DOI: 10.24996/ijs.2024.65.4.27

ISSN: 0067-2904

An Extended Subclass of Meromorphic Multivalent Functions Involving Ruscheweyh Derivative Operator

Nada Hameed Yassin , Kassim A. Jassim
Department of Mathematics, College of Science, University of Baghdad, Iraq

Received: 17/10/2022 Accepted: 18/4/2023 Published: 30/4/2024

Abstract:
In this paper, we introduce and discuss an extended subclass 〖 Ą〗 _p^*(λ,α,γ) of meromorphic multivalent functions involving Ruscheweyh derivative operator. Coefficients inequality, distortion theorems, closure theorem for this subclass are obtained.

Keywords: Meromorphic Multivalent Functions, Ruscheweyh Derivatives.

صنف جزئي موسع لدوال ميرومورفية متعددة التكافؤ متضمنة مؤثر مشتقة راشوية

ندى حميد ياسين ، قاسم عبد الحميد جاسم
قسم الرياضيات ، كلية العلوم ، جامعة بغداد، بغداد، العراق

الخلاصة
في هذا البحث قدمنا وناقشنا صنف جزئي موسع $Ą_p^*(\lambda,\alpha,\gamma)$ لدوال ميرومورفية متعددة التكافؤ متضمنة مؤثر مشتقة راشوية . تم الحصول على متراجحة المعاملات و نظريات التشوه و نظرية الانغلاق لهذا الصنف الجزئي.

1. Introduction

Let $Ą_p^*$ denoted the subclass of functions of the form

$$f(z) = z^{-p} + \sum_{n=p+1}^{\infty} a_{n-p} z^{n-p} \qquad ; (p = 1,2,3, \dots) , \tag{1}$$

which are analytic and p-valent in the punctured unit disk U*={ z : z ∈ Ç; $0 < |z| < 1$} for $f(z) \in Ą_p^*$ given by (1) and $g(z) \in Ą_p^*$ given by

$$(z) = z^{-p} + \sum_{n=p+1}^{\infty} b_{n-p} z^{n-p} \; ; \; (p = 1,2,3, \dots). \tag{2}$$

Some classes related to meromorphic functions are studied by Morga [1], Xu and Yang [2], Raina and srivastava [3] etc. The Hadamard product (or convolution product) of f and g is defined by

$$(f * g)(z) = z^{-p} + \sum_{n=p+1}^{\infty} a_{n-p} b_{n-p} z^{n-p} = (g * f)(z). \tag{3}$$

*Email:nadahameed1983@gmail.com

The extended linear derivative operator of Ruscheweyh type for the functions of the subclass A_{P}^{*}

$$D_{*}^{\lambda,p} : \mathrm{A}_{P}^{*} \rightarrow \mathrm{A}_{P}^{*}$$

is defined as

$$D_{*}^{\lambda,p} f(z) = \frac{1}{z^{p}(1-z)^{\lambda+1}} * f(z) \; ; (\lambda > -1 : f \in \mathrm{A}_{P}^{*}). \quad (4)$$

In terms of binomial coefficients, (4) can be written as

$$D_{*}^{\lambda,p} f(z) = z^{-p} + \sum_{n=p+1}^{\infty} \binom{\lambda+n}{n} a_{n} z^{n-p} ; (\lambda > -1 : f \in \mathrm{A}_{P}^{*}). \quad (5)$$

In particular where $\lambda = n$ ($n \in N$), it is easily observed from (4) and (5) that

$$D_{*}^{\lambda,p} f(z) = \frac{z^{-p}(z^{n+p} f(z))^{(n)}}{n!} \; (n \in N_{\circ} = N \cup \{0\}). \quad (6)$$

The definition 1.7 of linear operator $D_{*}^{\lambda,p}$ is motivated by Ruscheweyh operator D^{λ} [4]. Some linear operators analogous to $D_{*}^{\lambda,p}$ are considered by Raina and Srivastava [5] and Liu and Srivastava [6], using the operator $D_{*}^{\lambda,p}(\lambda > -1)$. Other related subclasses are studied in [7] and [8]. Now, we introduce an extended subclass $\mathrm{A}_{p}^{*}(\lambda, \alpha, \gamma)$ of meromorphically p-valent analytic function defined as follows:

Definition 1: A function $f(z) \in \mathrm{A}_{p}^{*}$ is said to be a mumber of the class $\mathrm{A}_{p}^{*}(\lambda, \alpha, \gamma)$ if and only if

$$\left| (1-\gamma)\left(\frac{z\left(D_{*}^{\lambda,p} f(z)\right)''}{(D_{*}^{\lambda,p} f(z))'} \right) + p + 1 \right| < p - \alpha \,, (\, z \in D \,, p \in N \,, \lambda > -1 \,,$$

$$0 \le \alpha < p \,, 0 \le \gamma < 1) \quad (7)$$

By taking $\gamma = 0$, we get the result is studied by Jitendra Awasthil [9].
The aim of this paper is to obtain some properties as coefficients inequality, Distortion Theorem, and Closure Theorem.

2. Coefficients Estimates

Theorem 2: A function $f(z)$ is in the subclass $\mathrm{A}_{p}^{*}(\lambda, \alpha, \gamma)$ if and only if

$$\sum_{n=p+1}^{\infty} \binom{\lambda+n}{n} (n-p)[n(1-\gamma) + p(1+\gamma) + \gamma - \alpha] \, a_{n-p} \le p[(p-\alpha) + \gamma(p+1)] \;. \quad (8)$$

The result is sharp.

Proof: let $f(z) \in \mathrm{A}_{p}^{*}$ then from (7), we have

$$\left| (1-\gamma)\left(\frac{z(D_{*}^{\lambda,p} f(z))''}{(D_{*}^{\lambda,p} f(Z))'} \right) + p + 1 \right| < p - \alpha.$$

Then

$$\left| (1-\gamma) \frac{z[(p^{2}-1)\, z^{-p-2} + \sum_{n=p+1}^{\infty} \binom{\lambda+n}{n} (n-p)(n-p-1) a_{n-p} z^{(n-p-2)}]}{-p\, z^{-p-1} + \sum_{n=p+1}^{\infty} \binom{\lambda+n}{n} (n-p) a_{n-p} z^{(n-p-1)}} + p + 1 \right| < p - \alpha.$$

Therefore,

$$\left|\frac{[(p^2-1)\,z^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)(n-p-1)a_{n-p}z^{(n-p-1)}]}{-p\,z^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)a_{n-p}z^{(n-p-1)}}\right.$$

$$\left.\frac{-\gamma[(p^2-1)\,z^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)(n-p-1)a_{n-p}z^{(n-p-1)}]}{-p\,z^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)a_{n-p}z^{(n-p-1)}}\right|+p+1<p-\alpha.$$

Thus

$$\left|\frac{[(p^2-1)\,z^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)(n-p-1)a_{n-p}z^{(n-p-1)}]}{-pz^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)a_{n-p}z^{(n-p-1)}}\right.$$

$$\left.\frac{-\gamma(p^2-1)z^{-p-1}-\gamma\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)(n-p-1)a_{n-p}z^{(n-p-1)}}{-pz^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)a_{n-p}z^{(n-p-1)}}+p+1\right|<p-\alpha.$$

Also,

$$\left|-\gamma p(p+1)(1+\gamma)z^{-p-1}+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}a_{n-p}[(n-p)((1-\gamma)(n-p-1)\right.$$
$$\left.+(p+1)]z^{n-p-1}\right|$$
$$\leq \gamma p(p+1)+\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}a_{n-p}[(n-p)((1-\gamma)(n-p-1)+(p+1).$$

Then

$$\sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(1+\gamma)+\gamma-\alpha]\,a_{n-p}$$
$$\leq p[(p-\alpha)+\gamma(p+1)]\quad .$$

Conversely, assuming that the inequality (8) hold true. Then from (7), we get

$$\left|(1-\gamma)\left(\frac{z(D_*^{\lambda,p}f(z))''}{(D_*^{\lambda,p}f(Z))'}\right)+p+1\right|<p-\alpha\,.$$

Since Re (z)≤ $|z|$,we get

$$Re\left\{(1-\gamma)\left(\frac{z(D_*^{\lambda,p}f(z))''}{(D_*^{\lambda,p}f(Z))'}\right)+p+1\right\}<p-\alpha.$$

Therefore, if z→ $1-$, we get (8).

Corollary 1: Let$f(z)\in A_{P}^{*}$. Then$f$(z)∈ A_{p}^{*}(λ,α,γ)if and only if

$$a_{n-p} \le \frac{p\left((p-\alpha)+\gamma(p+1)\right)}{\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(1+\gamma)+\gamma-\alpha]} \ . \qquad (10)$$

If we take γ =0 in Theorem 1, we get the following

Corollary 2: Let $f(z) \in A^*_{tp}$. Then $f(z) \in A^*_{tp}(\lambda,\alpha,0)$ if and only if

$$\sum_{n=p+1}^{\infty} \binom{\lambda+n}{n}(n-p)[n+p-\alpha]\, a_{n-p} \le p(p-\alpha) \, .$$

Corollary 3. Let $f(z) \in A^*_{tp}$. Then $f(z) \in A^*_{tp}(\lambda,\alpha,0)$ if and only if

$$a_{n-p} \le \frac{p(p-\alpha)}{\binom{\lambda+n}{n}(n-p)[n+p-\alpha]} \, .$$

3.Distortion Theorem

Theorem 3: If $f(z) \in A^*_{tp}(\lambda,\alpha,\gamma)$, then for $0 < |z| = r < 1$,

$$r^{-p} - \frac{p[(p-\alpha)+(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]} r \le |f(z)|$$

$$\le \ r^{-p} + \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1-\gamma)+\gamma-\alpha]} r.$$

Proof.

Since $f(z) \in A^*_{tp}(\lambda,\alpha,\gamma)$, then from the equation (8) it follows that

$$\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]\sum_{n=p+1}^{\infty} a_{n-p}$$

$$\le \sum_{n=p+1}^{\infty}\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(1+\gamma)+\gamma-\alpha]\, a_{n-p}$$

$$\le p[(p-\alpha)+\gamma(p+1)]$$

$$for \qquad \sum_{n=p+1}^{\infty} a_{n-p} \le \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]} .$$

So,

$$|f(z)| \ge |z|^{-p} - \sum_{n=p+1}^{\infty}|a_{n-p}||z|^{n-p} \ge |z|^{-p} - |z|\sum_{n=p+1}^{\infty}|a_{n-p}|$$

or

$$|f(z)| \ \ge r^{-p} - \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]} \qquad (10)$$

and

$|f(z)| \le |z|^{-p} + \sum_{n=p+1}^{\infty}|a_{n-p}||z|^{n-p} \le |z|^{-p} + |z|\sum_{n=p+1}^{\infty}|a_{n-p}|$,

or

$$|f(z)| \ \ge r^{-p} - \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]} . \quad (11)$$

Therefore,

$$r^{-p}-\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]}r\leq|f(z)|$$
$$\leq\ r^{-p}+\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}p[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]}.$$

If we take γ =0, we get the following

Corollary 4: If f(z)∈ A^*_{ip}(λ,α, 0),then

$$r^{-p}-\frac{p(p-\alpha)}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]}r\leq|f(z)|$$
$$\leq\ r^{-p}+\frac{p(p-\alpha)}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1+\gamma)+\gamma-\alpha]}.$$

Theorem 3: If $f(z)$∈ A^*_{ip}(λ,α,γ) , then for 0 <|z| = r < 1;

$$pr^{-p-1}-\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1-\gamma)+\gamma-\alpha]}\leq\left|\acute{f}(z)\right|$$
$$\leq pr^{-p-1}+\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1-\gamma)+\gamma-\alpha]}\ . \quad (12)$$

Proof: Since f(z)∈ A^*_{ip}(λ,α,γ), then

$$\sum_{n=p+1}^{\infty}a_{n-p}\leq\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1-\gamma)+\gamma-\alpha]}.$$

Also,

$$f(z)=\ z^{-p}+\textstyle\sum_{n=p+1}^{\infty}a_{n-p}z^{n-p}\qquad ,(p\,=\,1,2,3,\dots\dots). \quad (13)$$

Therefore,

$$|f'(z)|\geq\ pr^{-p-1}-\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1-\gamma)+\gamma-\alpha]} \quad (14)$$

and

$$\left|\acute{f}(z)\right|\leq pr^{-p-1}+\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+p+1}{p+1}[(p+1)(1-\gamma)+p(1-\gamma)+\gamma-\alpha]}. \quad (15)$$

From (14) and (15) we get the results.
If we take γ =0, we get the following
Corollary 5: If f(z)∈ A^*_{ip}(λ,α, 0),then

$$pr^{-p-1}-\frac{p(p-\alpha)}{\binom{\lambda+p+1}{p+1}[1+2p-\alpha]}\leq\left|\acute{f}(z)\right|\leq pr^{-p-1}+\frac{p(p-\alpha)}{\binom{\lambda+p+1}{p+1}[1+2p-\alpha]}.$$

4. Closure Theorem

Theorem 4: Letf_p(z) =$z^{-p}$and

$$f_p(z)=z^{-p}+\frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(\gamma+1)+\gamma-\alpha]}z^{n-p}\ ,(n\geq p+1). \quad (16)$$

Then

f(z)∈ A^*_p(λ,α, γ)if and only if it can be expressed in the form

$$f(z) = \sum_{n=p}^{\infty} M_n f_n(z)(17),$$

where $M_n \geq 0$ and $\sum_{n-p}^{\infty} n = 1$.

Proof: Suppose that f(z) can be expressed in the form (17). Then

$$f(z) = \sum_{n=p}^{\infty} M_n f_n(z) = M_p f_p(z) + \sum_{n=p+1}^{\infty} M_n f_n(z)$$

$$= z^{-p} + \sum_{n=p}^{\infty} \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(\gamma+1)+\gamma-\alpha]} z^{n-p}.$$

Since

$$\sum_{n=p+1}^{\infty} \binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(\gamma+1)+\gamma + \alpha] \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(\gamma+1)+\gamma-\alpha]}$$

$$= \sum_{n=p+1}^{\infty} p[(p-\alpha)+\gamma(p+1)]M_n$$

$= p[(p-\alpha)+\gamma(p+1)]\sum_{n=p+1}^{\infty} M_n \leq p[(p-\alpha)+\gamma(p+\gamma)] \leq p[(p-\alpha)+\gamma(p+\gamma)],$

So by Theorem 2 we have $f(z) \in A^*_p$(λ,α)

Conversely, let $f(z) \in A^*_p$(λ,α, γ).

Since

$$a_{n-p} \leq \frac{p[(p-\alpha)+\gamma(p+1)]}{\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(\gamma+1)+\gamma-\alpha]}$$

setting

$$M_n = \frac{\binom{\lambda+n}{n}(n-p)[n(1-\gamma)+p(\gamma+1)+\gamma-\alpha]}{p[(p-\alpha)+\gamma(p+1)]}$$

and

$$M_p = 1 - \sum_{n=p}^{\infty} M_n$$

it follows that

$$f(z) = \sum_{n=p+1}^{\infty} M_n f_n.$$

Conclusions:

We get an extended subclass A^*_p(λ,α,γ) of meromorphic multivalent functions involving Ruscheweyh derivative operator. Some geometric properties for this subclass are arrived.

References

[1] M.L.Mogra, "Hadamard product of certain meromorphic univalent functions", *J. Math. Anal. Appl.*, vol.157, pp.10-16, 1991,.

[2] N.Xu and D.Yang, "On starlikeness and close to convexity of certain meromorphic functions", *J. Korean Soc. Math. Edus. Ser B: Pure Appl. Math*., vol.10, pp. 566-581, 2003.
[3] R.K.Raina and H.M.Srivastava, "A new class of meromorphically multivalent functions with applications of generalized hypergeometric functions", *Math. Comput. Modelling*, vol.43, pp. 350-356, 2006.
[4] S.Rushweyh, "New criteria for univalent functions,Proc". *Amer. Math. Soc*., vol. 49 ,pp.109-115, 1975.
[5] R.K.Raina and H.M.Srivastava, "Inclusion and neighborhoods properties of some analytic and multivalent functions", *J. Inequal. Pure and Appl. Math*., vol.7 , no. 1 , 2006.
[6] J.L.Liu and H.M.Srivastava, "A linear operator and associated families of meromorphically multivalent functions", *J. Math. Anal. Appl*., vol.259, pp. 566-581, 2001.
[7] A. J. Kassim. A. Adnan, "Some Geometric Properties of GeneralizedClass of function associated with Higher Ruscheweyh Derivatives". *Iraq Journal of Science* , vol.60. no. 9, p, 2019p. 2036-2042 ,.
[8] A. J. Kassim andO. R. Reem, "New subclasses of meromorphic multivalent functions associated with a differential operator", *Journal of Interdisciplinary Mathematics* , vol. 22, no. 8 pp. 1443-1449, 2019.
[9] A. Jitendra , "A new class of meromorphic multivalent functions involving an extended linear derivative operator of Ruscheweyh" , *Int. J. Math.And Appl*.,vol.6, pp.369-375, 2018.