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Generalized Schultz and Modified Schultz Polynomials for Some Special Graphs

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Abstract

With simple and undirected connected graph Φ, the Schultz and modified Schultz polynomials are defined as $Sc(\Phi;x)=\sum(degv+degu)\,x^{d(v,u)}$ and $Sc^{*}(\Phi;x)=\sum(degu\times degv)\,x^{d(u,v)}$, respectively, where the summation is taken over all unordered pairs of distinct vertices in V(Φ), where V(Φ) is the vertex set of Φ, degu is the degree of vertex u and d(v,u) is the ordinary distance between v and u, u≠v. In this study, the Shultz distance, modified Schultz distance, the polynomial, index, and average for both have been generalized, and this generalization has been applied to some special graphs.

Keywords: generalized Schultz distance, generalized modified Schultz distance, indices, polynomials, special graphs.

متعددتي حدود شوالتز المعممة وشوالتز المعدلة المعممة لبعض البيانات الخاصة

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الخلاصة

مع رسم بياني متصل بسيط وغير موجه، يتم تعريف متعددة حدود شوالتز ومتعددة حدود شوالتز المعدلة كما يلي:

$Sc(\Phi;x)=\sum(degv+degu)\,x^{d(v,u)}, Sc^{*}(\Phi;x)=\sum(degu\times degv)\,x^{d(u,v)}$

على التوالي، حيث ان المجموع يؤخذ لكل الأزواج غير المرتبة من الرؤوس المختلفة $V(\Phi)$ ، حيث ان $V(\Phi)$ هي مجموعة رؤوس البيان Φ ،
$degu$ هي درجة الرأس u و $d(v,u)$ هي المسافة الاعتيادية بين الرأسين u و v ، حيث أن $u\neq v$.
في هذه الدراسة مسافة شوالتز ومسافة شوالتز المعدلة ومتعددات الحدود والدليل والمعدل لكليهما تم تعميما ومن و هذا التعميم طبق على بعض البيانات الخاصة.

1. Introduction

Topological graph indices are mathematical formulas that can be obtained for any graph which have some molecular structures. From these indices, it is possible to analyze mathematical values and further investigate some physicochemical properties of molecules.

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Therefore, it is an efficient method for avoiding expensive and time-consuming laboratory experiments.

Let Φ be a finite undirected connected graph without loops and multiple edges, denote the vertex and edge sets of Φ by $V(\Phi)$ and $E(\Phi)$, respectively, the order of Φ is the number of vertices in Φ, that is $|V(\Phi)| = p$ and the size of Φ is the number of edges in Φ, that is $|E(\Phi)| = q$, the degree of a vertex $u \in V(\Phi)$ is the number of vertices joining to u and denoted by $deg u$ (or $\delta(u)$), [1,2]. One of the topological indices is the Schultz index, this index was introduced by Schultz in 1989, [3] and it is defined as:
$\mathcal{S}c(\Phi) = \sum_{\{v,u\}\subseteq V(\Phi)}(deg v + deg u)d(v,u)$.
While the modified Schultz index was defined by Klavžar and Gutman in 1997, [4] as follows:
$\mathcal{S}c^*(\Phi) = \sum_{\{v,u\}\subseteq V(\Phi)}(deg v \cdot deg u)d(v,u)$.
Schultz polynomial and modified Schultz polynomial are defined by Gutman in 2005, [5] as, respectively:
$\mathcal{S}c(\Phi; x) = \sum_{\{u,v\}\subseteq V(\Phi)}(deg v + deg u)\, x^{d(u,v)}$.

$$\mathcal{S}c^*(\Phi; x) = \sum_{\{u,v\}\subseteq V(\Phi)} (deg v \cdot deg u)\, x^{d(u,v)}.$$

The average Schultz distance and modified Schultz distance are defined as, respectively [6]:
$\overline{\mathcal{S}c}(\Phi) = 2\mathcal{S}c(\Phi)/p(p-1)$.
$\overline{\mathcal{S}c^*}(\Phi) = 2\mathcal{S}c^*(\Phi)/p(p-1)$.
There are a lot of papers that have been done to compute Schultz and modified Schultz polynomials and the indices for many graphs, for more information, see the references [6-11]. In this paper, we generalized the Schultz polynomial and modified the Schultz polynomial by taking all vertices degrees of the path, provided that the product of the length of the path with the sum of the degrees is a minimum, because of the importance of degrees as chemical bonds located on atoms and they are effects on the stability of the chemical compound. Finally, there are many indices and many polynomials which are important in knowing the chemical and physical properties of chemical compounds, see [12-17].

2. Generalization Shultz and Modified Shultz Distance

In this section, we present generalizations to Shultz and modified Schultz distances and we give the definitions which related polynomials, indices, and averages for all Shultz and modified Schultz distances generalizations.

Definition 2.1: Let v and u be any distinct vertices of a connected graph Φ and let Q be a $(v,u)-$ path of a length $l(Q)$. The generalized Schultz distance $d^{GS}(u,v)$ between v and u of Φ is defined as:
$d^{GS}(v,u) = min_Q[\{deg v + \alpha \sum_{w\in V(Q)-\{v,u\}} deg w + deg u\}l(Q)]$,
where Q is any $(v,u)-$ path and the minimum are taken over all $(v,u)-$ paths Q and $deg y$ is a degree of any vertex y in $V(Q)$ and $\alpha \in \{0,1\}$.
If $\alpha = 0$, then the generalized Schultz distance is equal to Schultz distance.
We can rewrite the generalized Schultz distance $d^{GS}(v,u)$ as:

$$d^{GS}(v,u) = min_Q[\{\sum_{w\in V(Q)} deg w\}l(Q)], \quad \ldots (1.1)$$

where Q is any $(v,u)-$ path and the minimum are taken over all $(v,u)-$ paths Q, $deg y$ is a degree of vertex y. The generalized Shultz polynomial is defined as:
$GSc(\Phi; x) = \sum_{\{v,u\}\subseteq V(\Phi)} S_d(v,u,Q)\, x^{l(Q)}$, where $S_d(v,u,Q) = \frac{d^{GS}(v,u)}{l(Q)}$.
We can also write this polynomial in another form:

$$GSc(\Phi; x) = \sum_{k\geq 1} S_d(\Phi,k)\, x^k, \quad \ldots (1.2)$$

where $S_d(\Phi, k)$ is the sum degrees of all vertices in a $(v, u) -$ path Q for all $v, u \in V(\Phi)$ at k apart a distance such that $d^{GS}(v, u) = \{\sum_{w \in V(Q)} deg w\}k$.
Now, the generalized Schultz index is defined as:

$$GSc(\Phi) = \frac{d}{dx}\left(GSc(\Phi; x)\right)|_{x=1}$$
$$= \sum_{k \geq 1} kS_d(\Phi, k) = \sum_{\{u,v\} \subseteq V(\Phi)} d^{GS}(v, u). \quad \ldots (1.3)$$

The generalizations of the Schultz polynomial and Schultz index of a vertex v are defined as respectively:

$$GSc(v, \Phi; x) = \sum_{u \in V(\Phi) - \{v\}} S_d(v, u, Q) x^{l(Q)} = \sum_{k \geq 1} S_d(v, G, k)\, x^k,$$

where $S_d(v, \Phi, k)$ is the sum degrees of all vertices in a $(v, u) -$ path Q for all $u \in V(\Phi) - \{v\}$ at k apart a distance such that $d^{GS}(v, u) = \{\sum_{w \in V(Q)} deg w\}k$.

$$GSc(v, \Phi) = \frac{d}{dx}\left(GSc(v, \Phi; x)\right)|_{x=1}$$
$$= \sum_{k \geq 1} kS_d(v, G, k) = \sum_{u \in V(\Phi) - \{v\}} d^{GS}(v, u).$$

Hence,

$$GSc(\Phi; x) = \frac{1}{2} \sum_{v \in V(G)} GSc(v, \Phi; x).$$
$$GSc(\Phi) = \frac{1}{2} \sum_{v \in V(\Phi)} GSc(v, \Phi).$$

Definition 2.2: Let v and u be any distinct vertices of a connected graph Φ and let Q be a $(v, u) -$ path of a length $l(Q)$. The generalized modified Schultz distance $d^{GS^*}(v, u)$ between v and u of Φ is defined as:

$$d^{GS^*}(v, u) = min_Q[\{deg v \times \alpha \prod_{w \in V(Q) - \{u,v\}} deg w \times deg u\} l(Q)],$$

where Q is any $(v, u) -$ path and the minimum is taken over all $(v, u) -$ paths Q and $deg y$ is a degree of any vertex y in $V(Q)$ and $\alpha \in \{0,1\}$.
If $\alpha = 0$, then the generalized modified Schultz distance is equal modified Schultz distance.
We can rewrite the modified generalized Schultz distance $d^{GS^*}(v, u)$ as:

$$d^{GS^*}(v, u) = min_Q[\{\prod_{w \in V(Q)} deg w\} l(Q)]. \quad \ldots (2.1)$$

The generalized modified Shultz polynomial is defined as:

$GSc^*(\Phi; x) = \sum_{\{v,u\} \subseteq V(G)} S_d^*(v, u, Q)\, x^{l(Q)}$, where $S_d^*(v, u, Q) = \frac{d^{GS^*}(v,u)}{l(Q)}$.

We can also write this polynomial in another form as:

$$GSc^*(\Phi; x) = \sum_{k \geq 1} S_d^*(\Phi, k)\, x^k, \quad \ldots (2.2)$$

where $S_d^*(G, k)$ is the sum degrees of all vertices in a $(v, u) -$ path Q for all $v, u \in V(\Phi)$ at k apart distance such that $d^{GS^*}(v, u) = \{\prod_{w \in V(Q)} deg w\}k$.
Now, the generalized modified Schultz index is defined as:

$$GSc^*(\Phi) = \frac{d}{dx}\left(GSc^*(\Phi; x)\right)|_{x=1}$$
$$= \sum_{k \geq 1} kS_d^*(\Phi, k) = \sum_{\{v,u\} \subseteq V(\Phi)} d^{GS^*}(v, u)\,. \quad \ldots (2.3)$$

The generalizations of the modified Shultz polynomial and modified Schultz index of a vertex v are defined as respectively:

$$GSc^*(v, \Phi; x) = \sum_{u \in V(\Phi) - \{v\}} S_d^*(v, u, Q)\, x^{l(Q)} = \sum_{k \geq 1} S_d^*(v, \Phi, k)\, x^k.$$

where $S_d^*(v, \Phi, k)$ is the sum degrees of all vertices in a $(v, u) -$ path Q for all $u \in V(\Phi) - \{v\}$ at k apart a distance such that $d^{GS^*}(v, u) = \{\prod_{w \in V(Q)} deg w\}k$.

$$GSc^*(v, \Phi) = \frac{d}{dx}\left(GSc^*(v, \Phi; x)\right)|_{x=1}$$
$$= \sum_{k \geq 1} k\, S_d^*(v, \Phi, k) = \sum_{u \in V(\Phi) - \{v\}} d^{GS^*}(v, u).$$

Hence,

$GSc^*(\Phi;x) = \frac{1}{2}\sum_{v\in V(\Phi)} GSc^*(v,\Phi;x)$.
$GSc^*(\Phi) = \frac{1}{2}\sum_{v\in V(\Phi)} GSc^*(v,\Phi)$.

Definition 2.3: The averages of generalized Schultz distance and generalized modified Schultz distance are defined respectively:
$\overline{GSc}(\Phi) = 2GSc(\Phi)/p(p-1)$ and $\overline{GSc^*}(\Phi) = 2GSc^*(\Phi)/p(p-1)$.
Where p is an order of a connected graph G.

Remark 2.4:
If Φ is a tree graph, then the minimum in Definitions 2.1 and 2.2 are not significant, so it can be removed because for any two vertices in the tree graph, there is only one path between them, that is, $l(Q) = d(v,u)$, where Q is any $(v,u)-$ path.

3. New results:

In the following theorems, we find the polynomials, indices, and averages for all generalized Schultz and generalized modified Schultz for some special graphs such as: complete graph $K_p, p \geq 3$, star graph $S_p, p \geq 4$, fan graph $F_p, p \geq 5, p$ is an odd, path graph P_p and cycle graph $C_p, p \geq 3$.

Theorem 3.1: Let K_p, S_p and F_p be complete, star and fan graphs of order p, respectively, then
1. $GSc(K_p;x) = p(p-1)^2x, p \geq 3$,
$GSc^*(K_p;x) = \frac{1}{2}p(p-1)^3x, p \geq 3$.
2. $GSc(S_p;x) = p(p-1)x + \frac{1}{2}(p^2-1)(p-2)x^2, p \geq 4$,
$GSc^*(S_p;x) = (p-1)^2x + \frac{1}{2}(p-1)^2(p-2)x^2, p \geq 4$.
3. $GSc(F_p;x) = (p-1)(p+3)x + \frac{1}{2}(p-1)(p^2-9)x^2, p \geq 5, p$ is an odd.
$GSc^*(F_p;x) = 2p(p-1)x + 2(p-3)(p-1)^2x^2, p \geq 5, p$ is an odd.

Proof:

1. Since any two distinct vertices in $V(K_p)$ are adjacent and have $(p-1)$ degree, then
$GSc(K_p;x) = p(p-1)^2x$.
And
$GSc^*(K_p;x) = \frac{1}{2}p(p-1)^3x$.

2. Since the central vertex of S_p which has $(p-1)$ degree is only vertex adjacent to every end vertex of S_p and the distance between any two distinct vertices of the end vertex of S_p are two and that these paths must pass through the central vertex, then
$GSc(S_p;x) = p(p-1)x + \frac{1}{2}(p^2-1)(p-2)x^2$.
And
$GSc^*(S_p;x) = (p-1)^2x + \frac{1}{2}(p-1)^2(p-2)x^2$.

3. Since the central vertex u_1 of $F_p$which has $(p-1)$ degree adjacent to every vertex u_i of F_p such that $degu_i = 2$ for all $i = 2,3,\ldots,p, p \geq 5, p$ is odd. In addition to that every vertex u_{2i} is adjacent to the vertex u_{2i+1} where $i = 1,2,\ldots,(p-1)/2$, the distance between any two vertices that are not adjacent to each other is two, and the path connecting between them must pass from the central vertex u_1, then
$GSc(F_p;x) = (p-1)(p+3)x + \frac{1}{2}(p-1)(p^2-9)x^2$.
and
$GSc^*(F_p;x) = 2p(p-1)x + 2(p-1)^2(p-3)x^2$.

Corollary 3.2: Let K_p, S_p and F_p be complete, star and fan graphs of order p, respectively, then:

1. $GSc(K_p) = p(p-1)^2, p \geq 3,$

$GSc^*(K_p) = \frac{1}{2}p(p-1)^3, p \geq 3.$

2. $GSc(S_p) = (p-1)(p^2-2), p \geq 4,$

$GSc^*(S_p) = (p-1)^3, p \geq 4.$

3. $GSc(F_p) = (p+3)(p-1)(p-2), p \geq 5, p$ is an odd.

$GSc^*(F_p) = 2(p-1)(p-2)(2p-3), p \geq 5, p$ is an odd.

Proof: It is easy to get this result from equation (1.3) and (2.3).

Corollary 3.3: Let K_p, S_p and F_p be complete, star and fan graphs of order p, respectively, then

1. $\overline{GSc}(K_p) = 2(p-1), p \geq 3,$

$\overline{GSc^*}(K_p) = (p-1)^2, p \geq 3.$

2. $\overline{GSc}(S_p) = \frac{2(p^2-2)}{p}, p \geq 4,$

$\overline{GSc^*}(S_p) = \frac{2(p-1)^2}{p}, p \geq 4.$

3. $\overline{GSc}(F_p) = \frac{2(p+3)(p-2)}{p}, p \geq 5, p$ is odd.

$\overline{GSc^*}(F_p) = \frac{4(p-2)(2p-3)}{p}, p \geq 5, p$ is odd.

Proof: It is easy to get this result from Definition 2.3

Theorem 3.4: Let P_p be a path graph of order $p, p \geq 3$, then

1. $GSc(P_p;x) = 2\sum_{k=1}^{p-1}\{(p-1)(k+1)-k^2\}x^k.$

2. $GSc^*(P_p;x) = \sum_{k=1}^{p-2}(p-k-1)2^{k+1}x^k + 2^{p-2}x^{p-1}.$

Proof: Let $V(P_p) = \{u_1, u_2, \ldots, u_{p-1}, u_p\}$ such that $\delta(u_i) = deg u_i = 1$, $i = 1, p$ and $\delta(u_i) = deg u_i = 2$, $2 \leq i \leq p-1$. Then

1.
$$
\begin{aligned}
GSc(P_p;x) &= \sum_{k=1}^{p-1}\{\sum_{j=1}^{p-k}\sum_{i=j}^{j+k}\delta(u_i)\}x^k \\
&= \sum_{k=1}^{p-2}\{\sum_{j=1}^{p-k}\sum_{i=j}^{j+k}\delta(u_i)\}x^k + \sum_{j=1}^{1}\sum_{i=j}^{j+p-1}\delta(u_i)\,x^{p-1} \\
&= \sum_{k=1}^{p-2}\{\sum_{i=1}^{k+1}\delta(u_i) + \sum_{j=2}^{p-k-1}\sum_{i=j}^{j+k}\delta(u_i) + \sum_{i=p-k}^{p}\delta(u_i)\}x^k \\
&+ \sum_{i=1}^{p}\delta(u_i)\,x^{p-1} \\
&= \sum_{k=1}^{p-2}\{\delta(u_1) + \sum_{i=2}^{k+1}\delta(u_i) + \sum_{j=2}^{p-k-1}\sum_{i=j}^{j+k}\delta(u_i) \\
&+ \sum_{i=p-k}^{p-1}\delta(u_i) + \delta(u_p)\}x^k + \{\delta(u_1) + \sum_{i=2}^{p-1}\delta(u_i) + \delta(u_p)\}x^{p-1} \\
&= \sum_{k=1}^{p-2}\{1 + \sum_{i=2}^{k+1}2 + \sum_{j=2}^{p-k-1}\sum_{i=j}^{j+k}2 + \sum_{i=p-k}^{p-1}2 + 1\}x^k \\
&+ \{1 + \sum_{i=2}^{p-1}2 + 1\}x^{p-1} \\
&= \sum_{k=1}^{p-2}\{2 + 4k + 2(p-k-2)(k+1)\}x^k + 2(p-1)x^{p-1} \\
&= 2\sum_{k=1}^{p-1}\{(p-1)(k+1) - k^2\}x^k.
\end{aligned}
$$

2.
$$
\begin{aligned}
GSc^*(P_p;x) &= \sum_{k=1}^{p-1}\{\sum_{j=1}^{p-k}\prod_{i=j}^{j+k}\delta(u_i)\}\,x^k \\
&= \sum_{k=1}^{p-2}\{\sum_{j=1}^{p-k}\{\prod_{i=j}^{j+k}\delta(u_i)\}\}x^k + \sum_{j=1}^{1}\{\prod_{i=j}^{j+p-1}\delta(u_i)\}\,x^{p-1} \\
&= \sum_{k=1}^{p-2}\{\prod_{i=1}^{k+1}\delta(u_i) + \sum_{j=2}^{p-k-1}\{\prod_{i=j}^{j+k}\delta(u_i)\} + \prod_{i=p-k}^{p}\delta(u_i)\}x^k \\
&+ \prod_{i=1}^{p}\delta(u_i)\,x^{p-1} \\
&= \sum_{k=1}^{p-2}\{\prod_{i=2}^{k+1}\delta(u_i) + \sum_{j=2}^{p-k-1}\{\prod_{i=j}^{j+k}\delta(u_i)\} + \prod_{i=p-k}^{p-1}\delta(u_i)\}x^k \\
&+ \{\prod_{i=2}^{p-1}\delta(u_i)\}x^{p-1}
\end{aligned}
$$

$$= \sum_{k=1}^{p-2}\{\prod_{i=2}^{k+1} 2 + \sum_{j=2}^{p-k-1}\{\prod_{i=j}^{j+k} 2\} + \prod_{i=p-k}^{p-1} 2\}x^k + \{\prod_{i=2}^{p-1} 2\}x^{p-1}$$
$$= \sum_{k=1}^{p-2}\{2^{k+1} + (p-k-2)2^{k+1}\}x^k + 2^{p-2}x^{p-1}$$
$$= \sum_{k=1}^{p-2}(p-k-1)2^{k+1}x^k + 2^{p-2}x^{p-1}. \#$$

Corollary 3.5: Let P_p be a path of order p, $p \geq 3$, then

1. $GSc(P_p) = \frac{p(p-1)^2(p+4)}{6}$,

$\overline{GSc}(P_p) = \frac{(p-1)(p+4)}{3}$.

2. $GSc^*(P_p) = 2^{p-2}(9p-33) + 4(p+2)$,

$\overline{GSc^*}(P_p) = \frac{2^{p-1}(9p-33)+8(p+2)}{p(p-1)}$.

Proof:

1. $GSc(p_p) = \frac{d}{dx}\left(GSc(p_p;x)\right)|_{x=1}$

$$= 2\sum_{k=1}^{p-1} k[(p-1)k + (p-1) - k^2]$$
$$= 2(p-1)\sum_{k=1}^{p-1} k^2 + 2(p-1)\sum_{k=1}^{p-1} k - 2\sum_{k=1}^{p-1} k^3$$
$$= p(p-1)^2\left[\frac{2p-1}{3} + 1 + \frac{p}{2}\right] = \frac{p(p-1)^2(p+4)}{6}.$$

Now by multiplying the generalized Schultz index of path by $\frac{2}{p(p-1)}$ we get the average

$$\overline{GSc}(P_p) = \frac{\frac{2}{6}p(p-1)^2(p+4)}{p(p-1)} = \frac{(p-1)(p+4)}{3}.$$

2. $GSc^*(P_p) = \frac{d}{dx}\left(GSc^*(p_p;x)\right)|_{x=1}$

$$= \sum_{k=1}^{p-2} k(p-k-1)\,2^{k+1} + (p-1)2^{p-2}$$
$$= (p-1)\sum_{k=1}^{p-2} k2^{k+1} - \sum_{k=1}^{p-2} k^2 2^{k+1} + (p-1)2^{p-2}$$
$$= 2^p p^2 - 3p2^p - p2^p + (3)2^p + 4p - 4 - 2^p p^2 + 6p2^p - (11)2^p + 12 + (p-1)2^{p-2}$$
$$= 2^{p-2}(9p-33) + 4(p+2).$$

Now by multiplying the generalized modified Schultz index of the path by $\frac{2}{p(p-1)}$ we get the average

$$\overline{GSc^*}(P_p) = \frac{2\left[2^{p-2}(9p-33)+4(p+2)\right]}{p(p-1)} = \frac{\left[2^{p-1}(9p-33)+8(p+2)\right]}{p(p-1)}. \qquad \#$$

Theorem 3.6: Let C_p be a cycle graph of order p, $p \geq 3$, then

1. $GSc(C_p;x) = 2p\sum_{k=1}^{\lceil\frac{p}{2}\rceil-1}(k+1)\,x^k + \begin{cases} \frac{p(p+2)}{2}x^{\frac{p}{2}}, & p \text{ is even}, p \geq 4, \\ 0, & p \text{ is odd}, p \geq 3. \end{cases}$

2. $GSc^*(C_p;x) = p\sum_{k=1}^{\lceil\frac{p}{2}\rceil-1} 2^{k+1}x^k + \begin{cases} 2^{\frac{p}{2}}p\,x^{\frac{p}{2}}, & p \text{ is even}, p \geq 4, \\ 0, & p \text{ is odd}, p \geq 3. \end{cases}$

Proof: Let $u_i \in V(C_p)$ where $i = 1,2,\dots,p$, then

(i) If p is an odd, then:

1. $GSc(u_i,C_p;x) = \sum_{k=1}^{\frac{p-1}{2}} 2(2+2(k))\,x^k$, for all $i = 1,2,\dots,p$. Hence,

$GSc(C_p;x) = \frac{1}{2}\sum_{i=1}^{p} GSc(u_i,C_p;x)$

$$= \frac{1}{2}\sum_{i=1}^{p}\sum_{k=1}^{\frac{p-1}{2}} 2(2+2(k))\,x^k = 2p\sum_{k=1}^{\frac{p-1}{2}}(k+1)\,x^k.$$

2. $GSc^*(u_i,C_p;x) = \sum_{k=1}^{\frac{p-1}{2}} 2(2*2^k)\,x^k$, for all $i = 1,2,\dots,p$.

$= \sum_{k=1}^{\frac{p-1}{2}} 2^{k+2}x^k$, for all $i = 1,2,\dots,p$, Hence

$$GSc^*(C_p;x) = \frac{1}{2}\sum_{i=1}^{p} GSc^*(u_i, C_p;x)$$
$$= \frac{1}{2}\sum_{i=1}^{p}\sum_{k=1}^{\frac{p-1}{2}} 2^{k+2}\, x^k = p\sum_{k=1}^{\frac{p-1}{2}} 2^{k+1}\, x^k.$$

(ii) If p is an even, then:

1. $GSc(u_i, C_p;x) = \sum_{k=1}^{\frac{p}{2}-1} 2(2+2(k))\, x^k + \left(2+2\frac{p}{2}\right) x^{\frac{p}{2}}$, for all $i = 1,2,\dots,p$. Hence,

$$GSc(C_p;x) = \frac{1}{2}\sum_{i=1}^{p} GSc(u_i, C_p;x)$$
$$= \frac{1}{2}\sum_{i=1}^{p}[\sum_{k=1}^{\frac{p}{2}-1} 2(2+2(k))\, x^k + \left(2+2\frac{p}{2}\right) x^{\frac{p}{2}}]$$
$$= \sum_{i=1}^{p}\sum_{k=1}^{\frac{p}{2}-1}[(2+2(k))\, x^k + \left(1+\frac{p}{2}\right) x^{\frac{p}{2}}].$$
$$= 2p\sum_{k=1}^{\frac{p}{2}-1}(k+1)\, x^k + \frac{p(p+2)}{2} x^{\frac{p}{2}}.$$

2. $GSc^*(u_i, C_p;x) = \sum_{k=1}^{\frac{p}{2}-1} 2(2*2^k)\, x^k + \left(2*2^{\frac{p}{2}}\right) x^{\frac{p}{2}}$, $i = 1,2,\dots,p$. Hence,

$$GSc^*(C_p;x) = \frac{1}{2}\sum_{i=1}^{p} GSc^*(u_i, C_p;x)$$
$$= \frac{1}{2}\sum_{i=1}^{p}[\sum_{k=1}^{\frac{p}{2}-1} 2(2*2^k)\, x^k + \left(2*2^{\frac{p}{2}}\right) x^{\frac{p}{2}}]$$
$$= p\sum_{k=1}^{\frac{p}{2}-1} 2^{k+1}x^k + 2^{\frac{p}{2}}p\, x^{\frac{p}{2}}$$
$$= p\sum_{k=1}^{\frac{p}{2}-1} 2^{k+1}x^k + 2^{\frac{p}{2}}p\, x^{\frac{p}{2}}. \quad \#$$

Corollary 3.7: Let C_p be a cycle of order p, $p \geq 3$, then

1. $$GSc(C_p) = \begin{cases} \frac{p^2(p+2)(p+1)}{12} & , \ p \text{ is even}, p \geq 4, \\ \frac{p(p^2-1)(p+3)}{12} & , \ p \text{ is odd}, p \geq 3. \end{cases}$$

$$\overline{GSc}(C_p) = \begin{cases} \frac{p(p+2)(p+1)}{6(p-1)} & , \ p \text{ is even}, p \geq 4, \\ \frac{(p+1)(p+3)}{6} & , p \text{ is odd}, p \geq 3. \end{cases}$$

2. $$GSc^*(C_p) = \begin{cases} p(2^{\frac{p}{2}-1}(3p-8)+4) & , \ p \text{ is even}, p \geq 4, \\ p\left(2^{\frac{p+1}{2}}(p-3)+4\right) & , p \text{ is odd}, p \geq 3. \end{cases}$$

$$\overline{GSc^*}(C_p) = \begin{cases} \frac{2^{\frac{p}{2}}(3p-8)+8}{p-1} & , p \text{ is even}, p \geq 4, \\ \frac{2^{\frac{p+3}{2}}(p-3)+8}{p-1} & , p \text{ is odd}, p \geq 3. \end{cases}$$

Proof:

1. $GSc(C_p) = \frac{d}{dx}\left(GSc(C_p;x)\right)|_{x=1}$

i. If p is an even, then:

$$GSc(C_p) = 2\text{p}\sum_{k=1}^{\frac{p}{2}-1} k(k+1) + \frac{p}{2}\left(\frac{p(p+2)}{2}\right)$$
$$= 2p\sum_{k=1}^{\frac{p}{2}-1} k^2 + 2p\sum_{k=1}^{\frac{p}{2}-1} k + \frac{p^2}{4}(p+2)$$
$$= 2p\left[\frac{\left(\frac{p}{2}-1\right)\left(\frac{p}{2}\right)(p-1)}{6}\right] + 2p\left[\frac{\frac{p}{2}\left(\frac{p}{2}-1\right)}{2}\right] + \frac{p^2}{4}(p+2)$$
$$= \frac{p^2}{6}\left(\frac{p}{2}-1\right)(p+2) + \frac{p^2}{4}(p+2) = \frac{p^2}{12}(p+2)(p+1).$$

ii. If p is odd, then:

$$GSc(C_p) = 2p\sum_{k=1}^{\frac{p-1}{2}} k^2 + 2p\sum_{k=1}^{\frac{p-1}{2}} k$$
$$= 2p[\frac{\frac{p-1}{2}\left(\frac{p-1}{2}+1\right)p}{6}] + 2p[\frac{\frac{p-1}{2}\left(\frac{p-1}{2}+1\right)}{2}]$$
$$= \frac{p^2}{6}(p-1)\left(\frac{p+1}{2}\right) + p(\frac{p-1}{2})(\frac{p+1}{2}) = \frac{p}{12}(p^2-1)(p+3).$$

Now by multiplying the generalized Schultz index of cycle by $\frac{2}{p(p-1)}$ we get the average:

$$\overline{GSc}(C_p) = \begin{cases} \frac{p(p+2)(p+1)}{6(p-1)} & , p \text{ is even}, p \geq 4, \\ \frac{(p+1)(p+3)}{6} & , p \text{ is odd} \ , p \geq 3 . \end{cases}$$

2. $GSc^*(C_p) = \frac{d}{dx}\left(GSc^*(C_p; x)\right)|_{x=1}$

i.If p is an even, then:

$$GSc^*(C_p) = p\sum_{k=1}^{\frac{p}{2}-1} k2^{k+1} + \frac{p^2}{2}2^{\frac{p}{2}}$$
$$= \mathrm{p}\left[2^{\frac{p}{2}}(p-4)+4\right] + \frac{p^2}{2}2^{\frac{p}{2}} = p\left(2^{\frac{p}{2}-1}(3p-8)+4\right).$$

ii. If p is an odd, then:

$$GSc^*(C_p) = p\sum_{k=1}^{\frac{p-1}{2}} k2^{k+1} = p\left[2^{\frac{p+1}{2}}(p-3)+4\right].$$

Now by multiplying the generalized modified Schultz index of the cycle by $\frac{2}{p(p-1)}$ we get the average:

$$\overline{GSc^*}(C_p) = \begin{cases} \frac{2^{\frac{p}{2}}(3p-8)+8}{p-1} & , p \text{ is even}, p \geq 4, \\ \frac{2^{\frac{p+3}{2}}(p-3)+8}{p-1} & , p \text{ is odd} \ , p \geq 3 . \end{cases} \qquad \#$$

4. Conclusions:

In this paper, we have generalized the definitions of Schultz and modified Schultz distance, and the benefit of this generalization is to include all vertices degrees of paths with the arithmetic operations of addition and multiplication, and that the inclusion of all degrees of vertices is of importance in chemistry compared to the number of bonds that lie on the carbon atoms as well as the distance between carbon atoms.

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