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Characterization of P-Semi Homogeneous System of Difference Equations of Three dimension

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Abstract

The aim of this paper is to define new concepts, namely a homogenous system of difference equations $x(n+1) = Bx(n)$ where B is a matrix of real numbers, which is called P-semi homogenous of order m if there exists a non-zero matrix A and integer number m such that the following equation holds: $F\,(A(c)x(n)) = P(A(c))^m F(x(n))$, Where F is a function, m and P are integer numbers and c is a real number. This definition is a generalization to the (3×3)-semi-homogeneous system of difference equations of order m. Special cases are studied of this definition and illustrative examples are given and some characterizations of this definition are also given. The necessary and sufficient conditions for a homogenous system of difference Equations to be P-semi homogenous of order one or greater than one as well as some examples and theorems about there are given.

Keywords: Difference equations; Homogenous system; Semi-homogenous; P-semi-homogenous

توصيف نظام شبه متجانس لمعادلات الفروق من النمط P ثلاثي الابعاد

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الخلاصة

الهدف الأساسي من هذا البحث هو تعريف مفاهيم جديدة، يسمى النظام لمعادلات الفروق شبه متجانس من النمط P من الرتبة m إذا كانت هنالك مصفوفة غير صفرية A وعدد صحيح m بحيث تحقق المعادلة التالية:

$$F\,(A(c)x(n)) = P(A(c))^m F(x(n)),$$

، حيث F دالة , P هو عدد صحيح و c هو عدد حقيقي.

هذا التعريف المعمم على نظام شبه متجانس (3×3) من معادلات الفروق من الرتبة m ، حيث m عدد صحيح موجب. وتم دراسة جميع الحالات الخاصة بهذا التعريف وتم أعطاء بعض الأمثلة عليها.

أيضًا ، أعطيت بعض التوصيفات لتلك التعاريف ، الشروط الضرورية والكافية لنظام معادلات الفروق لتكون شبه متجانسة من النمط P من الرتبة الاولى والرتبةالاكبر من الاولى ، بالإضافة إلى بعض الأمثلة والنظريات حول ذلك.

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1- Introduction

Difference equations are existed since ancient times, there are two types of functions. First, the functions where the variable *x* can take every possible value in a given interval [9-11]. The general form of the difference equation is given by [1-4].

$$a(n+t) = f(a(n)) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where f is a function of a sequence $a(n)$ and n is a natural number.

In 2016, Al-Asadi [5] studied the properties of difference systems in his paper. In 2021, Abed and Al-Asadi [6] introduced the concept of a self-semi-homogenous system of difference equations, which they defined as the following:

A homogenous system of difference equations is called *self-semi homogenous* if there exists a non-zero, nonidentity real matrix M such that the following equation holds

$$f(Ma(n)) = M^m f(a(n)) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

In this paper new definitions of a homogeneous system are introduced that are generalized semi-homogenous systems defined as the following:

A homogenous system of difference equations is called generalized semi-homogenous if there exists a non-zero, real matrix M such that the following equation holds

$$F(Mx(n)) = P^k M^m F(x(n)), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where P, k, and m are integer numbers.
There are some special cases that are discussed in this work as well as the general case. Some examples and characteristics for definitions are given. We also prove some theorems and properties which can be summarized as follows:
1- If $P = 1$, then this system is called semi-homogenous of order m [6].
2- If $P \neq 1$, $k = 1$, then this system is called P-semi-homogenous of order m.
3- If $P \neq 1$, $k \neq 1$, then this system is called P^k-semi-homogenous of order m.
4- If $k = m$, then Equation (2) becomes like the following:

$$F\big(MX(n)\big) = (PM)^k F\big(X(n)\big)$$

and it is called P^k*-semi-homogenous* of order k.

2- *P*-semi-homogenous of order m

In this section P-semi-homogenous of order m is studied when $m = 1$ and in case $m > 1$. Consider the system

$$F(X) = BX, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

where $F = \begin{pmatrix} f \\ g \\ h \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $F(X(n)) = \begin{pmatrix} f(x(n)) \\ g(y(n)) \\ h(z(n)) \end{pmatrix} = \begin{pmatrix} x(n+1) \\ y(n+1) \\ z(n+1) \end{pmatrix}$

and $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$.

Definition 2.1.

System (4) is called P-semi homogenous of order m if there exists a non-zero matrix A and integer number m such that the following equation holds.

$$F\,(A(c)x(n)) = P(A(c))^m F(x(n)), \qquad \text{………………………(5)}$$

where $A(c)$ is a matrix, P is an integer number and c is a real number.

Example 2.2.

Consider the system $F(x) = Bx$ where $B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, then we can find a matrix $A = \begin{bmatrix} \frac{3}{13} & \frac{-2}{13} & 0 \\ \frac{2}{13} & \frac{3}{13} & 0 \\ \frac{-2}{13} & \frac{-3}{13} & 0 \end{bmatrix}$ with $P = 3$. Therefore, this system is p-semi-homogenous of order one.

The following theorem shows a matrix A exists in definition 2.1 and the general formula is given.

Theorem 2.3.

The necessary and sufficient conditions for a homogenous system of difference Equations (4) to be P-semi homogenous of order one is the A equal to

$$\begin{bmatrix} \frac{Pa_{12}b_{21} + Pa_{13}b_{31}}{b_{11} - Pb_{11}} & \frac{Pb_{11}b_{12} - b_{11}b_{12} + a_{13}(P^2b_{31}b_{12} + Pb_{11}b_{32} - P^2b_{11}b_{32})}{b_{11}^2 - Pb_{11}^2 - P^2b_{21}b_{12} - Pb_{11}b_{22} + P^2b_{11}b_{22}} & \frac{Pb_{11}b_{13} - b_{11}b_{13} + a_{12}(P^2b_{21}b_{13} + Pb_{11}b_{23} - P^2b_{11}b_{23})}{b_{11}^2 - Pb_{11}^2 - P^2b_{31}b_{13} - Pb_{11}b_{33} + P^2b_{11}b_{33}} \\ \frac{Pb_{22}b_{21} - b_{22}b_{21} + a_{23}(P^2b_{32}b_{21} + Pb_{22}b_{31} - P^2b_{22}b_{31})}{b_{22}^2 - Pb_{22}^2 - P^2b_{21}b_{12} - Pb_{11}b_{22} + P^2b_{11}b_{22}} & \frac{Pa_{21}b_{12} + Pa_{23}b_{32}}{b_{22} - Pb_{22}} & \frac{Pb_{22}b_{23} - b_{22}b_{23} + a_{21}(P^2b_{23}b_{12} + Pb_{22}b_{13} - P^2b_{22}b_{13})}{b_{22}^2 - Pb_{22}^2 - P^2b_{32}b_{23} - Pb_{22}b_{33} + P^2b_{22}b_{33}} \\ \frac{Pb_{33}b_{31} - b_{33}b_{31} + a_{32}(P^2b_{31}b_{23} + Pb_{33}b_{21} - P^2b_{33}b_{21})}{b_{33}^2 - Pb_{33}^2 - P^2b_{13}b_{31} - Pb_{33}b_{11} + P^2b_{33}b_{11}} & \frac{Pb_{33}b_{32} - b_{33}b_{32} + a_{31}(P^2b_{32}b_{13} + Pb_{33}b_{12} - P^2b_{33}b_{12})}{b_{33}^2 - Pb_{33}^2 - P^2b_{23}b_{32} - Pb_{33}b_{22} + P^2b_{33}b_{22}} & \frac{Pa_{31}b_{13} + Pa_{32}b_{23}}{b_{33} - Pb_{33}} \end{bmatrix}$$

Proof:

Necessary condition:

Since F is homogenous of degree one, then there exists a non-zero matrix A, such that the Equation (5) is held with $m = 1$; that is,

$$F\,(A(c)x(n)) = PA(c)F(x(n))$$

$$F\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = P\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$\begin{pmatrix} f(a_{11}x + a_{12}y + a_{13}z) \\ g(a_{21}x + a_{22}y + a_{23}z) \\ h(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} Pa_{11}f(x) + Pa_{12}g(y) + Pa_{13}h(z) \\ Pa_{21}f(x) + Pa_{22}g(y) + Pa_{23}h(z) \\ Pa_{31}f(x) + Pa_{32}g(y) + Pa_{33}h(z) \end{pmatrix}$$

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y + a_{13}z) + b_{12}y + b_{13}z \\ b_{21}x + b_{22}(a_{21}x + a_{22}y + a_{23}z) + b_{23}z \\ b_{31}x + b_{32}y + b_{33}(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} Pa_{11}f(x) + Pa_{12}g(y) + Pa_{13}h(z) \\ Pa_{21}f(x) + Pa_{22}g(y) + Pa_{23}h(z) \\ Pa_{31}f(x) + Pa_{32}g(y) + Pa_{33}h(z) \end{pmatrix}$$

$$b_{11}a_{11} = Pa_{11}b_{11} + Pa_{12}b_{21} + Pa_{13}b_{31}$$

$$b_{11}a_{12} + b_{12} = Pa_{11}b_{12} + Pa_{12}b_{22} + Pa_{13}b_{32}$$

$b_{11}a_{13} + b_{13} = Pa_{11}b_{13} + Pa_{12}b_{23} + Pa_{13}b_{33}$
$b_{21} + b_{22}a_{21} = Pa_{21}b_{11} + Pa_{22}b_{21} + Pa_{23}b_{31}$
$b_{22}a_{22} = Pa_{21}b_{12} + Pa_{22}b_{22} + Pa_{23}b_{32}$
$b_{22}a_{23} + b_{23} = Pa_{21}b_{13} + Pa_{22}b_{23} + Pa_{23}b_{33}$
$b_{31} + b_{33}a_{31} = Pa_{31}b_{11} + Pa_{32}b_{21} + Pa_{33}b_{31}$
$b_{32} + b_{33}a_{32} = Pa_{31}b_{12} + Pa_{32}b_{22} + Pa_{33}b_{32}$
$b_{33}a_{33} = Pa_{31}b_{13} + Pa_{32}b_{23} + Pa_{33}b_{33}$

It is easy to show the matrix A equal to

$$\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

where

$$z_{11} = \frac{Pa_{12}b_{21}+Pa_{13}b_{31}}{b_{11}-Pb_{11}}, \quad z_{21} = \frac{Pb_{22}b_{21}-b_{22}b_{21}+a_{23}(P^2b_{32}b_{21}+Pb_{22}b_{31}-P^2b_{22}b_{31})}{b_{22}^2-Pb_{22}^2-P^2b_{21}b_{12}-Pb_{11}b_{22}+P^2b_{11}b_{22}}$$

$$z_{31} = \frac{Pb_{33}b_{31}-b_{33}b_{31}+a_{32}(P^2b_{31}b_{23}+Pb_{33}b_{21}-P^2b_{33}b_{21})}{b_{33}^2-Pb_{33}^2-P^2b_{13}b_{31}-Pb_{33}b_{11}+P^2b_{33}b_{11}},$$

$$z_{12} = \frac{Pb_{11}b_{12}-b_{11}b_{12}+a_{13}\left(P^2b_{31}b_{12}+Pb_{11}b_{32}-P^2b_{11}b_{32}\right)}{b_{11}^2-Pb_{11}^2-P^2b_{21}b_{12}-Pb_{11}b_{22}+P^2b_{11}b_{22}}, \quad z_{22} = \frac{Pa_{21}b_{12}+Pa_{23}b_{32}}{b_{22}-Pb_{22}}$$

$$z_{32} = \frac{Pb_{33}b_{32}-b_{33}b_{32}+a_{31}\left(P^2b_{32}b_{13}+Pb_{33}b_{12}-P^2b_{33}b_{12}\right)}{b_{33}^2-Pb_{33}^2-P^2b_{23}b_{32}-Pb_{33}b_{22}+P^2b_{33}b_{22}},$$

$$z_{13} = \frac{Pb_{11}b_{13}-b_{11}b_{13}+a_{12}(P^2b_{21}b_{13}+Pb_{11}b_{23}-P^2b_{11}b_{23})}{b_{11}^2-Pb_{11}^2-P^2b_{31}b_{13}-Pb_{11}b_{33}+P^2b_{11}b_{33}},$$

$$z_{23} = \frac{Pb_{22}b_{23}-b_{22}b_{23}+a_{21}(P^2b_{23}b_{12}+Pb_{22}b_{13}-P^2b_{22}b_{13})}{b_{22}^2-Pb_{22}^2-P^2b_{32}b_{23}-Pb_{22}b_{33}+P^2b_{22}b_{33}} \text{ and } \quad z_{33} = \frac{Pa_{31}b_{13}+Pa_{32}b_{23}}{b_{33}-Pb_{33}}$$

Sufficient condition:

Suppose that there is a matrix A equal to $\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$

To show that the system (4) is P-semi-homogeneous that is

$$F\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = P\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

By substituting the value of the matrix A in (6) we have

$$F\left(\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = P\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

That is

$$F\left(\begin{bmatrix} \frac{Pa_{12}b_{21}+Pa_{13}b_{31}}{b_{11}-Pb_{11}} & \frac{Pb_{22}b_{21}-b_{22}b_{21}+a_{23}(P^2b_{32}b_{21}+Pb_{22}b_{31}-P^2b_{22}b_{31})}{b_{22}^2-Pb_{22}^2-P^2b_{21}b_{12}-Pb_{11}b_{22}+P^2b_{11}b_{22}} & \frac{Pb_{33}b_{31}-b_{33}b_{31}+a_{32}(P^2b_{31}b_{23}+Pb_{33}b_{21}-P^2b_{33}b_{21})}{b_{33}^2-Pb_{33}^2-P^2b_{13}b_{31}-Pb_{33}b_{11}+P^2b_{33}b_{11}} \\ \frac{Pb_{11}b_{12}-b_{11}b_{12}+a_{13}(P^2b_{31}b_{12}+Pb_{11}b_{32}-P^2b_{11}b_{32})}{b_{11}^2-Pb_{11}^2-P^2b_{21}b_{12}-Pb_{11}b_{22}+P^2b_{11}b_{22}} & \frac{Pa_{21}b_{12}+Pa_{23}b_{32}}{b_{22}-Pb_{22}} & \frac{Pb_{33}b_{32}-b_{33}b_{32}+a_{31}(P^2b_{32}b_{13}+Pb_{33}b_{12}-P^2b_{33}b_{12})}{b_{33}^2-Pb_{33}^2-P^2b_{23}b_{32}-Pb_{33}b_{22}+P^2b_{33}b_{22}} \\ \frac{Pb_{11}b_{13}-b_{11}b_{13}+a_{12}(P^2b_{21}b_{13}+Pb_{11}b_{23}-P^2b_{11}b_{23})}{b_{11}^2-Pb_{11}^2-P^2b_{31}b_{13}-Pb_{11}b_{33}+P^2b_{11}b_{33}} & \frac{Pb_{22}b_{23}-b_{22}b_{23}+a_{21}(P^2b_{23}b_{12}+Pb_{22}b_{13}-P^2b_{22}b_{13})}{b_{22}^2-Pb_{22}^2-P^2b_{32}b_{23}-Pb_{22}b_{33}+P^2b_{22}b_{33}} & \frac{Pa_{31}b_{13}+Pa_{32}b_{23}}{b_{33}-Pb_{33}} \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right)$$

$$= P\begin{bmatrix} \frac{Pa_{12}b_{21}+Pa_{13}b_{31}}{b_{11}-Pb_{11}} & \frac{Pb_{22}b_{21}-b_{22}b_{21}+a_{23}(P^2b_{32}b_{21}+Pb_{22}b_{31}-P^2b_{22}b_{31})}{b_{22}^2-Pb_{22}^2-P^2b_{21}b_{12}-Pb_{11}b_{22}+P^2b_{11}b_{22}} & \frac{Pb_{33}b_{31}-b_{33}b_{31}+a_{32}(P^2b_{31}b_{23}+Pb_{33}b_{21}-P^2b_{33}b_{21})}{b_{33}^2-Pb_{33}^2-P^2b_{13}b_{31}-Pb_{33}b_{11}+P^2b_{33}b_{11}} \\ \frac{Pb_{11}b_{12}-b_{11}b_{12}+a_{13}(P^2b_{31}b_{12}+Pb_{11}b_{32}-P^2b_{11}b_{32})}{b_{11}^2-Pb_{11}^2-P^2b_{21}b_{12}-Pb_{11}b_{22}+P^2b_{11}b_{22}} & \frac{Pa_{21}b_{12}+Pa_{23}b_{32}}{b_{22}-Pb_{22}} & \frac{Pb_{33}b_{32}-b_{33}b_{32}+a_{31}(P^2b_{32}b_{13}+Pb_{33}b_{12}-P^2b_{33}b_{12})}{b_{33}^2-Pb_{33}^2-P^2b_{23}b_{32}-Pb_{33}b_{22}+P^2b_{33}b_{22}} \\ \frac{Pb_{11}b_{13}-b_{11}b_{13}+a_{12}(P^2b_{21}b_{13}+Pb_{11}b_{23}-P^2b_{11}b_{23})}{b_{11}^2-Pb_{11}^2-P^2b_{31}b_{13}-Pb_{11}b_{33}+P^2b_{11}b_{33}} & \frac{Pb_{22}b_{23}-b_{22}b_{23}+a_{21}(P^2b_{23}b_{12}+Pb_{22}b_{13}-P^2b_{22}b_{13})}{b_{22}^2-Pb_{22}^2-P^2b_{32}b_{23}-Pb_{22}b_{33}+P^2b_{22}b_{33}} & \frac{Pa_{31}b_{13}+Pa_{32}b_{23}}{b_{33}-Pb_{33}} \end{bmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

Therefore,

$a_{11}b_{11} = Pa_{11}b_{11} + Pa_{12}b_{21} + Pa_{13}b_{31}$

We substitute the value $a_{11} = \frac{Pa_{12}b_{21}+Pa_{13}b_{31}}{b_{11}(1-P)}$ we have

$b_{11}\frac{Pa_{12}b_{21}+Pa_{13}b_{31}}{b_{11}(1-P)} = b_{11}\frac{P^2a_{12}b_{21}+P^2a_{13}b_{31}}{b_{11}(1-P)} + Pa_{12}b_{21} + Pa_{13}b_{31}$, therefore

$$Pa_{12}b_{21} + Pa_{13}b_{31} = P^2a_{12}b_{21} + P^2a_{13}b_{31} + Pa_{12}b_{21} - P^2a_{12}b_{21} + Pa_{13}b_{31} - P^2a_{13}b_{31}.$$

Hence,

$Pa_{12}b_{21} + Pa_{13}b_{31} = Pa_{12}b_{21} + Pa_{13}b_{31}$

The second equation

$$a_{12}b_{11}+b_{12}=Pa_{11}b_{12}+Pa_{12}b_{22}+Pa_{13}b_{32}$$

$$a_{12}b_{11}+b_{12}=\frac{P^2a_{12}b_{21}b_{12}+P^2a_{13}b_{31}b_{12}}{b_{11}-Pb_{11}}+Pa_{12}b_{22}+Pa_{13}b_{32}$$

$$a_{12}b_{11}^2-Pa_{12}b_{11}^2+b_{12}b_{11}-Pb_{12}b_{11} \\ =P^2a_{12}b_{21}b_{12}+P^2a_{13}b_{31}b_{12}+\ Pa_{12}b_{22}b_{11}-P^2a_{12}b_{22}b_{11}+\ Pa_{13}b_{32}b_{11} \\ -P^2a_{13}b_{32}b_{11}$$

$$a_{12}(b_{11}^2-Pb_{11}^2-P^2b_{21}b_{12}-\ Pb_{22}b_{11}+P^2b_{22}b_{11}) \\ =\ Pb_{12}b_{11}-b_{12}b_{11}+a_{13}(P^2b_{31}b_{12}+\ Pb_{32}b_{11}-P^2b_{32}b_{11})$$

$$\left(\frac{Pb_{12}b_{11}-b_{12}b_{11}+a_{13}(P^2b_{31}b_{12}+\ Pb_{32}b_{11}-P^2b_{32}b_{11})}{b_{11}^2-Pb_{11}^2-P^2b_{21}b_{12}-\ Pb_{22}b_{11}+P^2b_{22}b_{11}}\right)(b_{11}^2-Pb_{11}^2-P^2b_{21}b_{12} \\ -\ Pb_{22}b_{11}+P^2b_{22}b_{11}) \\ =\ Pb_{12}b_{11}-b_{12}b_{11}+a_{13}(P^2b_{31}b_{12}+\ Pb_{32}b_{11}-P^2b_{32}b_{11})$$

$$Pb_{12}b_{11}-b_{12}b_{11}+a_{13}(P^2b_{31}b_{12}+\ Pb_{32}b_{11}-P^2b_{32}b_{11})=Pb_{12}b_{11}-b_{12}b_{11}+ \\ a_{13}(P^2b_{31}b_{12}+\ Pb_{32}b_{11}-P^2b_{32}b_{11})$$

The third equation

$$a_{13}b_{11}+b_{13}=Pa_{11}b_{13}+Pa_{12}b_{23}+Pa_{13}b_{33}$$

$$a_{13}b_{11}+b_{13}=\frac{P^2a_{12}b_{21}b_{13}+P^2a_{13}b_{31}b_{13}}{b_{11}-Pb_{11}}+Pa_{12}b_{23}+Pa_{13}b_{33}$$

$$a_{13}b_{11}^2-Pa_{13}b_{11}^2+b_{13}b_{11}-Pb_{13}b_{11} \\ =P^2a_{12}b_{21}b_{13}+P^2a_{13}b_{31}b_{13}+\ Pa_{12}b_{23}b_{11}-P^2a_{12}b_{23}b_{11}+\ Pa_{13}b_{33}b_{11} \\ -P^2a_{13}b_{33}b_{11}$$

$$a_{13}(b_{11}^2-Pb_{11}^2-P^2b_{31}b_{13}-\ Pb_{33}b_{11}+P^2b_{33}b_{11}) \\ =\ Pb_{13}b_{11}-b_{13}b_{11}+a_{12}(P^2b_{21}b_{13}+\ Pb_{23}b_{11}-P^2b_{23}b_{11})$$

$$\left(\frac{Pb_{13}b_{11}-b_{13}b_{11}+a_{12}(P^2b_{21}b_{13}+\ Pb_{23}b_{11}-P^2b_{23}b_{11})}{b_{11}^2-Pb_{11}^2-P^2b_{31}b_{13}-\ Pb_{33}b_{11}+P^2b_{33}b_{11}}\right)(b_{11}^2-Pb_{11}^2-P^2b_{31}b_{13} \\ -\ Pb_{33}b_{11}+P^2b_{33}b_{11}) \\ =\ Pb_{13}b_{11}-b_{13}b_{11}+a_{12}(P^2b_{21}b_{13}+\ Pb_{23}b_{11}-P^2b_{23}b_{11})$$

$$Pb_{13}b_{11}-b_{13}b_{11}+a_{12}(P^2b_{21}b_{13}+\ Pb_{23}b_{11}-P^2b_{23}b_{11})=\ Pb_{13}b_{11}-b_{13}b_{11}+ \\ a_{12}(P^2b_{21}b_{13}+\ Pb_{23}b_{11}-P^2b_{23}b_{11})$$

$$a_{21}b_{22}+b_{21}=Pa_{21}b_{11}+Pa_{22}b_{21}+Pa_{23}b_{31}$$

$$a_{21}b_{22}+b_{21}=Pa_{21}b_{11}+\frac{P^2a_{21}b_{21}b_{12}+P^2a_{23}b_{21}b_{32}}{b_{22}-Pb_{22}}+Pa_{23}b_{31}$$

$$a_{21}b_{22}^2-Pa_{21}b_{22}^2+b_{21}b_{22}-Pb_{21}b_{22} \\ =\ Pa_{21}b_{11}b_{22}-P^2a_{21}b_{11}b_{22}+P^2a_{21}b_{21}b_{12}+P^2a_{23}b_{21}b_{32}+\ Pa_{23}b_{31}b_{22} \\ -P^2a_{23}b_{31}b_{22}$$

$$a_{21}(b_{22}^2-\ Pb_{22}^2-\ Pb_{11}b_{22}+P^2b_{11}b_{22}-P^2b_{21}b_{12}) \\ =\ Pb_{21}b_{22}-b_{21}b_{22}+a_{23}(P^2b_{21}b_{32}+\ Pb_{31}b_{22}-P^2b_{31}b_{22})$$

$$\left(\frac{Pb_{21}b_{22}-b_{21}b_{22}+a_{23}(P^2b_{21}b_{32}+\ Pb_{31}b_{22}-P^2b_{31}b_{22})}{b_{22}^2-\ Pb_{22}^2-\ Pb_{11}b_{22}+P^2b_{11}b_{22}-P^2b_{21}b_{12}}\right)(b_{22}^2-\ Pb_{22}^2-\ Pb_{11}b_{22} \\ +P^2b_{11}b_{22}-P^2b_{21}b_{12}) \\ =\ Pb_{21}b_{22}-b_{21}b_{22}+a_{23}(P^2b_{21}b_{32}+\ Pb_{31}b_{22}-P^2b_{31}b_{22})$$

$$Pb_{21}b_{22}-b_{21}b_{22}+a_{23}(P^2b_{21}b_{32}+\ Pb_{31}b_{22}-P^2b_{31}b_{22}) \\ =Pb_{21}b_{22}-b_{21}b_{22}+a_{23}(P^2b_{21}b_{32}+\ Pb_{31}b_{22}-P^2b_{31}b_{22})$$

$$a_{22}b_{22}=Pa_{22}b_{22}+Pa_{21}b_{12}+Pa_{23}b_{32}$$

$$b_{22}\frac{Pa_{21}b_{12}+Pa_{23}b_{32}}{b_{22}-Pb_{22}}=b_{22}\frac{P^2a_{21}b_{12}+P^2a_{23}b_{32}}{b_{22}-Pb_{22}}+Pa_{21}b_{12}+Pa_{23}b_{32}$$

$$Pa_{21}b_{12} + Pa_{23}b_{32} = P^2a_{21}b_{12} + P^2a_{23}b_{32} + Pa_{21}b_{12} - P^2a_{21}b_{12} + Pa_{23}b_{32} - P^2a_{23}b_{32}$$
$$Pa_{21}b_{12} + Pa_{23}b_{32} = Pa_{21}b_{12} + Pa_{23}b_{32}$$
$$a_{23}b_{22} + b_{23} = Pa_{21}b_{13} + Pa_{22}b_{23} + Pa_{23}b_{33}$$
$$a_{23}b_{22} + b_{23} = Pa_{21}b_{13} + \frac{P^2a_{21}b_{12}b_{23} + P^2a_{23}b_{32}b_{23}}{b_{22} - Pb_{22}} + Pa_{23}b_{33}$$
$$a_{23}b_{22}^2 - Pa_{23}b_{22}^2 + b_{23}b_{22} - Pb_{23}b_{22} = Pa_{21}b_{13}b_{22} - P^2a_{21}b_{13}b_{22} + P^2a_{21}b_{12}b_{23} + P^2a_{23}b_{32}b_{23} + Pa_{23}b_{33}b_{22} - P^2a_{23}b_{33}b_{22}$$
$$a_{23}(b_{22}^2 - Pb_{22}^2 - P^2a_{23}b_{32}b_{23} - Pa_{23}b_{33}b_{22} + P^2a_{23}b_{33}b_{22}) = Pb_{23}b_{22} - b_{23}b_{22} + a_{21}(Pb_{13}b_{22} - P^2b_{13}b_{22} + P^2b_{12}b_{23})$$
$$\left(\frac{Pb_{23}b_{22} - b_{23}b_{22} + a_{21}(Pb_{13}b_{22} - P^2b_{13}b_{22} + P^2b_{12}b_{23})}{b_{22}^2 - Pb_{22}^2 - P^2a_{23}b_{32}b_{23} - Pa_{23}b_{33}b_{22} + P^2a_{23}b_{33}b_{22}}\right)(b_{22}^2 - Pb_{22}^2 - P^2a_{23}b_{32}b_{23} - Pa_{23}b_{33}b_{22} + P^2a_{23}b_{33}b_{22}) = Pb_{23}b_{22} - b_{23}b_{22} + a_{21}(Pb_{13}b_{22} - P^2b_{13}b_{22} + P^2b_{12}b_{23})$$
$$Pb_{23}b_{22} - b_{23}b_{22} + a_{21}(Pb_{13}b_{22} - P^2b_{13}b_{22} + P^2b_{12}b_{23}) = Pb_{23}b_{22} - b_{23}b_{22} + a_{21}(Pb_{13}b_{22} - P^2b_{13}b_{22} + P^2b_{12}b_{23})$$
$$a_{31}b_{33} + b_{31} = Pa_{31}b_{11} + Pa_{32}b_{21} + Pa_{33}b_{31}$$
$$a_{31}b_{33} + b_{31} = Pa_{31}b_{11} + Pa_{32}b_{21} + \frac{P^2a_{31}b_{13}b_{31} + P^2a_{32}b_{23}b_{31}}{b_{33} - Pb_{33}}$$
$$a_{31}b_{33}^2 - Pa_{31}b_{33}^2 + b_{31}b_{33} - Pb_{31}b_{33} = Pa_{31}b_{11}b_{33} - P^2a_{31}b_{11}b_{33} + Pa_{32}b_{21}b_{33} - P^2a_{32}b_{21}b_{33} + P^2a_{31}b_{13}b_{31} + P^2a_{32}b_{23}b_{31}$$
$$a_{31}(b_{33}^2 - Pb_{33}^2 - Pb_{11}b_{33} + P^2b_{11}b_{33} - P^2b_{13}b_{31}) = Pb_{31}b_{33} - b_{31}b_{33} + a_{32}(Pb_{21}b_{33} - P^2b_{21}b_{33} + P^2b_{23}b_{31})$$
$$\left(\frac{Pb_{31}b_{33} - b_{31}b_{33} + a_{32}(Pb_{21}b_{33} - P^2b_{21}b_{33} + P^2b_{23}b_{31})}{b_{33}^2 - Pb_{33}^2 - Pb_{11}b_{33} + P^2b_{11}b_{33} - P^2b_{13}b_{31}}\right)(b_{33}^2 - Pb_{33}^2 - Pb_{11}b_{33} + P^2b_{11}b_{33} - P^2b_{13}b_{31}) = Pb_{31}b_{33} - b_{31}b_{33} + a_{32}(Pb_{21}b_{33} - P^2b_{21}b_{33} + P^2b_{23}b_{31})$$
$$Pb_{31}b_{33} - b_{31}b_{33} + a_{32}(Pb_{21}b_{33} - P^2b_{21}b_{33} + P^2b_{23}b_{31}) = Pb_{31}b_{33} - b_{31}b_{33} + a_{32}(Pb_{21}b_{33} - P^2b_{21}b_{33} + P^2b_{23}b_{31})$$
$$a_{32}b_{33} + b_{32} = Pa_{31}b_{12} + Pa_{32}b_{22} + Pa_{33}b_{32}$$
$$a_{32}b_{33} + b_{32} = Pa_{31}b_{12} + Pa_{32}b_{22} + \frac{P^2a_{31}b_{13}b_{32} + P^2a_{32}b_{23}b_{32}}{b_{33} - Pb_{33}}$$
$$a_{32}b_{33}^2 - Pa_{32}b_{33}^2 + b_{32}b_{33} - Pb_{32}b_{33} = Pa_{31}b_{12}b_{33} - P^2a_{31}b_{12}b_{33} + Pa_{32}b_{22}b_{33} - P^2a_{32}b_{22}b_{33} + P^2a_{31}b_{13}b_{32} + P^2a_{32}b_{23}b_{32}$$
$$a_{32}(b_{33}^2 - Pb_{33}^2 - Pb_{22}b_{33} + P^2b_{22}b_{33} - P^2b_{23}b_{32}) = Pb_{32}b_{33} - b_{32}b_{33} + a_{31}(Pb_{12}b_{33} - P^2b_{12}b_{33} + P^2b_{13}b_{32})$$
$$\left(\frac{Pb_{32}b_{33} - b_{32}b_{33} + a_{31}(Pb_{12}b_{33} - P^2b_{12}b_{33} + P^2b_{13}b_{32})}{b_{33}^2 - Pb_{33}^2 - Pb_{22}b_{33} + P^2b_{22}b_{33} - P^2b_{23}b_{32}}\right)(b_{33}^2 - Pb_{33}^2 - Pb_{22}b_{33} + P^2b_{22}b_{33} - P^2b_{23}b_{32}) = Pb_{32}b_{33} - b_{32}b_{33} + a_{31}(Pb_{12}b_{33} - P^2b_{12}b_{33} + P^2b_{13}b_{32})$$
$$Pb_{32}b_{33} - b_{32}b_{33} + a_{31}(Pb_{12}b_{33} - P^2b_{12}b_{33} + P^2b_{13}b_{32}) = Pb_{32}b_{33} - b_{32}b_{33} + a_{31}(Pb_{12}b_{33} - P^2b_{12}b_{33} + P^2b_{13}b_{32})$$
$$a_{33}b_{33} = Pa_{33}b_{33} + Pa_{31}b_{13} + Pa_{32}b_{23}$$

$$b_{33}\frac{Pa_{31}b_{13}+Pa_{32}b_{23}}{b_{33}-Pb_{33}}=b_{33}\frac{P^2a_{31}b_{13}+P^2a_{32}b_{23}}{b_{33}-Pb_{33}}+Pa_{31}b_{13}+Pa_{32}b_{23}$$
$$Pa_{31}b_{13}+Pa_{32}b_{23}=P^2a_{31}b_{13}+P^2a_{32}b_{23}+Pa_{31}b_{13}-P^2a_{31}b_{13}+Pa_{32}b_{23}-P^2a_{32}b_{23}$$
$$Pa_{31}b_{13}+Pa_{32}b_{23}=Pa_{31}b_{13}+Pa_{32}b_{23}$$
That is, the left hand is equal to the right hand, and the system (4) is P-semi-homogeneous of order one.

Corollary 2.4. A homogenous system of difference Equations (4) is P-semi homogenous of order one, if the following is held:
$$b_{11}a_{11} = Pa_{11}b_{11}+Pa_{12}b_{21}+Pa_{13}b_{31}$$
$$b_{11}a_{12}+b_{12}=Pa_{11}b_{12}+Pa_{12}b_{22}+Pa_{13}b_{32}$$
$$b_{11}a_{13}+b_{13}=Pa_{11}b_{13}+Pa_{12}b_{23}+Pa_{13}b_{33}$$
$$b_{21}+b_{22}a_{21}=Pa_{21}b_{11}+Pa_{22}b_{21}+Pa_{23}b_{31}$$
$$b_{22}a_{22}=Pa_{21}b_{12}+Pa_{22}b_{22}+Pa_{23}b_{32}$$
$$b_{22}a_{23}+b_{23}=Pa_{21}b_{13}+Pa_{22}b_{23}+Pa_{23}b_{33}$$
$$b_{31}+b_{33}a_{31}=Pa_{31}b_{11}+Pa_{32}b_{21}+Pa_{33}b_{31}$$
$$b_{32}+b_{33}a_{32}=Pa_{31}b_{12}+Pa_{32}b_{22}+Pa_{33}b_{32}$$
$$b_{33}a_{33}=Pa_{31}b_{13}+Pa_{32}b_{23}+Pa_{33}b_{33}$$
Proof: Direct from Theorem 2.3.

Example 2.5.

Consider the system $F(x)=Bx$ where $B=\begin{bmatrix}1 & -2 & 0\\ 1 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$, then there is a matrix A equal to $A=\begin{bmatrix}\frac{4}{9} & \frac{-2}{9} & 0\\ \frac{1}{9} & \frac{4}{9} & 0\\ \frac{-1}{3} & \frac{-4}{3} & 0\end{bmatrix}$ with $P=2$, then this system is P-semi homogenous of order one.

To show that by using Corollary 2.4.
$$b_{11}a_{11} = Pa_{11}b_{11}+Pa_{12}b_{21}+Pa_{13}b_{31}$$
that is $1\left(\frac{4}{9}\right)=2\left(\frac{4}{9}\right)1+2\left(\frac{-2}{9}\right)1+0=\frac{4}{9}$

In the same way, we complete the results

Corollary 2.6. The special case of 3

If $P\neq 1,\ k\neq 1$, then this system is called P^k-semi-homogenous of order m which is immediately held from case 2 because P and k are integers so $P^k$is an integer . To see that , the following example is given:

Example 2.7 Considers the system $F(x)=Bx$ where $B=\begin{bmatrix}1 & 0 & -1\\ 0 & 1 & -1\\ 1 & 0 & 1\end{bmatrix}$, then by Theorem 2.3, there is a matrix $A=\begin{bmatrix}\frac{4}{25} & 0 & \frac{-3}{25}\\ \frac{4}{25} & 0 & \frac{-3}{25}\\ \frac{3}{25} & 0 & \frac{4}{25}\end{bmatrix}$ with $P^k=2^2$.

It is clear that this system is 2^2-semi-homogenous or 4-semi-homogenous of order one.

3- *P*-semi-homogenous of order greater than one.

This section studies P-semi-homogenous of order greater than one. First, we need to state the general formula for the power of a matrix. The existence of this case is shown by giving an example.

Remark 3.1.

Since the matrix has 3 rows and 3 columns, then it has 3 eigenvalues. Assume that the eigenvalues are real, therefore, there are three cases:

Case 1. If the eigenvalues are distinct, say α_1, α_2 and $\alpha_3$then we can assume that:

Let $\alpha_{12} = (\alpha_1 - \alpha_2)$, $\alpha_{13} = (\alpha_1 - \alpha_3)$, $\alpha_{23} = (\alpha_2 - \alpha_3)$, $\alpha_{12}^n = (\alpha_1^n - \alpha_2^n)$, $\alpha_{13}^n = (\alpha_1^n - \alpha_3^n)$, $\alpha_{23}^n = (\alpha_2^n - \alpha_3^n)$, then

$A_{11} = [(\alpha_1^n \alpha_{12} + (a_{11} - \alpha_1)\alpha_{12}^n)\alpha_{13}\, \alpha_{23} + \big((a_{11} - \alpha_1)(a_{11} - \alpha_2) + a_{12}a_{21} + a_{13}a_{31}\big)\alpha_{23}\alpha_{13}^n - ((a_{11} - \alpha_1)(a_{11} - \alpha_2) + a_{12}a_{21} + a_{13}a_{31})\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$

$A_{12} = [a_{12}\alpha_{13}\, \alpha_{23}\alpha_{12}^n + \big((a_{11} - \alpha_1)a_{12} + a_{12}(a_{22} - \alpha_2) + a_{13}a_{32}\big)\alpha_{23}\alpha_{13}^n - \big((a_{11} - \alpha_1)a_{12} + a_{12}(a_{22} - \alpha_2) + a_{13}a_{32}\big)\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$ $A_{13} = [a_{13}\alpha_{13}\, \alpha_{23}\alpha_{12}^n + \big((a_{11} - \alpha_1)a_{13} + a_{12}a_{23} + a_{13}(a_{33} - \alpha_2)\big)\alpha_{23}\alpha_{13}^n - \big((a_{11} - \alpha_1)a_{13} + a_{12}a_{23} + a_{13}(a_{33} - \alpha_2)\big)\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$ $A_{21} = [a_{21}\alpha_{13}\, \alpha_{23}\alpha_{12}^n + \big((a_{11} - \alpha_2)a_{21} + (a_{22} - \alpha_1)a_{21} + a_{23}a_{31}\big)\alpha_{23}\alpha_{13}^n - \big((a_{11} - \alpha_2)a_{21} + (a_{22} - \alpha_1)a_{21} + a_{23}a_{31}\big)\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$ $A_{22} = [(\alpha_1^n \alpha_{12} + (a_{22} - \alpha_1)\alpha_{12}^n)\alpha_{13}\, \alpha_{23} + ((a_{12}a_{21} + (a_{22} - \alpha_1)(a_{22} - \alpha_2) + a_{23}a_{32})\alpha_{23}\alpha_{13}^n - ((a_{12}a_{21} + (a_{22} - \alpha_1)(a_{22} - \alpha_2) + a_{23}a_{32})\alpha_{13}\alpha_{23}^n]/\alpha_{12}\alpha_{13}\alpha_{23}$

$A_{23} = [a_{23}\alpha_{13}\, \alpha_{23}\alpha_{12}^n + (a_{21}a_{13} + (a_{22} - \alpha_1)a_{23} + (a_{33} - \alpha_2)a_{23})\alpha_{23}\alpha_{13}^n - (a_{21}a_{13} + (a_{22} - \alpha_1)a_{23} + (a_{33} - \alpha_2)a_{23})\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$ $A_{31} = [a_{31}\alpha_{13}\, \alpha_{23}\alpha_{12}^n + \big((a_{11} - \alpha_2)a_{31} + a_{32}a_{21} + (a_{33} - \alpha_1)a_{31}\big)\alpha_{23}\alpha_{13}^n - \big((a_{11} - \alpha_2)a_{31} + a_{32}a_{21} + (a_{33} - \alpha_1)a_{31}\big)\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$

$A_{32} = [a_{32}\alpha_{13}\, \alpha_{23}\alpha_{12}^n + (a_{31}a_{12} + a_{32}(a_{22} - \alpha_2) + (a_{33} - \alpha_1)a_{32})\alpha_{23}\alpha_{13}^n - (a_{31}a_{12} + a_{32}(a_{22} - \alpha_2) + (a_{33} - \alpha_1)a_{32})\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$ $A_{33} = [(\alpha_1^n \alpha_{12} + (a_{33} - \alpha_1)\alpha_{12}^n)\alpha_{13}\, \alpha_{23} + \big(a_{31}a_{13} + a_{32}a_{23} + (a_{33} - \alpha_2)(a_{33} - \alpha_2)\big)\alpha_{23}\alpha_{13}^n - \big(a_{31}a_{13} + a_{32}a_{23} + (a_{33} - \alpha_2)(a_{33} - \alpha_2)\big)\alpha_{13}\alpha_{23}^n]/\, \alpha_{12}\alpha_{13}\alpha_{23}$.

Case 2: If the eigenvalues are equal, let the eigenvalues equal to α, then

$A_{11} = \alpha^n + n\alpha^{n-1}(a_{11} - \alpha) + \frac{n(n-1)}{2}\alpha^{n-2}((a_{11} - \alpha)^2 + a_{12}a_{21} + a_{13}a_{31})$

$A_{12} = n\alpha^{n-1}a_{12} + \frac{n(n-1)}{2}\alpha^{n-2}((a_{11} - \alpha)a_{12} + a_{12}(a_{22} - \alpha) + a_{13}a_{32})$

$A_{13} = n\alpha^{n-1}a_{13} + \frac{n(n-1)}{2}\alpha^{n-2}((a_{11} - \alpha)a_{13} + a_{12}a_{23} + a_{13}(a_{33} - \alpha))$

$A_{21} = n\alpha^{n-1}a_{21} + \frac{n(n-1)}{2}\alpha^{n-2}(a_{21}(a_{11} - \alpha) + (a_{22} - \alpha)a_{21} + a_{23}a_{31})$

$A_{22} = \alpha^n + n\alpha^{n-1}(a_{22} - \alpha) + \frac{n(n-1)}{2}\alpha^{n-2}(a_{21}a_{12} + (a_{22} - \alpha)^2 + a_{23}a_{32})$

$A_{23} = n\alpha^{n-1}a_{23} + \frac{n(n-1)}{2}\alpha^{n-2}((a_{21}a_{13} + (a_{22} - \alpha)a_{23} + a_{23}(a_{32} - \alpha))$

$A_{31} = n\alpha^{n-1}a_{31} + \frac{n(n-1)}{2}\alpha^{n-2}(a_{31}(a_{11} - \alpha) + (a_{32}a_{21} + (a_{33} - \alpha)a_{31})$

$A_{32} = n\alpha^{n-1}a_{32} + \frac{n(n-1)}{2}\alpha^{n-2}((a_{31}a_{12} + a_{32}(a_{22} - \alpha) + (a_{33} - \alpha)a_{32})$

$A_{33} = \alpha^n + n\alpha^{n-1}(a_{33} - \alpha) + \frac{n(n-1)}{2}\alpha^{n-2}(a_{31}a_{13} + a_{32}a_{23} + (a_{33} - \alpha)^2)$

Case 3: if $\alpha = \alpha_1 = \alpha_2 \neq \alpha_3$, and let

$h = \frac{\alpha^{n-1}}{\alpha_3}[(\frac{\alpha}{\alpha_3}(1 - (\frac{\alpha}{\alpha_3})^n) - n(\frac{\alpha}{\alpha_3})^{n+1}(1 - \frac{\alpha}{\alpha_3}))/(1 - \frac{\alpha}{\alpha_3})^2]$

$A_{11} = \alpha^n + n\alpha^{n-1}(a_{11} - \alpha) + h((a_{11} - \alpha)(a_{11} - \alpha_3) + a_{12}a_{21} + a_{13}a_{31})$

$A_{12} = n\alpha^{n-1}a_{12} + h((a_{11} - \alpha)a_{12} + a_{12}(a_{22} - \alpha_3) + a_{13}a_{32})$

$A_{13} = n\alpha^{n-1}a_{13} + h((a_{11} - \alpha)a_{13} + a_{12}a_{23} + a_{13}(a_{33} - \alpha_3))$

$A_{21} = n\alpha^{n-1}a_{21} + h(a_{21}(a_{11} - \alpha) + (a_{22} - \alpha_3)a_{21} + a_{23}a_{31})$

$A_{22} = \alpha^n + n\alpha^{n-1}(a_{22} - \alpha) + h(a_{21}a_{12} + (a_{22} - \alpha)(a_{22} - \alpha_3) + a_{23}a_{32})$

$A_{23} = n\alpha^{n-1}a_{23} + h((a_{21}a_{13} + (a_{22} - \alpha)a_{23} + a_{23}(a_{32} - \alpha_3))$

$A_{31} = n\alpha^{n-1}a_{31} + h(a_{31}(a_{11} - \alpha_3) + (a_{32}a_{21} + (a_{33} - \alpha)a_{31})$
$A_{32} = n\alpha^{n-1}a_{32} + h((a_{31}a_{12} + a_{32}(a_{22} - \alpha_3) + (a_{33} - \alpha)a_{32})$
$A_{33} = \alpha^n + n\alpha^{n-1}(a_{33} - \alpha) + h(a_{31}a_{13} + a_{32}a_{23})(a_{33} - \alpha)(a_{33} - \alpha_3)$

Remark 3.2. If a matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ has the eigenvalues $\alpha_1, \alpha_2, and\ \alpha_3$, then $A^n = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where $A_{ij}, i, j = 1,2,3$ shown in Remark 3.1

Theorem 3.3 The necessary and sufficient conditions for a homogenous system of difference Equations (4) to be P-semi homogenous of order greater than one are the matrix A equal to

$$\begin{bmatrix} \frac{PA_{12}b_{21} + PA_{13}b_{31}}{a_{11} - PA_{11}} & \frac{PA_{11}a_{21} - a_{11}a_{21} + b_{31}(P^2A_{21}A_{13} + PA_{23}a_{11} - P^2A_{11}A_{23})}{a_{11} - PA_{11} - PA_{22}a_{11} + P^2A_{11}A_{22} - P^2A_{21}A_{12}} & \frac{PA_{11}a_{31} - a_{11}a_{31} + b_{21}(P^2A_{31}A_{12} + PA_{32}a_{11} - P^2A_{11}A_{32})}{a_{11} - PA_{11} - PA_{33}a_{11} + P^2A_{11}A_{33} - P^2A_{31}A_{13}} \\ \frac{PA_{22}a_{12} - a_{22}a_{12} + b_{32}(P^2A_{12}A_{23} + PA_{13}a_{22} - P^2A_{22}A_{13})}{a_{22} - PA_{22} - PA_{11}a_{22} + P^2A_{11}A_{22} - P^2A_{12}A_{21}} & \frac{PA_{21}b_{12} + PA_{23}b_{32}}{a_{22} - PA_{22}} & \frac{PA_{22}a_{32} - a_{22}a_{32} + b_{12}(P^2A_{32}A_{21} + PA_{31}a_{22} - P^2A_{22}A_{31})}{a_{22} - PA_{22} - PA_{33}a_{22} + P^2A_{33}A_{22} - P^2A_{32}A_{23}} \\ \frac{PA_{33}a_{13} - a_{33}a_{13} + b_{23}(P^2A_{13}A_{32} + PA_{12}a_{33} - P^2A_{33}A_{12})}{a_{33} - PA_{33} - PA_{11}a_{33} + P^2A_{11}A_{33} - P^2A_{13}A_{31}} & \frac{PA_{33}a_{23} - a_{33}a_{23} + b_{13}(P^2A_{31}A_{23} + PA_{21}a_{33} - P^2A_{33}A_{21})}{a_{33} - PA_{33} - PA_{22}a_{33} + P^2A_{22}A_{33} - P^2A_{32}A_{23}} & \frac{PA_{31}b_{13} + PA_{32}b_{23}}{a_{33} - PA_{33}} \end{bmatrix} \quad (7)$$

Proof: The necessary condition

$F(AX(n)) = PA^kF(X(n))$

$$F\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = P\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}\begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix}$$

$$\begin{pmatrix} f(a_{11}x + a_{12}y + a_{13}z) \\ g(a_{21}x + a_{22}y + a_{23}z) \\ h(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} PA_{11}f(x) + PAg(y) + PA_{13}h(z) \\ PA_{21}f(x) + PA_{22}g(y) + PA_{23}h(z) \\ PA_{31}f(x) + PA_{32}g(y) + PA_{33}h(z) \end{pmatrix}$$

$$\begin{pmatrix} b_{11}(a_{11}x + a_{12}y + a_{13}z) + b_{12}y + b_{13}z \\ b_{21}x + b_{22}(a_{21}x + a_{22}y + a_{23}z) + b_{23}z \\ b_{31}x + b_{32}y + b_{33}(a_{31}x + a_{32}y + a_{33}z) \end{pmatrix} = \begin{pmatrix} PA_{11}f(x) + PA_{12}g(y) + PA_{13}h(z) \\ PA_{21}f(x) + PA_{22}g(y) + PA_{23}h(z) \\ PA_{31}f(x) + PA_{32}g(y) + PA_{33}h(z) \end{pmatrix}$$

$b_{11}a_{11} = PA_{11}b_{11} + PA_{12}b_{21} + PA_{13}b_{31}$
$b_{11}a_{12} + b_{12} = PA_{11}b_{12} + PA_{12}b_{22} + PA_{13}b_{32}$
$b_{11}a_{13} + b_{13} = PA_{11}b_{13} + PAb_{23} + PA_{13}b_{33}$
$b_{21} + b_{22}a_{21} = PA_{21}b_{11} + PA_{22}b_{21} + PA_{23}b_{31}$
$b_{22}a_{22} = PA_{21}b_{12} + PA_{22}b_{22} + PA_{23}b_{32}$
$b_{22}a_{23} + b_{23} = PAb_{13} + PA_{22}b_{23} + PAb_{33}$
$b_{31} + b_{33}a_{31} = PA_{31}b_{11} + PA_{32}b_{21} + PA_{33}b_{31}$
$b_{32} + b_{33}a_{32} = PAb_{12} + PA_{32}b_{22} + PA_{33}b_{32}$
$b_{33}a_{33} = PA_{31}b_{13} + PA_{32}b_{23} + PA_{33}b_{33}$

It is easy to show the matrix A equal to

$$\begin{bmatrix} \frac{PA_{12}b_{21} + PA_{13}b_{31}}{a_{11} - PA_{11}} & \frac{PA_{11}a_{21} - a_{11}a_{21} + b_{31}(P^2A_{21}A_{13} + PA_{23}a_{11} - P^2A_{11}A_{23})}{a_{11} - PA_{11} - PA_{22}a_{11} + P^2A_{11}A_{22} - P^2A_{21}A_{12}} \\ \frac{PA_{22}a_{12} - a_{22}a_{12} + b_{32}(P^2A_{12}A_{23} + PA_{13}a_{22} - P^2A_{22}A_{13})}{a_{22} - PA_{22} - PA_{11}a_{22} + P^2A_{11}A_{22} - P^2A_{12}A_{21}} & \frac{PA_{21}b_{12} + PA_{23}b_{32}}{a_{22} - PA_{22}} \\ \frac{PA_{33}a_{13} - a_{33}a_{13} + b_{23}(P^2A_{13}A_{32} + PA_{12}a_{33} - P^2A_{33}A_{12})}{a_{33} - PA_{33} - PA_{11}a_{33} + P^2A_{11}A_{33} - P^2A_{13}A_{31}} & \frac{PA_{33}a_{23} - a_{33}a_{23} + b_{13}(P^2A_{31}A_{23} + PA_{21}a_{33} - P^2A_{33}A_{21})}{a_{33} - PA_{33} - PA_{22}a_{33} + P^2A_{22}A_{33} - P^2A_{32}A_{23}} \end{bmatrix}$$

$$\left.\begin{matrix} \dfrac{PA_{11}a_{31}-a_{11}a_{31}+b_{21}(P^2A_{31}A_{12}+PA_{32}a_{11}-P^2A_{11}A_{32})}{a_{11}-PA_{11}-PA_{33}a_{11}+P^2A_{11}A_{33}-P^2A_{31}A_{13}} \\ \dfrac{PA_{22}a_{32}-a_{22}a_{32}+b_{12}(P^2A_{32}A_{21}+PA_{31}a_{22}-P^2A_{22}A_{31})}{a_{22}-PA_{22}-PA_{33}a_{22}+P^2A_{33}A_{22}-P^2A_{32}A_{23}} \\ \dfrac{PA_{31}b_{13}+PA_{32}b_{23}}{a_{33}-PA_{33}} \end{matrix}\right]$$

Sufficient condition:

Suppose that there is a matrix A equal to (7).

After using the definition and substituting the matrix (7), we get

$$b_{11}a_{11} = PA_{11}b_{11}+PA_{12}b_{21}+PA_{13}b_{31}$$

$$\frac{PA_{12}a_{11}b_{21}+PA_{13}a_{11}b_{31}}{a_{11}-PA_{11}} = \frac{P^2A_{12}A_{11}b_{21}+P^2A_{13}A_{11}b_{31}}{a_{11}-PA_{11}}+PA_{12}b_{21}+PA_{13}b_{31}$$

$$\begin{aligned} &PA_{12}a_{11}b_{21}+PA_{13}a_{11}b_{31} \\ &\qquad = P^2A_{12}A_{11}b_{21}+P^2A_{13}A_{11}b_{31}+ PA_{12}a_{11}b_{21}-P^2A_{12}A_{11}b_{21} \\ &\qquad + PA_{13}a_{11}b_{31}-P^2A_{13}A_{11}b_{31} \end{aligned}$$

$$PA_{12}a_{11}b_{21}+PA_{13}a_{11}b_{31} = PA_{12}a_{11}b_{21}+PA_{13}a_{11}b_{31}$$

$$b_{11}a_{12}+b_{12} = PA_{11}b_{12}+PA_{12}b_{22}+PA_{13}b_{32}$$

$$b_{11}a_{12}+b_{12} = PA_{11}b_{12}+\frac{P^2A_{21}A_{12}b_{12}+P^2A_{23}A_{12}b_{32}}{a_{22}-PA_{22}}+PA_{13}b_{32}$$

And since these equations are independent we can assume $b_{11}=1$, so

$$a_{12}+b_{12} = PA_{11}b_{12}+\frac{P^2A_{21}A_{12}b_{12}+P^2A_{23}A_{12}b_{32}}{a_{22}-PA_{22}}+PA_{13}b_{32}$$

$$\begin{aligned} &b_{12}(a_{22}-PA_{22}-PA_{11}a_{22}+P^2A_{11}A_{22}-P^2A_{12}A_{21}) \\ &\qquad = PA_{22}a_{12}-a_{22}a_{12}+b_{32}(P^2A_{12}A_{23}+PA_{13}a_{22}-P^2A_{22}A_{13}) \end{aligned}$$

$$\begin{aligned} &\frac{PA_{22}a_{12}-a_{22}a_{12}+b_{32}(P^2A_{12}A_{23}+PA_{13}a_{22}-P^2A_{22}A_{13})}{a_{22}-PA_{22}-PA_{11}a_{22}+P^2A_{11}A_{22}-P^2A_{12}A_{21}}(a_{22}-PA_{22}-PA_{11}a_{22} \\ &\qquad +P^2A_{11}A_{22}-P^2A_{12}A_{21}) \\ &\qquad = PA_{22}a_{12}-a_{22}a_{12}+b_{32}(P^2A_{12}A_{23}+PA_{13}a_{22}-P^2A_{22}A_{13}) \end{aligned}$$

$$\begin{aligned} &PA_{22}a_{12}-a_{22}a_{12}+b_{32}(P^2A_{12}A_{23}+PA_{13}a_{22}-P^2A_{22}A_{13}) \\ &\qquad = PA_{22}a_{12}-a_{22}a_{12}+b_{32}(P^2A_{12}A_{23}+PA_{13}a_{22}-P^2A_{22}A_{13}) \end{aligned}$$

$$b_{11}a_{13}+b_{13} = PA_{11}b_{13}+PA_{12}b_{23}+PA_{13}b_{33}$$

$$b_{11}a_{13}+b_{13} = PA_{11}b_{13}+PA_{12}b_{23}+\frac{P^2A_{31}A_{13}b_{13}+P^2A_{32}A_{13}b_{23}}{a_{33}-PA_{33}}$$

$$b_{11}=1$$

$$a_{13}+b_{13} = PA_{11}b_{13}+PA_{12}b_{23}+\frac{P^2A_{31}A_{13}b_{13}+P^2A_{32}A_{13}b_{23}}{a_{33}-PA_{33}}$$

$$\begin{aligned} &b_{13}(a_{33}-PA_{33}-PA_{11}a_{33}+P^2A_{11}A_{33}-P^2A_{13}A_{31}) \\ &\qquad = PA_{33}a_{13}-a_{33}a_{13}+b_{23}(P^2A_{13}A_{32}+PA_{12}a_{33}-P^2A_{33}A_{12}) \end{aligned}$$

$$\begin{aligned} &\frac{PA_{33}a_{13}-a_{33}a_{13}+b_{23}(P^2A_{13}A_{32}+PA_{12}a_{33}-P^2A_{33}A_{12})}{a_{33}-PA_{33}-PA_{11}a_{33}+P^2A_{11}A_{33}-P^2A_{13}A_{31}}(a_{33}-PA_{33}-PA_{11}a_{33} \\ &\qquad +P^2A_{11}A_{33}-P^2A_{13}A_{31}) \\ &\qquad = PA_{33}a_{13}-a_{33}a_{13}+b_{23}(P^2A_{13}A_{32}+PA_{12}a_{33}-P^2A_{33}A_{12}) \end{aligned}$$

$$\begin{aligned} &PA_{33}a_{13}-a_{33}a_{13}+b_{23}(P^2A_{13}A_{32}+PA_{12}a_{33}-P^2A_{33}A_{12}) \\ &\qquad = PA_{33}a_{13}-a_{33}a_{13}+b_{23}(P^2A_{13}A_{32}+PA_{12}a_{33}-P^2A_{33}A_{12}) \end{aligned}$$

$$b_{21}+b_{22}a_{21} = PA_{21}b_{11}+PA_{22}b_{21}+PA_{23}b_{31}$$

$$b_{21}+b_{22}a_{21}=\frac{P^2A_{12}A_{21}b_{21}+P^2A_{13}A_{21}b_{31}}{a_{11}-PA_{11}}+PA_{22}b_{21}+PA_{23}b_{31}$$
$$b_{22}=1$$
$$b_{21}+a_{21}=\frac{P^2A_{12}A_{21}b_{21}+P^2A_{13}A_{21}b_{31}}{a_{11}-PA_{11}}+PA_{22}b_{21}+PA_{23}b_{31}$$
$$b_{21}(a_{11}-PA_{11}-PA_{22}a_{11}+P^2A_{11}A_{22}-P^2A_{21}A_{12}) = PA_{11}a_{21}-a_{11}a_{21}+b_{31}(P^2A_{21}A_{13}+PA_{23}a_{11}-P^2A_{11}A_{23})$$
$$\frac{PA_{11}a_{21}-a_{11}a_{21}+b_{31}(P^2A_{21}A_{13}+PA_{23}a_{11}-P^2A_{11}A_{23})}{a_{11}-PA_{11}-PA_{22}a_{11}+P^2A_{11}A_{22}-P^2A_{21}A_{12}}(a_{11}-PA_{11}-PA_{22}a_{11}+P^2A_{11}A_{22}-P^2A_{21}A_{12}) = PA_{11}a_{21}-a_{11}a_{21}+b_{31}(P^2A_{21}A_{13}+PA_{23}a_{11}-P^2A_{11}A_{23})$$
$$PA_{11}a_{21}-a_{11}a_{21}+b_{31}(P^2A_{21}A_{13}+PA_{23}a_{11}-P^2A_{11}A_{23}) = PA_{11}a_{21}-a_{11}a_{21}+b_{31}(P^2A_{21}A_{13}+PA_{23}a_{11}-P^2A_{11}A_{23})$$
$$b_{22}a_{22}=PA_{21}b_{12}+PA_{22}b_{22}+PA_{23}b_{32}$$
$$\frac{PA_{21}a_{22}b_{12}+PA_{23}a_{22}b_{32}}{a_{22}-PA_{22}}=PA_{21}b_{12}+\frac{P^2A_{21}A_{22}b_{12}+P^2A_{23}A_{22}b_{32}}{a_{22}-PA_{22}}+PA_{23}b_{32}$$
$$PA_{21}a_{22}b_{12}+PA_{23}a_{22}b_{32} = PA_{21}a_{22}b_{12}-P^2A_{21}A_{22}b_{12}+P^2A_{21}A_{22}b_{12}+P^2A_{23}A_{22}b_{32}+PA_{23}a_{22}b_{32}-P^2A_{23}A_{22}b_{32}$$
$$PA_{21}a_{22}b_{12}+PA_{23}a_{22}b_{32}=PA_{21}a_{22}b_{12}+PA_{23}a_{22}b_{32}$$
$$b_{22}a_{23}+b_{23}=PA_{21}b_{13}+PA_{22}b_{23}+PA_{23}b_{33}$$
$$b_{22}a_{23}+b_{23}=PA_{21}b_{13}+PA_{22}b_{23}+\frac{P^2A_{31}A_{23}b_{13}+P^2A_{32}A_{23}b_{23}}{a_{33}-PA_{33}}$$
$$b_{22}=1$$
$$a_{23}+b_{23}=PA_{21}b_{13}+PA_{22}b_{23}+\frac{P^2A_{31}A_{23}b_{13}+P^2A_{32}A_{23}b_{23}}{a_{33}-PA_{33}}$$
$$b_{23}(a_{33}-PA_{33}-PA_{22}a_{33}+P^2A_{22}A_{33}-P^2A_{32}A_{23}) = PA_{33}a_{23}-a_{33}a_{23}+b_{13}(P^2A_{31}A_{23}+PA_{21}a_{33}-P^2A_{33}A_{21})$$
$$\frac{PA_{33}a_{23}-a_{33}a_{23}+b_{13}(P^2A_{31}A_{23}+PA_{21}a_{33}-P^2A_{33}A_{21})}{a_{33}-PA_{33}-PA_{22}a_{33}+P^2A_{22}A_{33}-P^2A_{32}A_{23}}(a_{33}-PA_{33}-PA_{22}a_{33}+P^2A_{22}A_{33}-P^2A_{32}A_{23}) = PA_{33}a_{23}-a_{33}a_{23}+b_{13}(P^2A_{31}A_{23}+PA_{21}a_{33}-P^2A_{33}A_{21})$$
$$PA_{33}a_{23}-a_{33}a_{23}+b_{13}(P^2A_{31}A_{23}+PA_{21}a_{33}-P^2A_{33}A_{21}) = PA_{33}a_{23}-a_{33}a_{23}+b_{13}(P^2A_{31}A_{23}+PA_{21}a_{33}-P^2A_{33}A_{21})$$
$$b_{31}+b_{33}a_{31}=PA_{31}b_{11}+PA_{32}b_{21}+PA_{33}b_{31}$$
$$b_{31}+b_{33}a_{31}=\frac{P^2A_{12}A_{31}b_{21}+P^2A_{13}A_{31}b_{31}}{a_{11}-PA_{11}}+PA_{32}b_{21}+PA_{33}b_{31}$$
$$b_{33}=1$$
$$b_{31}+a_{31}=\frac{P^2A_{12}A_{31}b_{21}+P^2A_{13}A_{31}b_{31}}{a_{11}-PA_{11}}+PA_{32}b_{21}+PA_{33}b_{31}$$
$$b_{31}(a_{11}-PA_{11}-PA_{33}a_{11}+P^2A_{11}A_{33}-P^2A_{31}A_{13}) = PA_{11}a_{31}-a_{11}a_{31}+b_{21}(P^2A_{31}A_{12}+PA_{32}a_{11}-P^2A_{11}A_{32})$$
$$\frac{PA_{11}a_{31}-a_{11}a_{31}+b_{21}(P^2A_{31}A_{12}+PA_{32}a_{11}-P^2A_{11}A_{32})}{a_{11}-PA_{11}-PA_{33}a_{11}+P^2A_{11}A_{33}-P^2A_{31}A_{13}}(a_{11}-PA_{11}-PA_{33}a_{11}+P^2A_{11}A_{33}-P^2A_{31}A_{13}) = PA_{11}a_{31}-a_{11}a_{31}+b_{21}(P^2A_{31}A_{12}+PA_{32}a_{11}-P^2A_{11}A_{32})$$

$$PA_{11}a_{31} - a_{11}a_{31} + b_{21}(P^2A_{31}A_{12} + PA_{32}a_{11} - P^2A_{11}A_{32})$$
$$= PA_{11}a_{31} - a_{11}a_{31} + b_{21}(P^2A_{31}A_{12} + PA_{32}a_{11} - P^2A_{11}A_{32})$$
$$b_{32} + b_{33}a_{32} = PA_{31}b_{12} + PA_{32}b_{22} + PA_{33}b_{32}$$
$$b_{32} + b_{33}a_{32} = PA_{31}b_{12} + \frac{P^2A_{21}A_{32}b_{12} + P^2A_{23}A_{32}b_{32}}{a_{22} - PA_{22}} + PA_{33}b_{32}$$
$$b_{33} = 1$$
$$b_{32} + a_{32} = PA_{31}b_{12} + \frac{P^2A_{21}A_{32}b_{12} + P^2A_{23}A_{32}b_{32}}{a_{22} - PA_{22}} + PA_{33}b_{32}$$
$$b_{32}(a_{22} - PA_{22} - PA_{33}a_{22} + P^2A_{33}A_{22} - P^2A_{32}A_{23})$$
$$= PA_{22}a_{32} - a_{22}a_{32} + b_{12}(P^2A_{32}A_{21} + PA_{31}a_{22} - P^2A_{22}A_{31})$$
$$\frac{PA_{22}a_{32} - a_{22}a_{32} + b_{12}(P^2A_{32}A_{21} + PA_{31}a_{22} - P^2A_{22}A_{31})}{a_{22} - PA_{22} - PA_{33}a_{22} + P^2A_{33}A_{22} - P^2A_{32}A_{23}}(a_{22} - PA_{22} - PA_{33}a_{22}$$
$$+ P^2A_{33}A_{22} - P^2A_{32}A_{23})$$
$$= PA_{22}a_{32} - a_{22}a_{32} + b_{12}(P^2A_{32}A_{21} + PA_{31}a_{22} - P^2A_{22}A_{31})$$
$$PA_{22}a_{32} - a_{22}a_{32} + b_{12}(P^2A_{32}A_{21} + PA_{31}a_{22} - P^2A_{22}A_{31})$$
$$= PA_{22}a_{32} - a_{22}a_{32} + b_{12}(P^2A_{32}A_{21} + PA_{31}a_{22} - P^2A_{22}A_{31})$$
$$b_{33}a_{33} = PA_{31}b_{13} + PA_{32}b_{23} + PA_{33}b_{33}$$
$$\frac{PA_{31}a_{33}b_{13} + PA_{32}a_{33}b_{23}}{a_{33} - PA_{33}} = PA_{31}b_{13} + PA_{32}b_{23} + \frac{P^2A_{31}A_{33}b_{13} + P^2A_{32}A_{33}b_{23}}{a_{33} - PA_{33}}$$
$$PA_{31}a_{33}b_{13} + PA_{32}a_{33}b_{23}$$
$$= PA_{31}a_{33}b_{13} - P^2A_{31}A_{33}b_{13} + PA_{32}a_{33}b_{23} - P^2A_{32}A_{33}b_{23}$$
$$+ P^2A_{31}A_{33}b_{13} + P^2A_{32}A_{33}b_{23}$$
$$PA_{31}a_{33}b_{13} + PA_{32}a_{33}b_{23} = PA_{31}a_{33}b_{13} + PA_{32}a_{33}b_{23}.$$

Note: From the definition of P^k*-semi-homogenous* of order m, we can define another concept as follows:

Definition3.4. System 4 is called adjoint if there exist two non-zero matrices A and C such that the following equation holds
$F\left(A(c)x(n)\right) = CF(x(n))$.

Example 3.5. Considers the system $F(x) = Bx$ where $B = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -2 \\ -2 & 1 & 1 \end{bmatrix}$, then are $A = \begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & -2 \\ -2 & 1 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 4 & 4 \\ 4 & 6 & 4 \\ 4 & 4 & 6 \end{bmatrix}$ such that

$$F\left(A(c)x(n)\right) = CF(x(n))$$

Remark 3.6.
1- Every P^k-semi homogeneous system is an adjoint system. In general, Example 3.5 explains that the converse is not true,
2- In Definition 3.4, if the matrix C can be written as PA, then the converse of 1 will be true.

Conclusion In this paper, we have presented new concepts which are generalized to the (3×3)-semi-homogeneous system of difference equations of order m, where m is positive integer numbers, and study all special cases of there. That is
A homogenous system of difference equations is called generalized semi-homogenous if there exists a non-zero, real matrix M such that the following equation is held

$$F(Mx(n)) = P^kM^mF(x(n)), \dots\dots\dots\dots\dots\dots\dots\dots \quad (3)$$

where P, k, and m are integer numbers.

There are some special cases discussed in this work as well as the general case, and given some examples and characteristics for definitions and proved some theorems about there, which can be summarized as follows:
1- If $P = 1$, then this system is called semi-homogenous of order m [6].
2- If $P \neq 1$, $k = 1$, then this system is called P-semi-homogenous of order m.
3- If $P \neq 1$, $k \neq 1$, then this system is called P^k-semi-homogenous of order m.
4- If $k = m$, then Equation (2) becomes like the following:

$$F\big(MX(n)\big) = (PM)^k F\big(X(n)\big)$$

and it is called P^k*-semi-homogenous* of order k.

Future work
1- We can generalize the results to the $(n \times n)$-the semi-homogeneous system of difference equations of order m where $n \geq 4$
2- We can find the characterization of the adjoint system.
3- A homogenous system $x(n+1) = Bx(n)$ of difference equations are called P-self-semi homogenous of order m if there exists an integer number m such that the following equation holds.
$F\,(B(c)x(n)) = P(B(c))^m F(x(n))$, where p is an integer number and c is a real number, that is $F\,(B(c)x(n)) = P(B(c))^{m+1} x(n)$.
4- We can find the characterization of the called P-self-semi homogenous.

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