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Finding the Exact Solution of Kepler's Equation for an Elliptical Satellite Orbit Using the First Kind Bessel Function

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Abstract

In this study, the first kind Bessel function was used to solve Kepler equation for an elliptical orbiting satellite. It is a classical method that gives a direct solution for calculation of the eccentric anomaly. It was solved for one period from (M=0-360)° with an eccentricity of (e=0-1) and the number of terms from (N=1-10). Also, the error in the representation of the first kind Bessel function was calculated. The results indicated that for eccentricity of (0.1-0.4) and (N = 1-10), the values of eccentric anomaly gave a good result as compared with the exact solution. Besides, the obtained eccentric anomaly values were unaffected by increasing the number of terms (N = 6-10) for eccentricities (0.8 and 0.9). The Bessel function's solution appeared to be close to the exact solution for eccentricity of 1 and more than 10 number of terms. Finally, the representation of the first kind Bessel function $J_1(x)$ was closer to the exact representation only for eccentricity 0.5 and (N=1-10).

Keywords: Kepler Equation: The Classical Solution, Bessel Function of the First Kind, Elliptical Orbit, Eccentric Anomaly, Eccentricity.

ايجاد الحل المضبوط لمعادلة كبلر لمدار قمر اصطناعي بيضوي بأستخدام دالة بزل من النوع الاول

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الخلاصة

تم استخدام دالة بزل من النوع الاول لحل معادلة كبلر لاقمار اصطناعية في مدار بيضوي, تعتبر دالة بزل واحدة من الطرق الكلاسيكية للحل و تعطي حل مباشر لحساب قيمة الانحراف الشاذ. حيث تم حل معادلة كبلر خلال دورة واحده °(M=0−360) وشذوذ مركزي **(e=0−1)** و عدد تقسيمات **(N=1−10)**. ايضا تم حساب الخطأ في تمثيل دالة بزل من النوع الاول. أشارت نتائج قيم الانحراف الشاذ عند الشذوذ المركزي من (0.1 الى 0.4), وعدد التقسيمات من **(N=1−10)** بأنها متطابقة مع لحل المثالي. بالاضافة الى ان قيم الانحراف الشاذ عند الشذوذ المركزي 0.8 و 0.9 لا تتأثر بزيادة عدد التقسيمات عندما تكون **(N = 6−10)**. وتبين ان الحل بأستخدام دالة بزل مطابق للحل المثالي عند عند الشذوذ المركزي 1 و عدد التقسيمات اكبر من 10. واخيرا كان تمثيل دالة بزل من النوع الاول مقارب الى الحل المثالي فقط عند الشذوذ المركزي 0.5 و عدد التقسيمات **(N=1−10)**.

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1. Introduction

To solve Kepler's equation, different analytical and graphical solutions were proposed. From Kepler and Newton until the middle of the 20th century, almost every mathematician concentrated on this equation. After the beginning of the modern spaceflight era, new algorithms were being suggested and published. Many significant developments in mathematics were used to solve this equation, such as Fourier series, Bessel functions, and basic concepts of complex function theory, using many numerical analysis approaches [1, 2].

The Bessel function was found to be a direct solution to Kepler's equation and obtained the coefficient of the first kind, $J_1(x)$ in sine wave form. For all values of eccentricity and the periodic series of the mean anomaly, the coefficient was completely convergent [1]. Kepler's equation gives the relationship between the polar coordinates of a celestial object and the time required for this object to move around another object. It is a fundamental equation in celestial mechanics but cannot be directly reversed in terms of simple functions in order to determine where the object will be at a particular time. The solution of Kepler's equation by the Bessel function of the first kind depends on three parameters, which are: eccentricity, mean anomaly, and number of terms [3, 4, 5]. For many years, many researchers have presented their solutions. James provided a global solution by defining four M-e plane sub-domains [6]. Fouad and Anas investigated how the sun's position affects the length of astronomical twilight [7]. Daniele *et al.* used a predictor-corrector approach for orbit propagation. The value of the eccentric anomaly was estimated under a constant time interval constraint by linear or quadratic approximations and then corrected using a single Newton-Raphson or Halley iteration [8]. By using the Special Trans Functions Theory, Slavica *et al.* discovered an analytical solution for several families of Kepler's transcendental equations (STFT) [9]. By using the Maclaurin expansion, Aisha and Asrar were able to solve Kepler's equation without the requirement to decompose the associated nonlinearity as they would have to do with the differential transformation method and Adomian method [10]. Mohammed and Abdul Rahman evaluated the orbital maneuvers for changing from Low Earth Orbit to Geostationary Earth Orbit using the Newton-Raphson method [11]. For Baghdad city in 2019, Fouad calculated the variation in sunrise, sunset, and day length times [12]. Rasha and Abdul-Rahman used iterative and non-iterative methods to find the value of the eccentric anomaly and then calculated the state vectors for the satellite [13].

2. Bessel function of the first kind to solve Kepler's equation

It is dependent on the direct application of the series expansion and is considered one of the classical methods. The first kind of Bessel solution to Kepler's equation can be expressed as [3, 14]:

$$E = M_e + \sum_{n=0}^{\infty} \frac{2}{n} J_n\,(ne)\,sin(nM_e) \quad (1)$$

Where E = eccentric anomaly, which has a range of 0° to 360°, M_e = mean anomaly, measured in degrees from 0° to 360°, e= eccentricity, which has a range from 0 to 1, J_n= coefficient, named a Bessel function, described by the following equation:

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{n+2k} \quad (2)$$

Here, $J_n(x)$ is represented by $J_1(x)$ to $J_5(x)$, which are plotted in Figure (1).

In fact, equation (1) was truncated to a limited number of terms (N), as a result, it will be re-written as [3]:

$$E = M_e + \sum_{n=0}^{N} \frac{2}{n} J_n\,(ne)\,sin(nM_e) \quad (3)$$

Where:
N: the number of terms, which has a value from (1-10).
Eq. (3) can be applied to all values of the eccentricity (0-1) for an elliptical orbiting satellite. Kepler's equation solution by the iterative method is given in Figure (2). As the eccentricity increases, the curves start to shift more form the circular orbit [13, 14]. In this study, the used value for the eccentricity is 0.8, because this value within eccentricity ranges for elliptical orbit. Obviously, from Figure (1), all curves appear oscillatory and go closer to zero as x increases. The reference curve in Figure (1) is J_1, it has a sine wave form with a range from -0.5 to 0.5. The dummy variable x is a numerical variable whose range is from 0 to 6.4. In this research, the x-axis represents the eccentric anomaly [3, 15, 16].

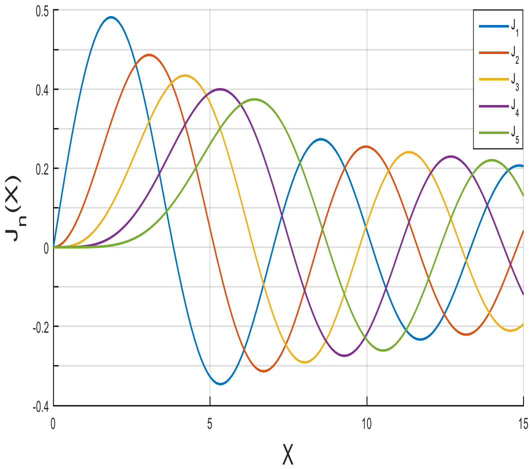

Figure 1: Bessel functions of the first kind [3].

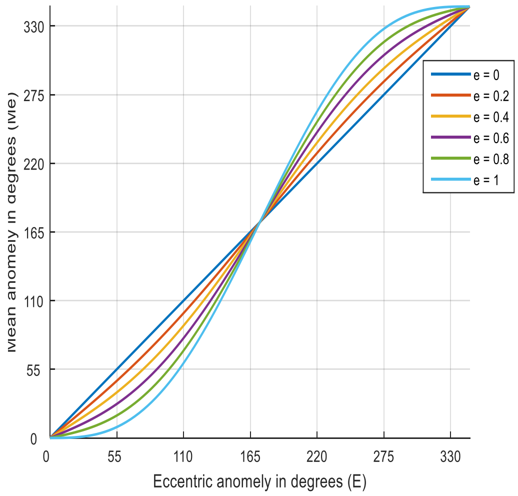

Figure 2: The eccentric anomaly for different values of eccentricity [3].

Results and Discussion

The first kind of Bessel function was used to solve the Kepler's equation for an eccentricity from (0 to 1) and a mean anomaly from (0° to 360°). The results of the mean anomaly and eccentric anomalies showed a monotonic increase for various values of eccentricity. Eccentric anomaly has a linear relationship for all curves when eccentricity is very small. Also, those curves were indistinguishable from Kepler exact solution, as displayed in Figures (3 and 4). As the eccentricity increased, as shown in Figure (5), one of the curves began to deviate from the exact solution but was also indistinguishable. When eccentricity became 0.4, the eccentric anomaly started to diverge clearly from the exact solution, as obtained from Figure (6). As eccentricity grew to (0.5 and 0.6), the curves for (N = 1 and 2) started to oscillate around the exact solution, as noticed from Figures (7 and 8), but were fairly close to the exact solution. Figure (9) showed that for an eccentricity of 0.7, three curves for (N = 1-3) diverged from the exact solution. Figure (10) illustrated that for an eccentricity of 0.8 and N = 1-5, the eccentric anomaly gave a poor approximation to Kepler's equation solution, but when the number of terms was higher than 5, the curves became similar to the exact solution. For greater eccentricity and (N = 1-5), the curves continued oscillating around the exact solution, but for (N = 6-10), the curves were indistinguishable from the exact solution, as clarified in Figure (11). Figures (12a and 12b) indicated that for an eccentricity of 1 and a number of terms (N=1-5), the curves diverged from the exact solution. For the same eccentricity but for (N = 6-10), the curves oscillated a little above and beyond the exact solution. Using a greater number of terms (greater than 10), with the same eccentricity, produced a good result as compared to the exact solution.

Notice that the curves were not affected by the number of terms more than 10. They were indistinguishable from the exact curve. In addition, the error for eccentric anomaly was investigated in this study. The curves in Figure (13) were periodically oscillated with the eccentric anomaly; they had a value between -0.1 and 0.1. Figure (14) oscillated above and below zero in a periodic uniform manner; the curves had a value between -0.2 and 0.2. In Figure (15), the curves started to shift towards zero with increasing eccentricity. The same appearance occurred for the curves when the eccentricity became 0.4, as shown in Figure (16). Figures (17 and 18) were shifted more towards zero as the eccentric anomaly increased. Figure (19) had different peaks with different amplitudes. For an eccentricity of 0.8, the curves appeared oscillatory and tended towards zero as the eccentric anomaly increased, as displayed in Figure (20). Finally, Figures (21 and 22) had periodic behavior with the eccentric anomaly; the curves had values from -1 to 1 and -1.2 to 1.2, respectively. This means that there was less error for an eccentricity of (0.5-0.7).

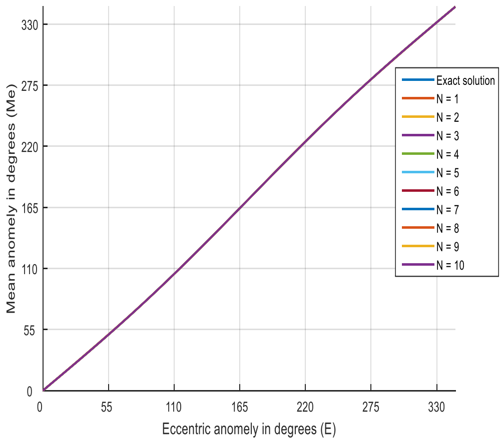

Figure 3: The eccentric anomaly for eccentricity 0.1.

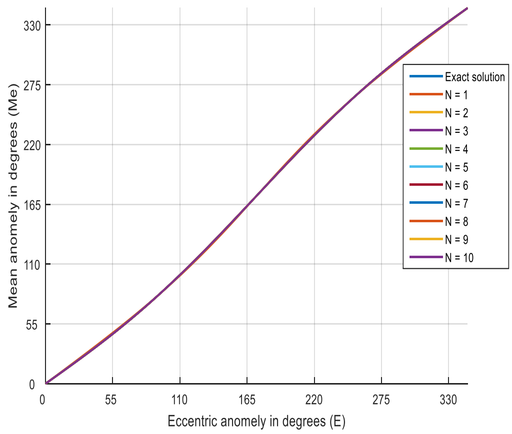

Figure 4: The eccentric anomaly for eccentricity 0.2.

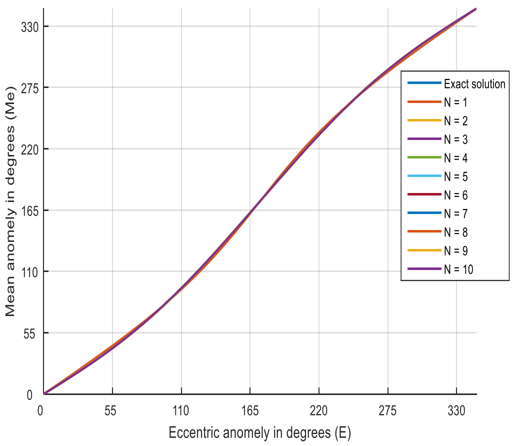

Figure 5: The eccentric anomaly for eccentricity 0.3.

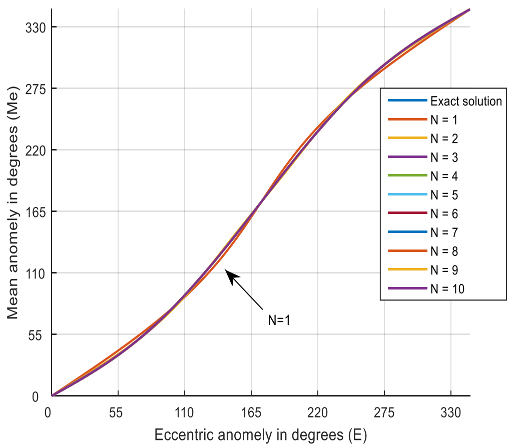

Figure 6: The eccentric anomaly for eccentricity 0.4.

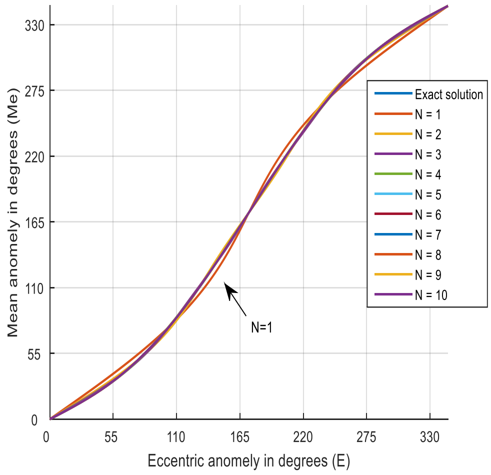

Figure 7: The eccentric anomaly for eccentricity 0.5.

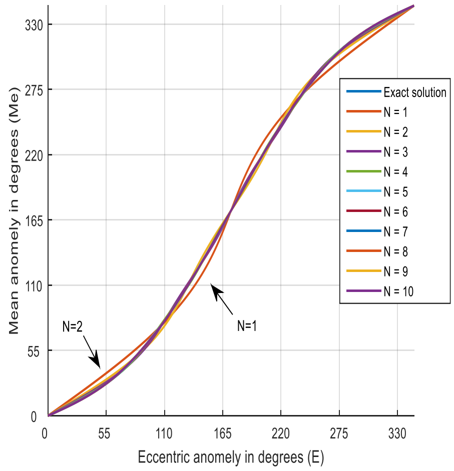

Figure 8: The eccentric anomaly for eccentricity 0.6.

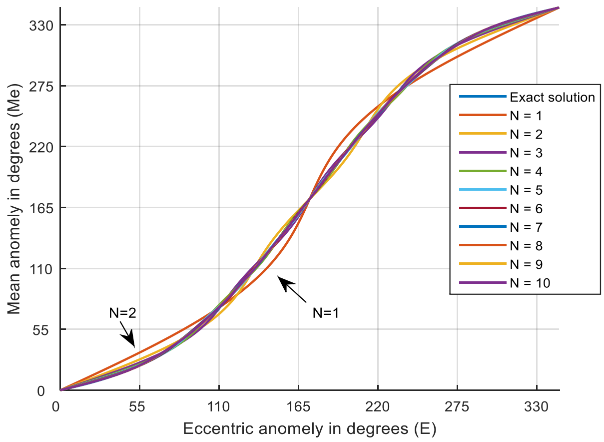

Figure 9: The eccentric anomaly for eccentricity 0.7.

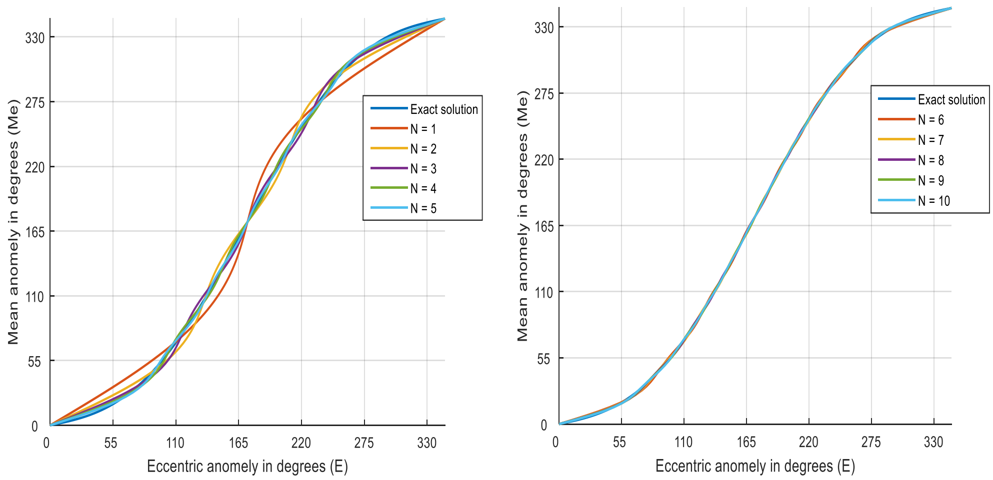

Figure 10: The eccentric anomaly for eccentricity 0.8.

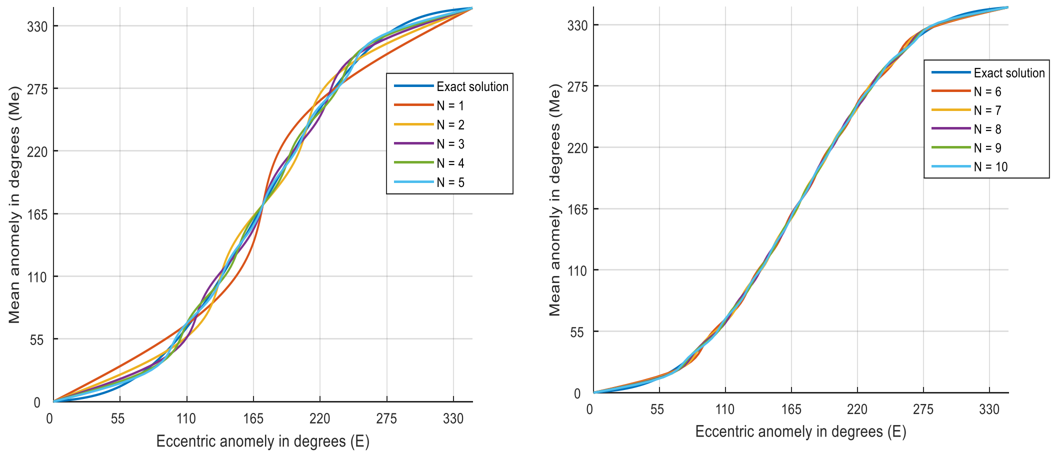

Figure 11: The eccentric anomaly for eccentricity 0.9.

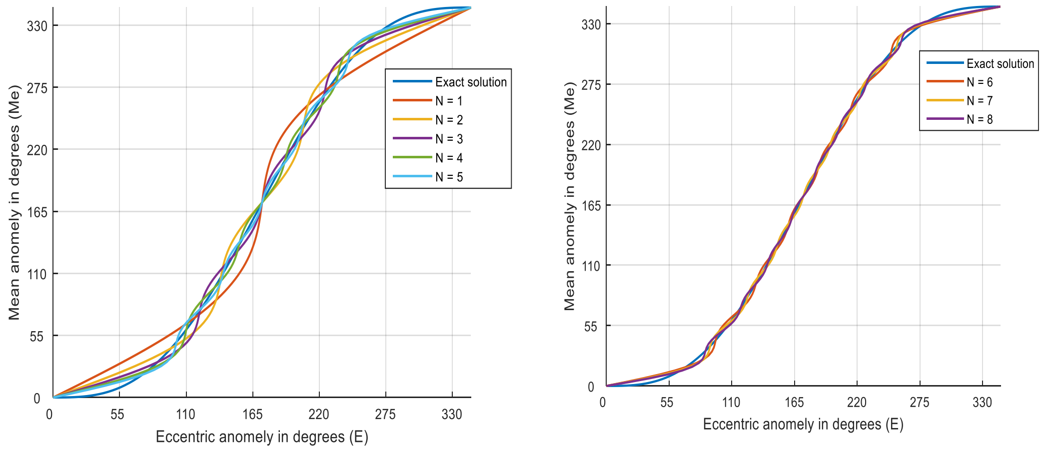

Figure 12a: The eccentric anomaly for eccentricity 1.

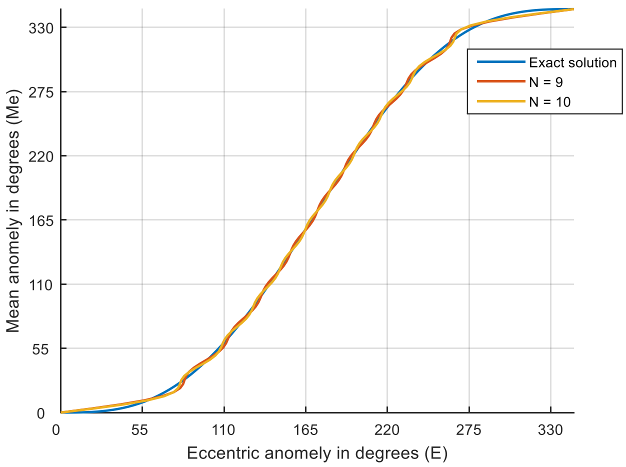

Figure 12b: The eccentric anomaly for eccentricity 1 for N= 9 and 10.

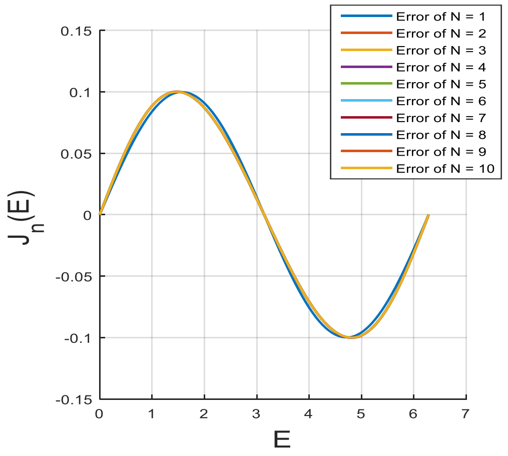

Figure 13: The error for eccentricity 0.1.

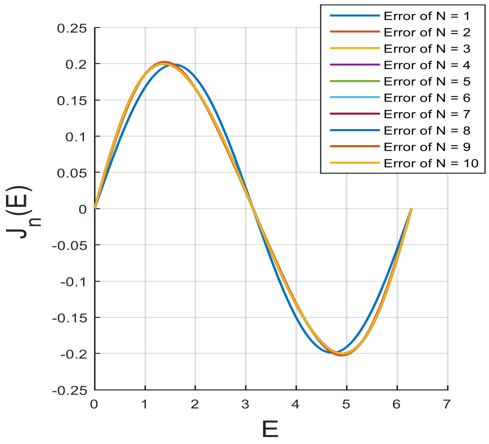

Figure 14: The error for eccentricity 0.2.

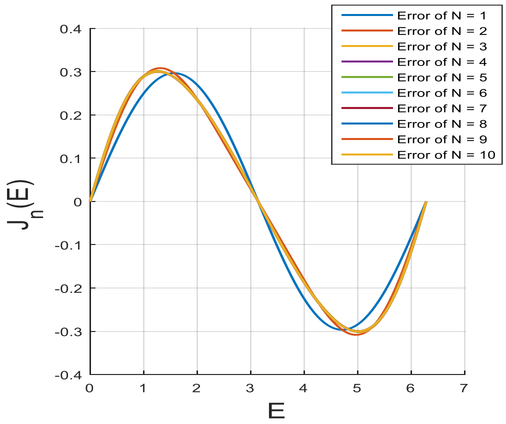

Figure 15: The error for eccentricity 0.3.

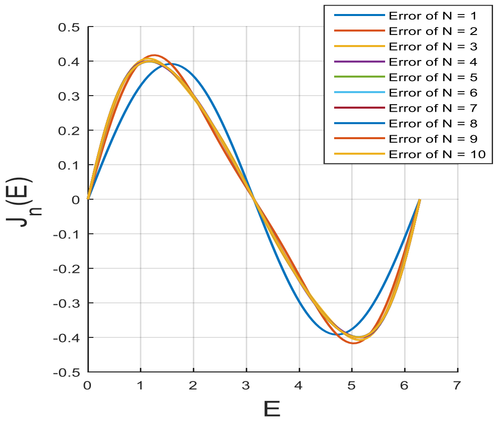

Figure 16: The error for eccentricity 0.4

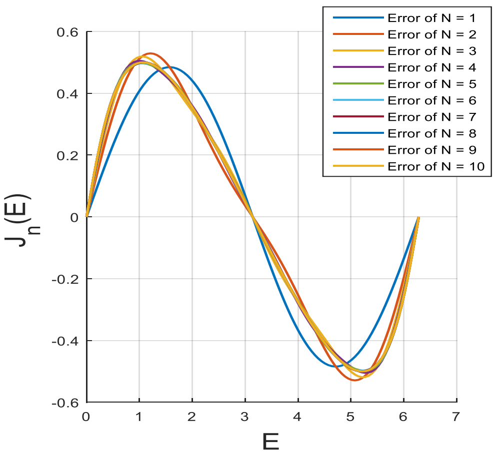

Figure 17: The error for eccentricity 0.5.

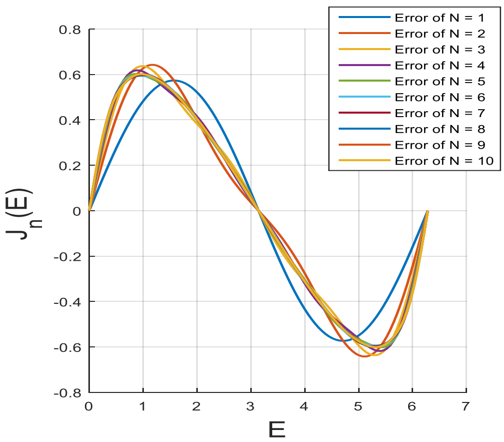

Figure 18: The error for eccentricity 0.6.

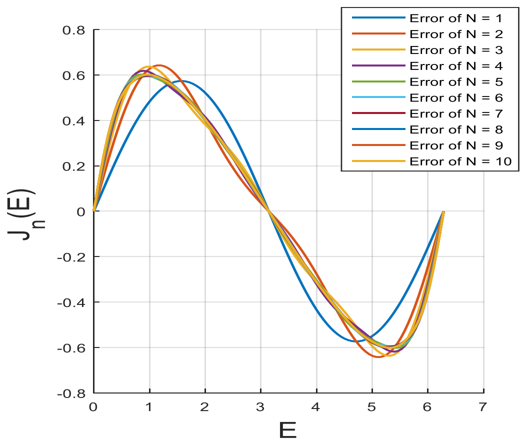

Figure 19: The error for eccentricity 0.7.

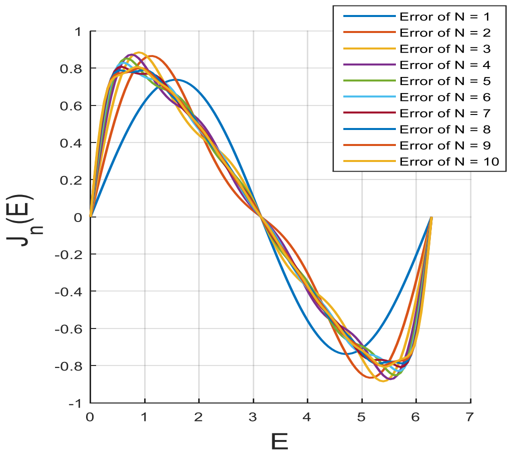

Figure 20: The error for eccentricity 0.8.

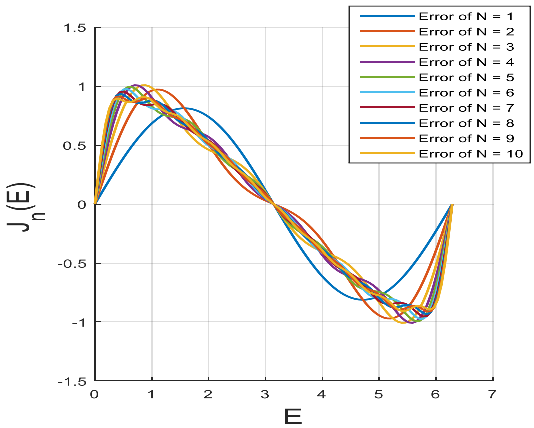

Figure 21: The error for eccentricity 0.9.

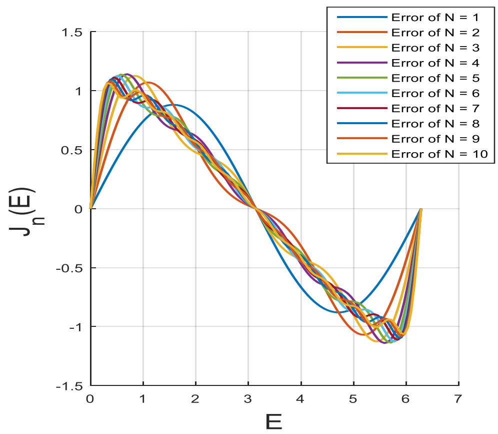

Figure 22: The error for eccentricity 1.

3. Conclusions

The results of this study reveal several essential points, which are:

1. For eccentricity of (0.1-0.4) and (N = 1-10), the values of eccentric anomaly give a present value of goodness result as compared with the exact solution. For eccentricity of (0.5-0.7) and (N = 3-10), the values of eccentric anomaly by Bessel function are similar to the exact solution. The obtained eccentric anomaly values are unaffected by increasing the number of terms for eccentricities (0.8 and 0.9) and (N = 6-10).
2. The number of terms (N=1-5) continues to have an impact on the values of the eccentric anomaly for eccentricity equal to 1.
3. When the number of terms grows to more than 5 and the eccentricity is 1, the Bessel function demonstrate a good agreement when compared to the Kepler's equation solution and other published studies in this subject.
4. The Bessel function solution appears to be close to the exact solution for eccentricity of 1 and more than 10 number of terms.
5. The representation of the first kind Bessel function $J_1(x)$ is closer to the exact depiction in Figure (1) with eccentricity 0.5 and (N=1-10).

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