balancing position and figuring out the time-dependent changing boundary position are the goals of this method. See [1, 2, 3] for further information on the time-fractional oxygen diffusion problem. Fractional differential equations have gained a lot of interest in recent years. Differential equations of fractional order provide a clear description of many significant phenomena in physics, engineering, finance, mathematics, transport dynamics, and hydrology in particular. See, for example, references [4, 5], authors where studied some problems and solve them. In the theory of complex systems and in the modelling of the so-called anomalous transport phenomena, fractional differential equations are crucial. When contrasted, these fractional derivatives function more appropriately using the common integer-order models.

The Caputo fractional derivative distinguished from other definitions of derivation, and the most important of these advantages, which is that the Caputo derivative of a constant is zero, that is ${}_{a}^{c}D_{t}^{\alpha}\theta = {}_{t}^{c}D_{b}^{\alpha}\theta = 0$, where $\theta$is a constant [6]. For the simplicity of dealing with this definition of fractional derivatives for mathematical calculations. This model of the oxygen diffusion problem has been presented as defined by Caputo. Find the analytic solution of this type of problems is hard, so may be obtain a corresponding formula of it and solve it by approximate method [7]. We will be using a modified approach for Magri's approach to find a variational corresponding formula of the problem, after that solve the last formula numerically. In Section 2 of this work we introduce some of the definitions and properties in fractional calculus. The model of oxygen problem presented in Section 3. After that, we find the variational formulation of this problem in next section. In Section 5, present the results graphically after find the numerical solution of the last formula of problem. Finally, the conclusion in Section 6.

2. Basic concepts

Basic definitions, concepts and properties of fractional order differentiation and integration in this section.

Definition 2.1 [6]: The left and right Riemann-Liouville Fractional Integrals of order $\alpha > 0$ of a function $\omega(t) \in C[a, b]$ is defined as

$$ {}_{a}I_{t}^{\alpha}\omega(t) = \frac{1}{\Gamma(\alpha)}\int_{a}^{t}(t-\tau)^{\alpha-1}\omega(\tau)d\tau \qquad \text{……… (1)}$$

and

$$ {}_{t}I_{b}^{\alpha}\omega(t) = \frac{1}{\Gamma(\alpha)}\int_{t}^{b}(\tau-t)^{\alpha-1}\omega(\tau)d\tau, \qquad \text{……… (2)}$$

respectively.

Definition 2.2 [6]: The left and right Caputo Fractional derivatives of order $\alpha > 0$ of a function $\omega(t) \in C^{m}[a, b]$,$m \in \mathbb{N}$ is defined as

$${}_{a}^{c}D_{t}^{\alpha}\omega(t) = \frac{1}{\Gamma(m-\alpha)}\int_{a}^{t}(t-\tau)^{m-\alpha-1}\omega^{(m)}(\tau)d\tau, \quad \text{……… (3)}$$

and

$${}_{t}^{c}D_{b}^{\alpha}\omega(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t}^{b}(\tau-t)^{m-\alpha-1}\omega^{(m)}(\tau)d\tau, \quad \text{……… (4)}$$

$m-1<\alpha<m.$

Proposition 2.3 [6]: For $\omega(t) \in C^m[a,b], \alpha,\gamma \geq 0, n-1<\alpha \leq n, \alpha+\gamma \leq m,$ $\mu \geq -1$ and $t>0,$ we mention the following:

1. $I^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}t^{\mu+\alpha}.$
2. ${}^{c}D_{a}^{\alpha}t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}.$

Proposition (Fractional Leibniz Formulation) 2.4 [6]: Let $\omega(t) \in C^m[a,b], \alpha>0, m-1<\gamma \leq m$ *and* $m \in \mathbb{N}$, then:

$$I_{a}^{\gamma}\left({}^{c}D_{a}^{\gamma}\omega(t)\right) = \omega(t) - \sum_{j=0}^{m-1}\frac{\omega^{(j)}(a)}{\Gamma(j+1)}(t-a)^{j}.$$

Definition 2.5 [8]: Let U and V are two linear spaces over a real numbers R, we call the functional $\mathcal{L}: U \times V \to$ R bilinear form.

Definition 2.6 [8]: Let $\mathcal{L}(.,.)$ be a bilinear form on $U \times V$, then:

i. .... $\mathcal{L}(.,.)$ is said to be symmetric if

$\mathcal{L}(u,v) = \mathcal{L}(v,u)$ for all $u \in U$ and $v \in V$.

ii. .... $\mathcal{L}(.,.)$ is said to be nondegenerate if

(a) $\mathcal{L}(u,\bar{v})=0 \Rightarrow \bar{v}=0,$ forall $u \in U.$

(b) $\mathcal{L}(\bar{u},v)=0 \Rightarrow \bar{u}=0,$ forall $v \in V.$

Note: Usually, denoted bilinear form by $<.,.>$ instead of $\mathcal{L}(.,.)$.

Definition (Magri's definition of symmetric bilinear form) 2.7 [9]: Let U and V be two normed linear spaces, and let $<.,.>$ be a bilinear form, then $\mathcal{L}(\omega,u) =< u,\omega >$ is said to be symmetric if:

$<\omega,u> = <u,\omega>$ for all $<\omega,u> \in U \times V.$

Definition 2.8 [9]: Let U and V be two normed linear spaces and $\mathcal{L}: U \to R$ be a linear operator, then $\mathcal{L}$ is called symmetric operator with respect to choosing bilinear form $<.,.>$ if

$<\mathcal{L}\omega,u> = <\mathcal{L}u,\omega>$ for all $\omega \in U$ and $u \in V$

Theorem (Magri's Theorem) 2.9 [9]: There is a variational problem corresponding to linear equation $\mathcal{L}(\omega)=f$, if and only if the operator $\mathcal{L}$ is symmetric relative to the bilinear form which is non- degenerate, where the functional is given by:

$$\Im(\omega) = \frac{1}{2} < \mathcal{L}\omega,\omega > - < f,\omega > \quad \text{………. (5)}$$

3. One-dimensional oxygen diffusion problem:

Consider the following oxygen diffusion problem:

$${}^{c}D_{t}^{\alpha}w(x,t) = \frac{\partial^2}{\partial x^2}w(x,t) - 1,\ 0 \leq x \leq s(t), 0 \leq t \quad \text{………. (6)}$$

with the initial conditions:

$$w(x,0) = \frac{1}{2}(1-x)^2,\ 0 \leq x \leq 1, t=0, \quad \text{……… (7)}$$

$$\frac{\partial}{\partial x}w(x,t) = 0, x=0, t>0, \quad \text{……… (8)}$$

and boundary conditions:

$$w(x,t) = \frac{\partial}{\partial x}w(x,t), x=s(t), t>0, \quad \text{……… (9)}$$

$$s(t) = 0, t = 0. \quad \ldots\ldots\ldots (10)$$

where $0 < \alpha \leq 1$. Such that ${}^{c}D^{\alpha}$ is the Caputo fractional derivative operator of order α.
In this work we will establish the variational formulation of the presented problem

It is clear that the differential operator related to Equation (6) is given by:

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} - \frac{\partial^\alpha}{\partial t^\alpha}. \quad \ldots\ldots\ldots (11)$$

For simplicity rewrite ${}^{c}D_t^{\alpha}$ as $\frac{\partial^\alpha}{\partial t^\alpha}$, the first part of the operator $\mathcal{L}$ is the Laplacian operator, and the second part is the diffusion operator.

In order to find a variational formulation related to problem (6) and the conditions (7)-(10), we have to use the inverse problem of calculus of variation (Theorem 2.9), and to that objective, we have to prove that $\mathcal{L}$ is symmetric relative to the chosen nondegenerate bilinear form.

4. Variational formulation of the problem

4.1 The symmetry of the bilinear form

Consider the following operator:

$$< u, v >= I_R \, {}_0I_t^{\alpha}[u(x,s)\bar{v}(x,s)], \quad \ldots\ldots\ldots\ldots\ldots (12)$$

where:

$$\bar{v}(x,s) = {}_0I_{t-\tau}^{\alpha}[\kappa(t,s)v(x,s)],$$

${}_0I_t^{\alpha}$ and ${}_0I_{t-\tau}^{\alpha}$ are the Riemann-Liouville fractional integral operators of order $\alpha > 0$ and $\kappa(t,s)$ is the kernel function which is chosen to satisfy the symmetry and nondegeneracy of $< u, v >$. Also, $\kappa(t,s)$ is assumed to be symmetric and smooth function, which is defined on the square region, as in Figure (4.1):

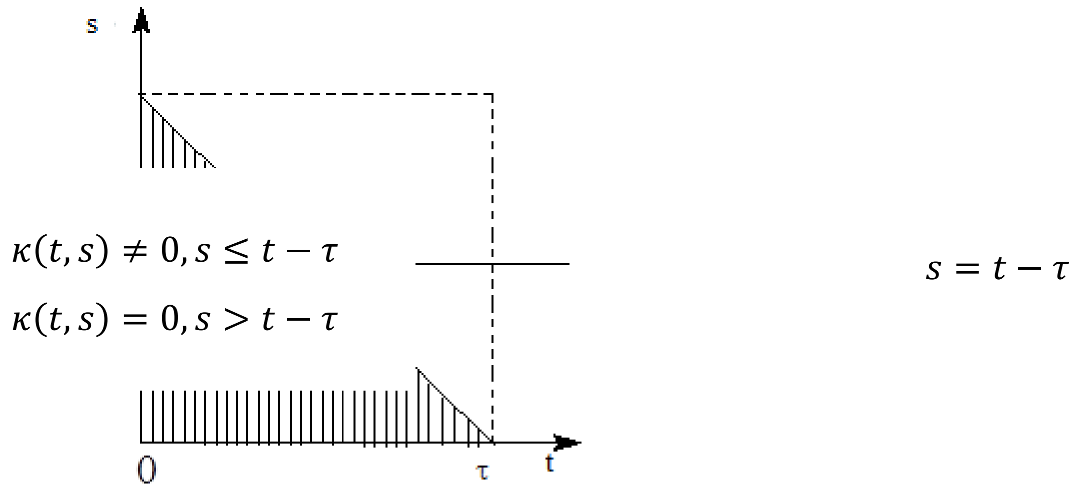

Figure 1: The square region of definition of the kernel $\kappa(t,s)$

This operator is bilinear form, because the integration operator is bilinear [9]. To prove the symmetry of the operator $<\cdot,\cdot>$ given in (12),

$$< u, v >= \frac{1}{\Gamma(1+\alpha)} \int_R \int_{\tau=0}^{\tau=t} u(x,\tau)\Big[\frac{1}{\Gamma(\alpha)} \int_{s=0}^{s=t-\tau} (t-s)^{\alpha-1}\kappa(\tau,s)v(x,s)ds\Big]d\tau dx. \quad \ldots\ldots (13)$$

Notice that, the first integration over R represents the integration over the region of the problem and the second integration from 0 to t because of the initial conditions related to the problem.

By changing the variables of integrations from s to τ and vise a versa and according to vertical slice instead of horizontal one, we get:

$$< u, v >= \frac{1}{\Gamma(1+\alpha)} \int_R \int_{s=0}^{s=t-\tau} [\frac{1}{\Gamma(\alpha)} \int_{\tau=0}^{\tau=t} (t-s)^{\alpha-1}\ u(x,s)\kappa(s,\tau)v(x,s)d\tau]dsdx$$

Since, one can choose $\kappa(\tau, s)$ to be symmetric, so:

$$< u, v >= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^{t-\tau} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(x,s)\kappa(s,\tau)v(x,s)d\tau\right] dsdx$$
$$= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^{t-\tau} \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(x,s)\kappa(s,\tau)u(x,s)d\tau\right] dsdx$$
$$= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t v(x,\tau) \left[\frac{1}{\Gamma(\alpha)} \int_0^{t-\tau} (\tau-s)^{\alpha-1} \kappa(\tau,s)u(x,s)ds\right] d\tau dx$$
$$=< v, u >.$$

Implies that $< u, v >$ is symmetric bilinear form, which means the first condition is satisfied.

4.2 Nondegeneracy of bilinear form

To show that $< u, v >$ is nondegenerate by using the definition of nondegenerate bilinear forms and the relation (4.2), fixe v_0 then:

$$< u, v_0 >= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t u(x,\tau) \left[\frac{1}{\Gamma(\alpha)} \int_0^{t-\tau} (t-s)^{\alpha-1} \kappa(\tau,s)v_0(x,s)ds\right] d\tau dx \qquad \text{...…….. (14)}$$

To prove $< u, v >$ is nondegenerate, assumed that the relation (14) equal to zero. From properties of the kernel function definition and its domain one can be define:

$$\kappa(\tau, s) = \begin{cases} h(\tau, s), & 0 \le s \le t \\ h(s, \tau), & 0 \le \tau \le t \end{cases} \qquad \text{…….. (15)}$$

this selection is to make the boundaries of integrations in (14) are invertible and by suitable define of the function $h(\tau, s)$ in Equation (15). Therefore, by making a kernel function of the Volterra kind, which is defined as:

$$h(\tau, s) = \begin{cases} \eta(\tau, s), & 0 \le s \le t-\tau \\ 0, & t-\tau \le \tau \le t \end{cases},$$

since Volterra equation defined as:

$$\int_0^{t-\tau} \eta(\tau, s)(t-s)^{\alpha-1} v_0(x,s)ds = 0.$$

Then, relation (14) will become:

$$< u, v_0 >= \frac{1}{\Gamma(1+\alpha)} \int_R \int_{\tau=0}^{\tau=t} u(x,\tau)$$
$$\left[\frac{1}{\Gamma(\alpha)} \int_{s=0}^{s=t-\tau} \eta(\tau,s)(t-s)^{\alpha-1} v_0(x,s)ds\right] d\tau dx,$$

and according to Volterra kernel, this problem has no solution unless $(t-s)^{\alpha-1} v_0(x,s) = 0$, and since $(t-s)^{\alpha-1} \neq 0$, then $v_0(x,s) = 0$.
Therefore, $< u, v_0 >= 0$, just when $v_0 = 0$.

Also, by same procedure, we can be proved: $< u_0, v >= 0$ which is only true when $u_0 = 0$.
Hence, the second condition is satisfied.

4.3 The symmetry of the operator $\mathcal{L} = \mathcal{L}^s + \mathcal{L}^t$

To prove $\mathcal{L}$ symmetric operator relative to the nondegenerate bilinear form $<\cdot,\cdot>$, we must prove that:

$$< \mathcal{L}u, v >=< \mathcal{L}v, u >. \qquad \text{………..… (16)}$$

Now, by applying modified Magri's approach (relation (12)) to be as follows:

$$< \mathcal{L}u, v >= (\mathcal{L}u, \mathcal{L}^t v). \quad \text{............ (17)}$$

Therefore,

$$\begin{aligned} < \mathcal{L}u, v > &= (\mathcal{L}u, \mathcal{L}^t v) \\ &= ((\mathcal{L}^s + \mathcal{L}^t)u, \mathcal{L}^t v) \\ &= (\mathcal{L}^s u + \mathcal{L}^t u, \mathcal{L}^t v) \quad \text{(since } \mathcal{L}^s \text{ linear operator)} \\ &= (\mathcal{L}^s u, \mathcal{L}^t v) + (\mathcal{L}^t u, \mathcal{L}^t v). \end{aligned} \quad \text{............. (18)}$$

Using the same approach on the right hand side of (16), we get:

$$< \mathcal{L}v, u > = (\mathcal{L}^s v, \mathcal{L}^t u) + (\mathcal{L}^t v, \mathcal{L}^t u). \quad \text{........... (19)}$$

Since $< , >$ symmetric bilinear form, then:

$$(\mathcal{L}^t u, \mathcal{L}^t v) = (\mathcal{L}^t v, \mathcal{L}^t u), \quad \text{........... (20)}$$

hence, the problem here is to show that:

$$(\mathcal{L}^s u, \mathcal{L}^t v) = (\mathcal{L}^s v, \mathcal{L}^t u). \quad \text{.......... (21)}$$

Let us take $(\mathcal{L}^s u, \mathcal{L}^t v)$ and suppose that $\mathcal{L}^s u = w$, therefore:

$$(\mathcal{L}^s u, \mathcal{L}^t v) = (w, \mathcal{L}^t v), \quad \text{.......... (22)}$$

but

$$\begin{aligned} (w, \mathcal{L}^t v) &=< w , v > \\ &=< \mathcal{L}^s u, v > \\ &=< \mathcal{L}^s v, u > \quad \text{(Since } \mathcal{L}^s \text{ symmetric operator)} \\ &= (\mathcal{L}^s v, \mathcal{L}^t u). \end{aligned} \quad \text{........... (23)}$$

Implies that

$$(\mathcal{L}^s u, \mathcal{L}^t v) = (\mathcal{L}^s v, \mathcal{L}^t u).$$

This means that relation (16) fulfil.

Now, we must prove that:

$$(w, \mathcal{L}^t v) = (v, \mathcal{L}^t w), \quad \text{.......... (24)}$$

this means to prove that $\mathcal{L}^t$ is a symmetric operator relative to the chosen bilinear form $<\cdot,\cdot>$.

To prove the relation (24), we will use the relation (12), hence:

$$\begin{aligned} (w, \mathcal{L}^t v) &= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t w(x,\tau) \left[{}_0I_{t-\tau}^{\alpha} [\kappa(\tau,s) \mathcal{L}^t v(x,s)] \right] d\tau dx \\ &= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t w(x,\tau) \left[{}_0I_{t-\tau}^{\alpha} \left[\kappa(\tau,s) \frac{\partial^\alpha}{\partial s^\alpha} v(x,s) \right] \right] d\tau dx, \end{aligned}$$

it can be supposed that $\kappa(\tau,t) \equiv 1$, for simplicity, then:

$$\begin{aligned} (w, \mathcal{L}^t v) &= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t w(x,\tau) \left[{}_0I_{t-\tau}^{\alpha} \left[\frac{\partial^\alpha}{\partial s^\alpha} v(x,s) \right] \right] d\tau dx \\ &= \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t w(x,\tau) \, [v(x,t-\tau) - v(x,0)] d\tau dx, \end{aligned}$$

by fractional Leibniz formulation, then:

$$(w, \mathcal{L}^t v) = \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t w(x,\tau) \, v(x,t-\tau) d\tau dx. \quad \text{......... (25)}$$

In the same procedure, we get:

$$(v, \mathcal{L}^t w) = \frac{1}{\Gamma(1+\alpha)} \int_R \int_0^t v(x,\tau) \, w(x,t-\tau) d\tau dx. \quad \text{......... (26)}$$

According to the convolution theorem, then the relations (25) and (26) are equals. Hence, $\mathcal{L}^t$ is a symmetric operator relative to the bilinear form $<\cdot,\cdot>$, which implies that relation (24) is true, i.e., $\mathcal{L}$ symmetric operator relative to the chosen nondegenerate bilinear form $<\cdot,\cdot>$.

Now, can be find the variational formulation corresponding to the fractional oxygen diffusion equation as follows:

The non-homogeneous oxygen diffusion equation with time-fractional derivative order has the form:

$$\frac{\partial^2}{\partial x^2}w(x,t)-\frac{\partial^\alpha}{\partial t^\alpha}w(x,t)=f(x,t),\ 0<\alpha\leq 1. \quad \text{.......... (27)}$$

where $f(x,t)$ a given function as the nonhomogeneous term.

The operator of this problem is $\mathcal{L}=\frac{\partial^2}{\partial x^2}-\frac{\partial^\alpha}{\partial t^\alpha}$, which is symmetric operator relative to the nondegenerate bilinear form $<\cdot,\cdot>$. Implies that we can be obtain the variational formulation to the problem depending on the inverse problem of calculus of variation (Magri's Theorem), which has the following general form:

$$\Im(w)=\frac{1}{2}<\mathcal{L}w,w>-<f,w>, \quad \text{.......... (28)}$$

depending on Magri's approach on the operator $\mathcal{L}^t$, that is $<u,v>=(u,\mathcal{L}^t v)$, then the relation 28 becomes:

$$\Im(w)=\frac{1}{2}(\mathcal{L}w,\mathcal{L}^t w)-(f,\mathcal{L}^t w). \quad \text{.......... (29)}$$

Now, the bilinear form given in Equation (12) may be used to give:

$$\begin{aligned}\Im(w)&=\frac{1}{2\Gamma(1+\alpha)}\int_R\int_0^t \mathcal{L}w(x,\tau)\overline{\mathcal{L}^t w}(x,\tau)d\tau dx\\&\quad-\frac{1}{\Gamma(1+\alpha)}\int_R\int_0^t f(x,\tau)\overline{\mathcal{L}^t w}(x,\tau)d\tau dx\\&=\frac{1}{2\Gamma(1+\alpha)}\int_R\int_0^t \mathcal{L}w(x,\tau)\{{}_0I_{t-\tau}^{\alpha}[\kappa(\tau,t)\mathcal{L}^t w(x,t)]\}d\tau dx\\&\quad-\frac{1}{\Gamma(1+\alpha)}\int_R\int_0^t f(x,\tau)\{{}_0I_{t-\tau}^{\alpha}[\kappa(\tau,t)\mathcal{L}^t w(x,t)]\}d\tau dx,\end{aligned}$$

for simplicity, let $\kappa(\tau,t)=1$, then:

$$\begin{aligned}\Im(w)&=\frac{1}{2\Gamma(1+\alpha)}\int_R\int_0^t \mathcal{L}w(x,\tau)\{{}_0I_{t-\tau}^{\alpha}[\mathcal{L}^t w(x,t)]\}d\tau dx\\&\quad-\frac{1}{\Gamma(1+\alpha)}\int_R\int_0^t f(x,\tau)\{{}_0I_{t-\tau}^{\alpha}[\mathcal{L}^t w(x,t)]\}d\tau dx\\&=\frac{1}{2\Gamma(1+\alpha)}\int_R\int_0^t \mathcal{L}w(x,\tau)\{w(x,t-\tau)-w(x,0)\}d\tau dx\\&\quad-\frac{1}{\Gamma(1+\alpha)}\int_R\int_0^t f(x,\tau)\{w(x,t-\tau)-w(x,0)\}d\tau dx\\&=\frac{1}{2\Gamma(1+\alpha)}\int_R\int_0^t\Big[\frac{\partial^2}{\partial x^2}w(x,\tau)\{u(x,t-\tau)-w(x,0)\}\\&\quad-\frac{\partial^\alpha}{\partial t^\alpha}w(x,\tau)\{w(x,t-\tau)-w(x,0)\}\Big]d\tau dx\\&\quad-\frac{1}{\Gamma(1+\alpha)}\int_R\int_0^t f(x,\tau)\{w(x,t-\tau)-w(x,0)\}d\tau dx.\end{aligned} \quad \text{......... (30)}$$

When the initial condition $w(x,0)$ is homogenous, then the relation (30) becomes:

$$\begin{aligned}\Im(w)&=\frac{1}{2\Gamma(1+\alpha)}\int_R\int_0^t\Big[\frac{\partial^2}{\partial x^2}w(x,\tau)w(x,t-\tau)\\&\quad-\frac{\partial^\alpha}{\partial t^\alpha}w(x,\tau)w(x,t-\tau)\Big]d\tau dx\\&\quad-\frac{1}{\Gamma(1+\alpha)}\int_R\int_0^t f(x,\tau)w(x,t-\tau)d\tau dx.\end{aligned} \quad \text{......... (31)}$$

5. Numerical solution of the problem

In order to use the direct Ritz method [10], we must approximate the solution $w(x,t)$ of the presented problem (6) with the conditions (7)-(10) as in the following decomposition:

$$w(x,t)=\psi(x,t)+\omega(x,t,a_i),\ i=0,1,\dots,k \quad \text{.........(32)}$$

where $\psi(x,t)$ is any function that satisfies the non-homogenous boundary conditions while the function $\omega(x,t,a_i)$, for all i = 0, 1, …, k satisfies the homogenous boundary conditions.

By the initial condition (7), one can be chosen

$$\psi(x,t)=\frac{1}{2}(s(t)-x)^2.$$

While to satisfy the another conditions, let

$$\omega(x,t,a_i) = \frac{1}{2}(s(t)-x)^2 \sum_{i=0}^{k} a_{i+1} t^{(i+1)\alpha}.$$

Therefore, the approximate solution becomes

$$w(x,t) = \frac{1}{2}(s(t)-x)^2[1 + \sum_{i=0}^{k} a_{i+1} t^{(i+1)\alpha}].$$

For simplicity of calculations, suppose that k=3, then:

$$w(x,t) = \frac{1}{2}(s(t)-x)^2[1 + a_1 t^{\alpha} + a_2 t^{2\alpha} + a_3 t^{3\alpha} + a_4 t^{4\alpha}]. \quad \ldots\ldots\ldots (33)$$

With respect to the moving boundary $s(t)$ which satisfy the desired condition on the boundary, may have the definition:

$$s(t) = \sqrt{1-4bt^{\alpha}}\,. \quad \ldots\ldots\ldots (34)$$

Now, we have to find the partial derivatives of (33) with respect to x and t, as follows:

$$\frac{\partial}{\partial x} w = \big(x - s(t)\big)[1 + a_1 t^{\alpha} + a_2 t^{2\alpha} + a_3 t^{3\alpha} + a_4 t^{4\alpha}],$$

$$\frac{\partial^2}{\partial x^2} w = [1 + a_1 t^{\alpha} + a_2 t^{2\alpha} + a_3 t^{3\alpha} + a_4 t^{4\alpha}]$$

and

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} w = \frac{1}{2}(s(t)-x)^2\left[a_1\Gamma(1+\alpha) + a_2\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} t^{\alpha} + a_3\frac{\Gamma(1+3\alpha)}{\Gamma(1+2\alpha)} t^{2\alpha} + a_4\frac{\Gamma(1+4\alpha)}{\Gamma(1+3\alpha)} t^{3\alpha}\right]$$
$$+\big(x - s(t)\big)[1 + a_1 t^{\alpha} + a_2 t^{2\alpha} + a_3 t^{3\alpha} + a_4 t^{4\alpha}]\frac{\partial^{\alpha}}{\partial t^{\alpha}} s.$$

Then substituting these derivatives in (30) to get the final form of the functional $\Im(w)$ which has to be minimized. So, by used the computer software Mathcad 14, the final results are presented of the critical points a_1, a_2, a_3, a_4 and b are follows:

$$a_1 = 0.05, a_2 = 0.26, a_3 = 0.25, a_4 = 1.0 \text{ and } b = 0.25.$$

Bottom, are presented some of figures of the concentration distributions for the oxygen and position of moving surface at deferent values of time with respect to deferent value of fractional derivative order α. The Figure 5.1 shows the change in the limits of diffusion oxygen with time when the derivate is 1. The Figures 5.2 and 5.3 show the distribution for the oxygen with the change of time and the fractional derivative α. Position of the surface is varying with the value of *t* as in Figure 5.4.

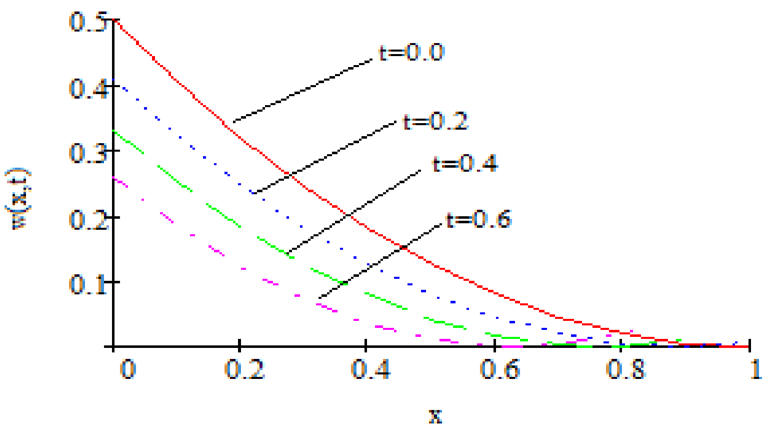

Figure 2: Concentration distributions for the oxygen at the steady-state (t=0) and deferent values of t, where $\alpha = 1$.

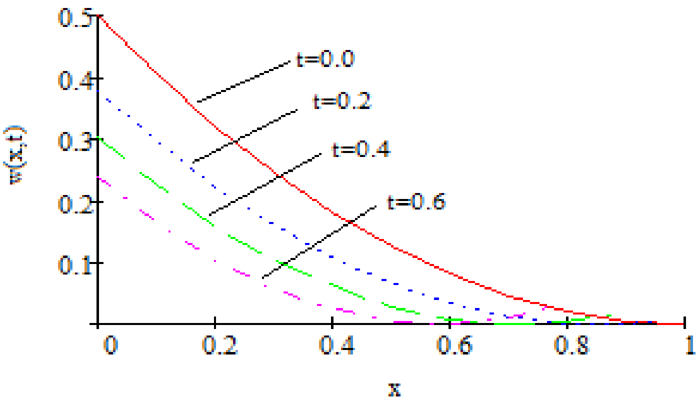

Figure 3: Concentration distributions for the oxygen at the steady-state (t=0) and deferent values of t, where $\alpha = 0.8$.

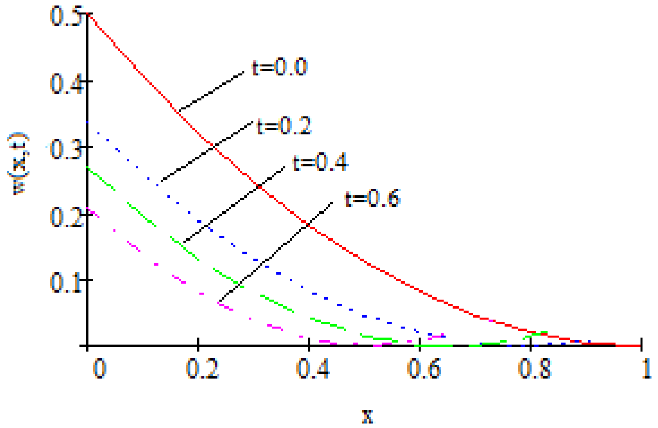

Figure 4: Concentration distributions for the oxygen at the steady-state (t=0) and deferent values of t, where $\alpha = 0.6$.

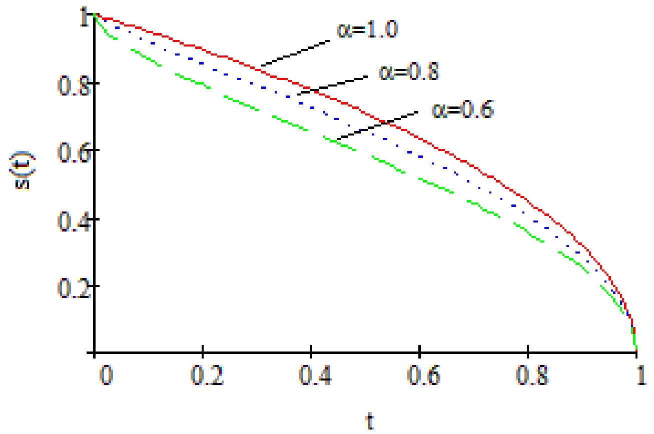

Figure 5: Position of the moving surface where $\alpha = 1, 0.8$ and 0.6, respectively.

6. Conclusions

In this work, we solved the moving boundary value problem (oxygen diffusion problem) with time fractional order, by using variational approach, this approach is modified of

Magri’s approach. First, we present a modified formula of Magrie’s approach and proof the last verification the required conditions to find corresponding model of the problem. In order to use this method requires to find corresponding model of the problem by using specific procedure. Second solving the result model numerically to obtain approximate analytic solution to the problem. Rits method one of the direct variational approximation methods, which can used to find the solution, then display the results as a figures to show the change in the boundary of diffusion with the change of time and the value of derivative. So, Magri's approach can be used as a variational approach to solving free and moving boundary value problems, after developing the approach, such that the differential operator is linear and not required to be symmetric.

References

[1] J. Crank and R. S. Gupta, “A moving boundary problem arising from the diffusion of oxygen in absorbing tissue,” *IMA Journal of Applied Mathematics,* vol. 10, no. 1, pp. 19-33, 1972.

[2] V. Gülkaç, “The New Approximate Analytic Solution for Oxygen Diffusion Problem with Time-Fractional Derivative,” *Mathematical Problems in Engineering,* 2016.

[3] S. L. Mitchell, “An accurate application of the integral method applied to the diffusion of oxygen in absorbing tissue,” *Applied Mathematical Modelling*, vol. 38, no. 17-18, pp. 4396-4408, 2014.

[4] B. Ahmed, B. M. Iqbal, A. Mazin, J. H. Iqbal, H. Mohammed and K. Roshdi, “Atomic exact solution for some fractional partial differential equations in Banach spaces,” *Partial Differential Equations in Applied Mathematics*, vol. 9, pp. 01-08, 2024.

[5] S. Mohammed, M. Shqair, I. M. Batiha, M. H. Abu-Sei’leek, S. Alshorm, A. Abdelnebi and I. H. Jebril, “Numerical Solutions for Fractional Multi-Group Neutron Diffusion System of Equations,” *International Journal of Neutrosophic Science*, vol. 24, pp. 08-38, 2024.

[6] A. B. Malinowska, T. Odzijewicz and D. F. Torres, *Advanced Methods in the Fractional Calculus of Variations,* Cham: Springer, 2015.

[7] V. Daftardar-Gejji, "Numerics of Fractional Differential Equations," *Fractional Calculus and Fractional Differential Equations*, Birkhäuser, Singapore, pp. 1-16, 2019.

[8] J. Muscat, *Functional analysis: an introduction to metric spaces, Hilbert spaces, and Banach algebras,* Springer, 2014.

[9] F. Magri, “Variational formulation for every linear equation,” *International Journal of Engineering Science*, vol. *12, no.* 6, pp.537-549, 1974.

[10] J. N. Reddy and J. N. Reddy, *Applied functional analysis and variational methods in engineering*, New York: McGraw-Hill, 1986.