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Goldie Rationally Extending Modules

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Abstract
In this work, we introduce a new generalization of both Rationally extending and Goldie extending which is Goldie Rationally extending module which is known as follows: if for any submodule K of an R-module M there is a direct summand U of M (denoted by $U \subseteq_{\oplus} M$) such that $K \beta_r U$. A β_r is a relation of $K \subseteq M$ and $U \subseteq M$, which defined as $K \beta_r U$ if and only if $K \cap U \subseteq_r K$ and $K \cap U \subseteq_r U$.

Keywords: Rationally Extending Modules, Goldie Rationally Extending Modules, Goldie Extending Modules.

مقاسات التوسعة الراشدة من النمط (Goldie)

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الخلاصة
في هذا العمل قدمنا اعماماً جديداً لمقاسات التوسعة الراشدة و مقاسات التوسعة من النمط (Goldie) و هو مقاسات التوسعة الراشدة من النمط (Goldie) و معرف بالشكل التالي، اذا كان لأي مقاس جزئي K من مقاس M يوجد مركبة جمع مباشر U من M بحيث $K \beta_r U$. β_r هي علاقة بين المقاسات الجزئية K و U من المقاس M، و معرفة في الشكل الاتي $K \beta_r U$ اذا و فقط اذا $K \cap U \subseteq_r K$ و $K \cap U \subseteq_r U$.

1. Introduction

Throughout this work, all modules are unitary left R-module over a commutative ring with identity.

In [1], a submodule $V \subseteq M$ is rational (symbolize by $V \subseteq_r M$). If for any $t, h \in M$ and $h \neq 0$ there exists $b \in R$, such that $bt \in V$ and $bh \neq 0$. A submodule $0 \neq V_1$ is called an essential submodule (symbolize by $V_1 \subseteq_e M$), if $V_1 \cap V_2 \neq 0$ for each $0 \neq V_2 \subseteq M$.

In [2], M. S. Abbas and M. A. Ahmad introduced the definition of rationally closed submodule as a submodule V of M which has no proper rational extension (symbolize by rc-submodule or $V \subseteq_{rc} M$). And if every submodule V of M such that $V \subseteq_{rc} M$ is a summand, then M is rationally extending module (symbolize by RCS module).

In [3], the relation β and α between the submodule V and K of an R-module has been introduced M by: $V \beta K$ if and only if, $V \cap K$ is an essential in V and in K. $V \alpha K$ if and only if, V and K are essential in $D \subseteq M$. And if for any $V \subseteq M$ there is $U \subseteq_{\oplus} M$ such that $V \beta U$, then

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M is Goldie extending module (symbolize by $\mathcal{G}$-CS). Every extending module is $\mathcal{G}$-CS module.

In this work we present a new relations which are β_r and α_r in term of rational submodule. And a new concept that is UR-closure which is known as follows: a module M is called an UR-closure if every submodule of M has a unique rat-closure. Also, we introduce a stronger generalize of $\mathcal{G}$-CS module which is $\mathcal{G}$-RCS module. For more details about the generalizations of $\mathcal{G}$-CS and RCS module see [2-5].

2. Basic properties of relations α_r , β_r

Definition 2.1: let $S \subseteq M$ and $E \subseteq M$ when M an R-module. And α_r , β_r are relations between S, E. Then:

1. $S\ \alpha_r\ E$ if and only if there is $C \subseteq M$ such that $S \subseteq_r C$ and $E \subseteq_r C$.
2. $S\ \beta_r\ E$ if and only if $S \cap E \subseteq_r S$ and $S \cap E \subseteq_r E$.

Remarks and Examples 2.2: Let M be an R-module:

1. A relation α_r is reflexive and symmetric, but it is not be transitive.
2. A relation β_r is reflexive, symmetric and now we prove transitive:

Proof: Let $E_1, E, P \subseteq M$ such that $E_1 \beta_r E$ and $E\ \beta_r P$. Now to proof $E_1 \beta_r P$ let $x, e \in E_1$ and $e \neq 0$, but $E_1 \cap E \subseteq_r E_1$, then there exists $r \in R$ such that $rx \in E_1 \cap E$ and $re \neq 0$. Since $E \cap P \subseteq_r E$ and $re \in E$ there is $r_1 \in R$ such that $r_1 rx \in E \cap P$ and $r_1 re \neq 0$. Let $t = r_1 r$, this means there is $t \in R$ such that $tx \in E_1 \cap P$ and $te \neq 0$. Then $E_1 \cap P \subseteq_r E_1$. Similarly, $E_1 \cap P \subseteq_r P$. Hence, $E_1\ \beta_r P$, that is β_r is a transitive. □

3. $A_1\ \beta_r\ M$ if and only if $A_1 \subseteq_r M$.
4. $S\ \beta_r\ \{0\}$ if and only if $S = 0$.
5. If $E_1 \alpha_r\ E$ when $E_1, E \subseteq M$, then $E_1\ \beta_r\ E$.

Proof: Let $E_1\ \alpha_r\ E$ and $E_1, E \subseteq M$. then there is $P \subseteq M$ such that $E_1 \subseteq_r P$ and $E \subseteq_r P$, thus by [1, p.55, Proposition 2.25], $E_1 \cap E \subseteq_r P$. So $E_1 \cap E \subseteq_r E_1$ and $E_1 \cap E \subseteq_r E$ by [1, p.55, Proposition 2.25]. Hence $E_1\ \beta_r\ E$.□

6. If $W_1\ \beta_r\ E_1$ and $W_2\ \beta_r\ E_2$ when $W_1, W_2, E_1, E_2 \subseteq M$, then $(W_1 \cap W_2)\beta_r\ (E_1 \cap E_2)$.

Proof: Let $W_1\ \beta_r\ E_1$ and $W_2\ \beta_r\ E_2$ when $W_1, W_2, E_1, E_2 \subseteq M$, so $W_1 \cap E_1 \subseteq_r W_1$ and $W_1 \cap E_1 \subseteq_r E_1$, also $W_2 \cap E_2 \subseteq_r W_2$ and $W_2 \cap E_2 \subseteq_r E_2$. Then by [1, p.55, Proposition 2.25], $(W_1 \cap E_1) \cap (W_2 \cap E_2) \subseteq_r W_1 \cap W_2$ and $(W_1 \cap E_1) \cap (W_2 \cap E_2) \subseteq_r E_1 \cap E_2$. Hence, $(W_1 \cap W_2)\beta_r\ (E_1 \cap E_2)$.□

7. If $W_i\ \beta_r\ E_i$ $(i = 1,2, \dots, n)$ when $W_i, E_i \subseteq M$, then $(\bigcap_{i=1}^{n} W_i)\ \beta_r\ (\bigcap_{i=1}^{n} E_i)$ when $(i = 1,2, \dots, n)$.
8. In Z as Z-module. $4Z\ \alpha_r\ 8Z$ since there is $2Z \subseteq Z$ such that $4Z \subseteq_r 2Z$ and $8Z \subseteq_r 2Z$. By (5) $4Z\ \beta_r\ 8Z$.
9. Let $D, E \subseteq M$

I.If $D\ \beta_r\ E$, then $D\ \beta\ E$.

Proof: let $D, E \subseteq M$ and $D\ \beta_r\ E$. Then $D \cap E \subseteq_r D$ and $D \cap E \subseteq_r E$, so we have $D \cap E \subseteq_e D$ and $D \cap E \subseteq_e E$. Hence, $D\ \beta\ E$.□

II.If $D\ \alpha_r\ E$, then $D\ \alpha\ E$.

Proof: let $D, E \subseteq M$ and $D\ \alpha_r\ E$. Then there is $P \subseteq M$ such that $D \subseteq_r P$ and $E \subseteq_r P$, so we have $D \subseteq_e P$ and $E \subseteq_e P$. Hence, $D\ \alpha\ E$.□

The opposite direction is satisfied if a module M be a non-singular.

Let M be an R-module and $H \subseteq_{rc} M$, if $E \subseteq_r H \subseteq_{rc} M$ when $E \subseteq M$, then we called H is rational closure (rat-closure) of E [2]. Now, we introduce the following definition:

Definition 2.3: A module M is called an UR-closure if every submodule of M has a unique rat-closure.

So, an UR-closure definition is a necessarily condition to make α_r transitive.

Proposition 2.4: M is an UR-closure R-module if and only if α_r transitive.

Proof: Assume M be a UR-closure and $S_1, E, P \subseteq M$ such that $S_1\ \alpha_r\ E$ and $E\ \alpha_r\ P$. Then there is $E_1, V \subseteq M$ such that $S_1 \subseteq_r E_1$, $E \subseteq_r E_1$, $E \subseteq_r V$ and $P \subseteq_r V$. Assume $S_1 \subseteq_r J$, $E \subseteq_r D$ and $P \subseteq_r B$ when $J, D, B \subseteq_{rc} M$. By [1, p.55, Proposition 2.25], $S_1 \cap E \subseteq_r E_1$ and $E \cap P \subseteq_r V$. Hence, $S_1 \cap E \subseteq_r S_1 \subseteq_r J$, $S_1 \cap E \subseteq_r E \subseteq_r D$, $E \cap P \subseteq_r E \subseteq_r D$ and $E \cap P \subseteq_r P \subseteq_r B$. But M is UR-closure, then $J = D = B$ thus $S_1 \subseteq_r J$ and $P \subseteq_r J$. Therefore, $S_1 \alpha_r P$ and hence α_r is transitive. Opposite direction, suppose that α_r is transitive. Let $H \subseteq M$ and let $E, P \subseteq_{rc} M$ such that $H \subseteq_r E$ and $H \subseteq_r P$. Since $E \subseteq_r E$ and $P \subseteq_r P$, so $E\ \alpha_r\ H$ and $H\ \alpha_r\ P$ then $E\ \alpha_r\ P$, there is $V \subseteq M$ such that $E \subseteq_r V$ and $P \subseteq_r V$. But $E, P \subseteq_{rc} M$, hence $E = P = V$.□

Proposition 2.5: M is an UR-closure R-module if and only if $\alpha_r = \beta_r$.

Proof: First direction. Let $\alpha_r = \beta_r$, by Remark 2.2 (2) we have β_r is transitive. And hence α_r is transitive. Then by Proposition 2.4, M is an UR-closure. Conversely, suppose M be an UR-closure. By Remark 2.2 (5), we have every α_r is β_r . Now to prove every β_r is α_r. For this, let $F, F_1 \subseteq M$ such that $F\ \beta_r\ F_1$ then $F \cap F_1 \subseteq_r F$ and $F \cap F_1 \subseteq_r F_1$. Let $V_1, V_2 \subseteq_{rc} M$ such that $F \subseteq_r V_1$ and $F_1 \subseteq_r V_2$, then $F \cap F_1 \subseteq_r V_1$ and $F \cap F_1 \subseteq_r V_2$. But M is an UR-closure, then $V_1 = V_2$. Thus $F\ \alpha_r\ F_1$ and hence $\alpha_r = \beta_r$.□

Proposition 2.6: Let M and W be R-modules, and $\theta: M \to W$ be a monomorphism. Then the following condition are holds:

1. If $P\ \beta_r\ E$, then $\theta(P)\ \beta_r\ \theta(E)$ where $P, E \subseteq M$.
2. If $P\ \beta_r\ E$, then $\theta^{-1}(P)\ \beta_r\ \theta^{-1}(E)$ where $P, E \subseteq W$.
3. If $P\ \alpha_r\ E$, then $\theta(P)\ \alpha_r\ \theta(E)$ where $P, E \subseteq M$.
4. If $P\ \alpha_r\ E$, then $\theta^{-1}(P)\ \alpha_r\ \theta^{-1}(E)$ where $P, E \subseteq W$.

Proof: (1) and (3) are clear by Lemma 2.10 [5].
(2) and (4) are clear by [6].

Lemma 2.7: [7] Let M be an R-module and $V \subseteq_e M$ if and only if for any $0 \neq c \in M$ there is $c_1 \in R$ such that $c_1 c \in V$.

Now, the next proposition is equivalent to definition of RCS module.

Proposition 2.8: M is an RCS R −module if and only if, for any $S \subseteq M$ there is $S_1 \subseteq_{\oplus} M$ such that $S\ \alpha_r\ S_1$.

Proof: Let S a submodule of an RCS module M. Then there is $S_1 \subseteq_{\oplus} M$ such that $S \subseteq_r S_1$, but $S_1 \subseteq_r S_1$. Hence $S\ \alpha_r\ S_1$.

Conversely, let $S \subseteq M$ then by hypothesis there is $S_1 \subseteq_{\oplus} M$ such that $S\ \alpha_r S_1$, this means there is $V \subseteq M$ such that $S \subseteq_r V$ and $S_1 \subseteq_r V$. Now we prove that $V \subseteq_{\oplus} M$, since $S_1 \subseteq_{\oplus} M$, then $S_1 + E = M$ for some $E \subseteq M$. But $M = S_1 + E \subseteq V + E$, hence $V + E = M$. Now, let $0 \neq a \in V \cap E$, then $a \in V$ and $a \in E$. Since $S_1 \subseteq_r V$ then $S_1 \subseteq_e V$, so by Lemma 2.7 there is $r \in R$ such that $ra \in S_1$ but $ra \in E$. Hence $ra \in S_1 \cap E$ which is a contradiction since $S_1 \subseteq_{\oplus} M$. So $V \cap E = 0$ this means $V \subseteq_{\oplus} M$ such that $S \subseteq_r V$.□

3. $\mathcal{G}$-Rationally extending modules

Definition 3.1: If any submodule W of an R-module M there is $D \subseteq_{\oplus} M$ such that $W\ \beta_r\ D$, then M is named by Goldie Rationally extending module (symbolize by $\mathcal{G}$-RCS module).

A ring R is called a $\mathcal{G}$-RCS module, if R is $\mathcal{G}$-RCS module R-module.

Proposition 3.2: M is a $\mathcal{G}$-RCS R-module if and only if for each $W \subseteq_{rc} M$ there is $W_1 \subseteq_{\oplus} M$ such that $W\ \beta_r\ W_1$.

Proof: Assume that M is a $\mathcal{G}$-RCS and let $C \subseteq M$, there is $W \subseteq_{rc} M$ such that $C \subseteq_r W$ by [2], and by hypothesis there is $W_1 \subseteq_{\oplus} M$ such that $W\ \beta_r W_1$. Since $C = C \cap W \subseteq_r W$ and $C \subseteq_r C$, then $C\ \beta_r\ W$. But a relation β_r is transitive. So, we have $C\ \beta_r\ W_1$ and hence M is a $\mathcal{G}$-RCS. Conversely, let $W \subseteq_{rc} M$ and by definition of $\mathcal{G}$-RCS module, there is $W_1 \subseteq_{\oplus} M$ such that $W\ \beta_r\ W_1$.□

Proposition 3.3: An R-module M is $\mathcal{G}$-RCS if and only if for any $W \subseteq M$ there are $Y \subseteq M$ and $V \subseteq_{\oplus} M$ such that $Y \subseteq_r W$ and $Y \subseteq_r V$.

Proof: Let $W \subseteq M$ and by definition of $\mathcal{G}$-RCS module, there is $Y_1 \subseteq_{\oplus} M$ such that $W\ \beta_r\ Y_1$ ($W \cap Y_1 \subseteq_r W$ and $W \cap Y_1 \subseteq_r Y_1$). Take $W \cap Y_1 = Y$, so $Y \subseteq_r W$ and $Y \subseteq_r Y_1$.

Conversely, let $W \subseteq M$ then by hypothesis there is $Y \subseteq M$ and $Y_1 \subseteq_{\oplus} M$ such that $Y \subseteq_r W$ and $Y \subseteq_r Y_1$. Since $Y \subseteq W \cap Y_1 \subseteq W$, then $W \cap Y_1 \subseteq_r W$. And $Y \subseteq W \cap Y_1 \subseteq Y_1$, then $W \cap Y_1 \subseteq_r Y_1$. Hence M is a $\mathcal{G}$-RCS. □

Remarks and Examples 3.4:

1. Any $\mathcal{G}$-RCS module is a $\mathcal{G}$-CS module. The opposite is not necessarily true in general. For example, let $M = Z_4$ as Z-module is a $\mathcal{G}$-CS, since it is extending, but it is not a $\mathcal{G}$-RCS, since $<\bar{2}> \subseteq M$ and the direct summand of M are $M \subseteq_{\oplus} M$ and $<\bar{0}> \subseteq_{\oplus} M$ but $<\bar{2}> \cap M \not\subseteq_r M$ and $<\bar{2}> \cap <\bar{0}> \not\subseteq_r <\bar{2}>$.
2. Any RCS module is a $\mathcal{G}$-CS. The opposite is not necessarily true in general. For example, let M=Z_{25} as Z-module is a $\mathcal{G}$-CS since it is CS, but it is not an RCS module, since $<\bar{5}> \subseteq_{rc} M$ but $<\bar{5}>$ is not summand of M.
3. Every RCS module is $\mathcal{G}$-RCS module.
4. Every monoform module is a $\mathcal{G}$-RCS module, in fact every monoform is an RCS module. But the opposite is not necessarily true in general. For example, let $M = Z_{30}$ as Z-module, M is a $\mathcal{G}$-RCS module (since it is RCS), but M is not monoform (Since $<\bar{2}> \subseteq M$ But $<\bar{2}> \not\subseteq_r M$ since $<\bar{2}>$ is not essential in M .
5. Every semisimple module is $\mathcal{G}$-RCS module (since every semisimple is RCS). The opposite is not necessarily true in general. For example, Z as Z-module.
6. Each integral domain is a $\mathcal{G}$-RCS module.

Proof: Let $0 \neq I$ be an ideal of an integral domain R. Let $0 \neq c \in R$ and $0 \neq b \in R$ for any $0 \neq a \in I$, $ca \in I$ and $cb \neq 0$, also $ba \neq 0$ (since if $ba = 0$ with R has no zero divisor elements, then $a = 0$ is contradiction as $0 \neq a$). So, we have $I \subseteq_r R$, then R is a monoform and by (4) R is a $\mathcal{G}$-RCS module. □

Rationally extending	$\Longrightarrow$	Extending
$\Downarrow$		$\Downarrow$
$\mathcal{G}$-Rationally extending	$\Longrightarrow$	$\mathcal{G}$-extending

Proposition 3.5: If M is a $\mathcal{G}$-RCS module and UR-closure, then M is RCS.

Proof: Suppose that M be a $\mathcal{G}$-RCS. And let $W \subseteq M$ then there is $W_1 \subseteq_{\oplus} M$ such that $W\ \beta_r\ W_1$. But M is UR-closure then by Proposition 2.5 $\beta_r = \alpha_r$. So, for any $W \subseteq M$ then there is $W_1 \subseteq_{\oplus} M$ such that $W\ \alpha_r\ W_1$. Then by Proposition 2.8 we have M is an RCS a module. □

Proposition 3.6: If an indecomposable R-module M be a $\mathcal{G}$-RCS, then M is a monoform module.
Proof: Assume that M is a $\mathcal{G}$-RCS, and let $0 \neq W \subseteq M$ then there is $W_1 \subseteq_{\oplus} M$ such that $W \beta_r W_1$. That's mean $W \cap W_1 \subseteq_r W$ and $W \cap W_1 \subseteq_r W_1$, but M is an indecomposable then either $W_1 = 0$ or $W_1 = M$. If $W_1 = 0$, then $< 0 > \subseteq_r W$ is a contradiction. So, $W_1 = M$ and $W \subseteq_r M$ ($W = W \cap W_1 = W \cap M \subseteq_r W_1 = M$). Hence, M is a monoform module. □

Corollary 3.7: In an indecomposable R-module M, the following statement are equivalent:
1. An R-module M is an RCS module;
2. An R-moduleM is a $\mathcal{G}$-RCS module;
3. An R-module M is a monoform module.

The following condition is necessarily to make a submodule of $\mathcal{G}$-RCS module is a $\mathcal{G}$-RCS:
(1#): Let $J \subseteq M$, if $J \cap J_1 \subseteq_{\oplus} J$ for each $J_1 \subseteq_{\oplus} M$. Then we say that M has condition (1#).

Proposition 3.8: Let $W_1 \subseteq M$ and M be a $\mathcal{G}$-RCS R-module. If W_1 satisfies the condition (1#), then W_1 is $\mathcal{G}$-RCS module.
Proof: Let $B \subseteq W_1$ and M is $\mathcal{G}-$RCS, then there is $W \subseteq_{\oplus} M$ such that $B \beta_r W$. So, we have $B \cap W \subseteq_r B$, $B \cap W \subseteq_r W$ and by condition (1#) we have $W \cap W_1 \subseteq_{\oplus} W_1$. Then by [1] we obtain $B \cap (W \cap W_1) = (B \cap W) \cap W_1 \subseteq_r B \cap W_1 = B$, and $B \cap (W \cap W_1) = (B \cap W) \cap W_1 \subseteq_r W \cap W_1$. Then $B \beta_r (W \cap W_1)$ and hence W_1 is $\mathcal{G}$-RCS module. □
(2#): Let J be a summand of M, if $J \cap J_1 \subseteq_{\oplus} J$ for each $J_1 \subseteq_{\oplus} M$. Then we say that M has condition (2#).

Corollary 3.9: Let $M = N_1 \oplus N_2$ be $\mathcal{G}$-RCS module where $N_1, N_2 \subseteq M$ and M has a condition (2#), then N_1 is $\mathcal{G}$-RCS module.
It is well known where M be an R-module and $W \subseteq M$, then W is called fully invariant if $f(W) \subseteq W$ for each endomorphism f of M [8]. Moreover, a module M is called duo, if any $L \subseteq M$ is fully invariant [9]. Furthermore, $H \subseteq M$ is a distributive submodule, if for any $V_1, V \subseteq M$, $H \cap (V_1 + V) = (H \cap V_1) + (H \cap V)$. And if all submodule is distributive, then M is called a distributive module [10].

Proposition 3.10: Let W be a fully invariant submodule of an R-module M. If M is a $\mathcal{G}$-RCS module, then W is $\mathcal{G}$-RCS module.
Proof: Let M be a $\mathcal{G}$-RCS and $V_1 \subseteq W \subseteq M$, then there is A , $A_1 \subseteq_{\oplus} M$ such that $V_1 \beta_r A$. Then the projection map $\pi_1: M \to A$, $\pi_2: M \to A_1$. For each $y \in W$, $y = a + b$ where $a \in A$ and $b \in A_1$, so $\pi_1(y) = a$ and $\pi_2(y) = b$. Since W is a fully invariant and $\tau_{1\circ} \pi_1 \in End(M)$ hence $a = \pi_1(y) = \tau_{1\circ} \pi_1(y) \in \tau_{1\circ} \pi_1(M) \cap W$, that is $a \in \pi_1(M) \cap W$, where $\tau_1: A \to M$ is an inclusion map. And by the same way of $a \in \pi_1(M) \cap W$,we have $b \in \pi_2(M) \cap W$. Therefore, $y = a + b \in (\pi_1(M) \cap W) \oplus (\pi_2(M) \cap W)$, and hence $W = (A \cap W) \oplus (A_1 \cap W)$. Since $V_1 \beta_r A$, then with the same few steps of Proposition 3.8 we have $V_1 \beta_r (A \cap W)$ and $A \cap W \subseteq_{\oplus} W$. Hence W is a $\mathcal{G}$-RCS module. □

Proposition 3.11: Let W be a distributive submodule of an R-module M. If M is a $\mathcal{G}$-RCS module, then W is a $\mathcal{G}$-RCS module.
Proof: Assume that M is $\mathcal{G}$-RCS module and let $E \subseteq W \subseteq M$. Then there is $B \subseteq_{\oplus} M$ such that $E \beta_r B$ and for some $B_1 \subseteq M$, $M = B \oplus B_1$. But W is a distributive then $W = (B \cap W) \oplus (B_1 \cap W)$. Hence W satisfy (condition 1#). So, we have $E \beta_r (B \cap W)$, ($E \cap (B \cap W) = (E \cap B) \cap W \subseteq_r E \cap W$, and $E \cap (B \cap W) = (E \cap B) \cap W \subseteq_r B \cap W$). Then W is a $\mathcal{G}$-RCS-module.

Corollary 3.12: If an R-module M is distributive (or duo) and $\mathcal{G}$-RCS, then any $W \subseteq M$ is a $\mathcal{G}$-RCS module.

Recall that, in an R-module M. If for any $W \subseteq M$ there is an ideal U of R such that $W = UM$, then M is called a multiplication, [11].

Corollary 3.13: If an R-module M is a multiplication and $\mathcal{G}$-RCS, then any $W \subseteq M$ is a $\mathcal{G}$-RCS module.

The that result that showing the class of $\mathcal{G}$-RCS module is an isomorphic property has been proved in the following proposition:

Proposition 3.14: Let $M \cong M_1$ and a module M is $\mathcal{G}$-RCS, then M_1 is a $\mathcal{G}$-RCS module.

Recall that, $Z(M) = \{a \in M | La = 0;$ for some essential ideal L of $R\}$ is singular submodule of a module M. If $Z(M) = M$, then M is a singular module and if $Z(M) = 0$, then M is non-singular module, [12].

Theorem 3.15: Let M be a non-singular R-module. Then the following statements are equivalent:

1. M is a $\mathcal{G}$-CS module;
2. M is a $\mathcal{G}$-RCS module;
3. M is an RCS module;
4. M is an extending module.

An R-module M has a rat-closed property, if $V_1 \cap V_2 \subseteq_{rc} M$ when $V_1, V_2 \subseteq_{rc} M$.

Proposition 3.16: Let an R-module M be a $\mathcal{G}$-RCS. If M is an UR-closure module, then M has the rat-closed property also has condition 2#.

Proof: Suppose that M is an UR-closure module, then by Proposition 3.5, M is an RCS. Let W and $V_1 \subseteq_{\oplus} M$, so by [2], we have W and V_1 are RCS modules. Since $W \cap V_1 \subseteq W$, there is $V \subseteq_{\oplus} W$ such that $W \cap V_1 \subseteq_r V$. And $W \cap V_1 \subseteq_r B$ for some $B \subseteq_{\oplus} V_1$. Then $W \cap V_1 \subseteq_r V \subseteq_{rc} W$ and $W \cap V_1 \subseteq_r B \subseteq_{rc} V_1$, so V and B are rat-closures of $W \cap V_1$, but M is an UR-closure then $V = B$. Then we have $V = B \subseteq W \cap V_1$, and hence $V = B = W \cap V_1$. But $V, B \subseteq_{\oplus} M$, then M has a condition 2#. And by (every direct summand is rat-closed), then we have $W, V_1 \subseteq_{rc} M$ and $V, B \subseteq_{rc} M$. So, $V = B = W \cap V_1$ is a rat-closed of M. Hence, M has a rat-closed property. □

4. The direct sum of $\mathcal{G}$ -RCS module

In this section, we will study the direct sum of $\mathcal{G}$-RCS module.

The direct sum of $\mathcal{G}$-RCS module need not be a $\mathcal{G}$-RCS, by the following example below:

Example 4.1: We know that Z is an integral domain, then Z[x] is an integral domain. By **Remarks 3.4,** (6), Z[x] is $\mathcal{G}$-RCS. But $M = Z[x] \oplus Z[x]$ as Z[x] is not a $\mathcal{G}$-RCS, since is not a $\mathcal{G}$-CS, [3].

Now we take a condition to make the direct sum of a $\mathcal{G}$-RCS to be $\mathcal{G}$-RCS. Firstly, we need to define a new concept which named an Rat-direct sum.

Definition 4.2: Let M be an R-module and $\{X_\rho\}, \{Y_\rho\}$ be collections of submodules of M. If for any ρ , $X_\rho \subseteq_r Y_\rho$ and $\oplus X_\rho \subseteq_r \oplus Y_\rho$, then M is called an Rat-direct sum.

Lemma 4.3: If $X_\rho \beta_r Y_\rho$ of M_ρ for any $\rho \in \Lambda$ where $\{M_\rho : \rho \in \Lambda\}$ be a family of an Rat-direct sum modules, then $(\bigoplus X_\rho)\beta_r(\bigoplus Y_\rho)$,

Proof: Assume that $X_\rho \beta_r Y_\rho$ of module M_ρ for any $\rho \in \Lambda$. Then $(X_\rho \cap Y_\rho) \subseteq_r X_\rho$ and $(X_\rho \cap Y_\rho) \subseteq_r Y_\rho$ for any $\rho \in \Lambda$. But M_ρ is an Rat-direct sum, then $\bigoplus (X_\rho \cap Y_\rho) \subseteq_r \bigoplus X_\rho$ and $\bigoplus (X_\rho \cap Y_\rho) \subseteq_r \bigoplus Y_\rho$, then $(\bigoplus X_\rho) \cap (\bigoplus Y_\rho) \subseteq_r \bigoplus X_\rho$ and $(\bigoplus X_\rho) \cap (\bigoplus Y_\rho) \subseteq_r \bigoplus Y_\rho$. Hence $(\bigoplus X_\rho)\beta_r(\bigoplus Y_\rho)$.□

Proposition 4.4: Let W_1 and W_2 be modules such that $M = W_1 \oplus W_2$ be a duo Rat-direct sum R-module. Then W_1 and W_2 are $\mathcal{G}$-RCS if and only if M is a $\mathcal{G}$-RCS.

Proof: Let M be a $\mathcal{G}$-RCS, so by Corollary 3.12 W_1 and W_2 are $\mathcal{G}$-RCS. Conversely, assume that W_1 and W_2 are $\mathcal{G}$-RCS and $P \subseteq M$. But $M = W_1 \oplus W_2$ is a duo module so by [9], we have $P = (P \cap W_1) \oplus (P \cap W_2)$. Since W_i is $\mathcal{G}$-RCS for $(i = 1,2)$, and $P \cap W_i \subseteq W_i$, then there is $V_i \subseteq_\oplus W_i$ such that $(P \cap W_i)\, \beta_r\, V_i$. Then by Lemma 4.3, we obtain $P = (P \cap W_1) \oplus (P \cap W_2)\beta_r(V_1 \oplus V_2)$. Since M is an Rat-direct sum, therefore, $V_1 \oplus V_2 \subseteq_\oplus M$. Hence M is a $\mathcal{G}$-RCS. □

Proposition 4.5: Let W_1 and W_2 be modules such that $M = W_1 \oplus W_2$ be a distributive Rat-direct sum R-module. Then M is a $\mathcal{G}$-RCS if and only if W_1 and W_2 are $\mathcal{G}$-RCS.

Proof: Let M be a $\mathcal{G}$-RCS, then by Corollary 3.12 W_1 and W_2 are $\mathcal{G}$-RCS. Conversely, assume that W_1 and W_2 are $\mathcal{G}$-RCS and $P \subseteq M$,then $P = P \cap M = P \cap (W_1 \oplus W_2)$, since M is a disteibutive module. By a similar steps of Proposition 4.4, we have M is a $\mathcal{G}$-RCS module.□

Proposition 4.6: Let W_1 and W_2 be modules such that $M = W_1 \oplus W_2$ be a Rat-direct sum R-module and $\mathrm{ann}(W_1) + \mathrm{ann}(W_2) = R$. If W_1 and W_2 are $\mathcal{G}$-RCS, then M is a -RCS module.

Proof: Let $0 \neq P \subseteq M$, since $\mathrm{ann}(W_1) + \mathrm{ann}(W_2) = R$, then by [13], we obtain $P = X \oplus Y$, where $X \subseteq W_1$ and $Y \subseteq W_2$. But $P \neq 0$ then we have three cases:

Case 1, if $X = 0$ and $Y \neq 0$, then $P = Y \subseteq W_2$. But W_2 is $\mathcal{G}$-RCS module, then there is $V \subseteq_\oplus W_2$ such that $P\beta_r V$. So, $V \subseteq_\oplus M$.

Case 2, if $X \neq 0$ and $Y = 0$, then we get by same way of Case 1, that $J \subseteq_\oplus W_1$ such that $P\, \beta_r\, J$, then $J \subseteq_\oplus M$.

Case 3, if $X \neq 0$ and $Y \neq 0$, then there is $S \subseteq_\oplus W_1$ and $F \subseteq_\oplus W_2$, such that $X\, \beta_r\, S$ and $Y\, \beta_r\, F$. But M is a Rat-direct sum, then by Lemma 4.3 we have $P = (X \oplus Y)\beta_r (S \oplus F)$ and $(S \oplus F) \subseteq_\oplus M$ such that $P\, \beta_r\, (S \oplus F)$.

Then by Cases 1, 2 and 3 M is a -RCS module.□

Proposition 4.7: Let $M = \bigoplus_{j\in J} W_j$ and every Rat-closed submodule is fully invariant. If W_j is a $\mathcal{G}$-RCS module for any $j \in J$, then M is a $\mathcal{G}$-RCS module.

Proof: Let $P \subseteq_{rc} M$, then P is a fully invariant of M. So, by [9], we have $P = \bigoplus (P \cap W_j)$ since $(P \cap W_j) \subseteq_{rc} P$, so by [6], we obtain $(P \cap W_j) \subseteq_{rc} M$ then $(P \cap W_j) \subseteq_{rc} W_j$. But W_j for any $j \in J$ is $\mathcal{G}$-RCS module, then there is $V_j \subseteq_\oplus W_j$ such that $(P \cap W_j)\beta_r V_j$, $((P \cap W_j) \cap V_j \subseteq_r (P \cap W_j)$ and $(P \cap W_j) \cap V_j \subseteq_r V_j$). Take $V = \bigoplus V_j \subseteq_\oplus M$. But by Lemma 4.3, $\bigoplus (P \cap W_j)\beta_r (\bigoplus V_j)$. Then M is a $\mathcal{G}$-RCS module.□

5. Conclusions

Through this paper, we reached to the following conclusions: any RCS module is a $\mathcal{G}$-RCS module. And any $\mathcal{G}$-RCS is a $\mathcal{G}$-CS. So, we have any RCS module is a $\mathcal{G}$-CS. And a summand of $\mathcal{G}$-RCS need not be a $\mathcal{G}$-RCS . And a direct sum of $\mathcal{G}$-RCS need not be a $\mathcal{G}$-RCS, and we add a condition to make a direct sum of $\mathcal{G}$-RCS is -RCS.

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