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Novel Definitions of α-Fractional Integral and Derivative of the Functions

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Abstract

An α-fractional integral and derivative of real function have been introduced in new definitions and then, they compared with the existing definitions. According to the properties of these definitions, the formulas demonstrate that they are most significant and suitable in fractional integrals and derivatives. The definitions of α-fractional derivative and integral coincide with the existing definitions for the polynomials for $0 \leq \alpha < 1$. Furthermore, if $\alpha = 1$, the proposed definitions and the usual definition of integer derivative and integral are identical. Some of the properties of the new definitions are discussed and proved, as well, we have introduced some applications in the α- fractional derivatives and integrals. Moreover, α-power series and α–rule of integration by parts have been proposed and implemented in this study.

Keywords: α-Fractional Integral; α-Fractional Derivative ;α-power series; α–rule of integration; FrDEs.

تعريفات جديده للتكاملات والمشتقات الفا– كسورية للدوال

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الخلاصة

تم تقديم تعريفات جديدة الفا– كسورية للتكامل والمشتقة للدوال الحقيقية وقورنت بالتعاريف الكسورية المعروفة. ووفقًا لخصائص هذه التعريفات ، توضح الصيغ أنها الأكثر أهمية وملاءمة في التكاملات والمشتقات الكسورية. تتطابق تعريفات المشتق والتكامل الكسري مع التعريفات الحالية لكثيرات الحدود لـ $0 \leq \alpha < 1$. علاوة على ذلك ، إذا كانت $\alpha = 1$ ، فإن التعريفات المقترحة والتعريف الكلاسيكي للمشتقة والتكاملات الصحيحة تكون متطابقة. تمت مناقشة بعض خصائص التعريفات الجديدة وإثباتها ، كما قدمنا

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بعض التطبيقات في المشتقات والتكاملات الكسورية. علاوة على ذلك في هذه الدراسة، تم اقتراح وتطبيق متسلسلة القوى وقاعدة تكامل التجزئة الفا– كسورية .

1. Introduction

Fractional calculus was a hot topic in the 20th and 21th centuries. Evidently, over the past two decades, the use of fractional calculus has become increasingly prevalent in both pure and applied fields of science and engineering. Discrete versions of fractional calculus and their properties have been studied by many researchers. For example, some authors studied and introduced the properties and methods of solving fractional differential equations (FrDEs), as well as some concepts in FrDEs theory and applications [1-4]. Some authors proposed new definitions of fractional derivatives. For example, R. Khalil et al. introduced a new definition of the fractional derivative and fractional integral and then, they show the new definition is the most fruitful and natural definition [5] while Zheng et al. proposed a new fractional derivative of the Caputo type and then, some of the basic properties this definition has been studied [6]. In the review of the literature of the authors which are used the new definitions for solving FrDEs, namely Anderson and Avery have reformulated the second-order conjugate boundary value problem(BVP) using the new conformable fractional derivative [7], Cenesiz and Kurt discovered the precise answers to the time fractional heat differential equations (DEs) [8], Unal and Gan presented the conformable fractional differential transform method and its use with conformable FrDEs [9]. Hammad and Khalil have studied the Legendre conformable FrDEs and the basic properties of such fractional polynomials [10], Abdel Hakim has proved the existence of the conformable fractional [11], Khalil and Abu-Hammad have gave the exact solution of the heat conformable FrDE. They also discussed some other differential equations [12], Khalil et al. studied the geometrical meaning of the conformable fractional derivative and the fractional orthogonal trajectories are also introduced [13]. Unal et al. solved the variable coefficients, homogeneous sequential linear conformable FrDEs of order two using the power series around a regular point, also, they introduced the conformable fractional Hermite DEs [14], Abdel Jawad developed the definition of the fractional conformable derivative and he set the basic concepts in this fractional calculus [15]. In addition, Ortega and Rosales introduced the properties of fractional conformable derivatives [16]. Moreover, Qasim and Holel, studied the solution of some types of composition fractional order DEs corresponding to optimal control problems [17]. Lastly, Mechee and Senu studied the numerical solutions of FrDEs of Lane-Emden type by the method of collocation, [18]. On the other hand, many authors studied the properties and the applications of the definitions of solving fractional definitions derivative or integration [19]-[27]. In this study, we introduced novel definitions of α-fractional integrals and derivatives. Indeed, their properties have been studied. Moreover, α-power series and α–rule of integration by parts have been proposed and implemented in this study.

2. Preliminary

The background related to this study has been introduced in this section.

2.1 Gamma Function

Albert Einstein created the non-integral factorial function known as the Gamma function (Gf). Gf is the most significant notation that is used in classical fractional calculus.

Definition 2.1 Gamma Function [28]

The function of Gamma Γ(x) is a function of a positive real number x, defined

1. $$\Gamma(x) = \int_0^\infty y^{x-1}\, e^{-y} dy,\ x > 0. \quad (1)$$

In the following, some of the most important properties of the Gamma function are given: for x > 0; $\forall n \in N^{+}$.

2. $\Gamma(x+1) = x\Gamma(x) = x!$.
3. $\Gamma(jx) = \frac{j(j+1)}{2}\frac{\Gamma(x+1)}{x} - \frac{(1-j)(j-1)}{2}\frac{\pi\, csc(\pi x)}{\Gamma(x+1)}$, $j = -1, 1$.
4. $\Gamma\left(\frac{1}{2}+jx\right) = \frac{j(j+1)}{2}\frac{(2n)!\sqrt{\pi}}{4^n n!} - \frac{(1-j)(j-1)}{2}\frac{(-4)^n\, n!\sqrt{\pi}}{4^n (2n)!}, j = -1, 1.$
5. $\Gamma(nx) = \sqrt{\frac{2\pi}{n}}\ \frac{n^x}{\sqrt{2\pi}}\ \pi_{k=0}^{n-1}\left(n+\frac{k}{n}\right),\ n \in N^{+}.$
6. $2\Gamma\left(\frac{1}{2}\right) = \Gamma\left(-\frac{1}{2}\right) = 2\sqrt{\pi}.$
7. $lim_{x\to -1}\,\Gamma(x) = lim_{x\to 0}\,\Gamma(x) = \infty.$

2.2 The Fractional Derivatives

Since the beginning of calculus, there have been fractional integrals and derivatives. L'Hospital wondered what $\frac{d^n f(t)}{dt^n}$ does it mean if $n = \frac{1}{2}$. Many researchers have attempted to define the fractional derivative since then. For the fractional derivative, most of them are used in an integral form. Two of them are the most well-liked. Since that time, numerous researchers have attempted to define a fractional derivative. For the fractional derivative, the majority of them employed an integral form as follows.

Definition 2.2 Riemann–Liouville Derivative [5]

Let a function $f: [a,\infty) \to \mathcal{R}$, and $a > 0$. Then, the Riemann-Liouville α-derivative definition of the function f, for $\alpha \in [n-1, n)$, where n is an integer number n ∈ N, is defined as follows:

$$D_a^{\alpha} f(\tau) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^{\tau}\frac{f(\tau)}{(\tau-x)^{n-\alpha+1}}dx. \quad (2)$$

Indeed, the Caputo derivative definition of f, for α ∈ [n – 1, n), is defined as follows:

Definition 2.3 Caputo Derivative [5]

For α ∈ [n – 1, n), the Caputo α-derivative of the function $f: [a,\infty) \to \mathcal{R}$, and $a > 0$ is defined as follows:

$$D_a^{\alpha} f(\tau) = \frac{1}{\Gamma(n-\alpha)}\int_a^{\tau}\frac{f^{(n)}(\tau)}{(\tau-x)^{n-\alpha+1}}dx, \quad (3)$$

in case that n is an integer number where $n - 1 < \alpha < n$, $n \in N$.

Definition 2.4 Conformable Fractional Derivative [5].

Given a function $f: [a,\infty) \to \mathcal{R}$, Then, the conformable fractional α –derivative of the function f of order α is defined by

$$T_\alpha\big(f(\tau)\big) = f^{(\alpha)}(\tau) = lim_{\epsilon\to 0}\frac{f(x+\epsilon x^{\,1-\alpha}) - f(x)}{2\epsilon},$$

for all $x>0$ and $\alpha \in (0, 1]$.

Definition 2.5 a-Conformable Fractional Derivative [27]

Sarikaya et al. have defined a-conformable fractional derivative for the function $f : [a,\infty) \to \mathcal{R}$ with $0 \le a < b$ of order α is as follows: [5]

$$D_{\alpha}^{a}\big(f(\tau)\big) = f^{(\alpha)}(\tau) = \lim_{\epsilon\to 0}\frac{f\big(\tau+\epsilon\tau^{\,1-\alpha}(\tau-a)\big)-f(\tau)}{\epsilon(1-a\tau^{-\alpha})}. \quad (5)$$

All of the above definitions, including Equations (2)-(5), satisfy the fractional derivative's linear property. All of the definitions share this property with the first derivative. Some of the disadvantages of the other definitions are given in the following:

1. The Riemann Liouville derivative does not meet the requirement for property $D_{\tau}^{\alpha}(1) = 0$.
2. The fractional Riemann Liouville derivative formula fails to meet the requirement for being the derivative of the product of two functions for integer derivatives. $D_{\zeta}^{\alpha}(fg)(\zeta) = f(\zeta)D_{\tau}^{\alpha}(g(\zeta)) + g(\zeta)\, D_{\tau}^{\alpha}(f(\zeta))$.
3. The fractional Riemann Liouville derivative formula does not satisfy the property of the derivative of the quotient of two functions with integer order:

$$D_{\zeta}^{\alpha}\left(\frac{f}{g}(\zeta)\right) \neq \frac{g(\zeta)D_{\zeta}^{\alpha}\big(f(\zeta)\big) - f(\zeta)D_{\zeta}^{\alpha}\big(g(\zeta)\big)}{g^2(\zeta)}.$$

4. The fractional Riemann Liouville derivative formula does not satisfy the property of the derivative by the chain rule of two functions with integer order:

$$D_{\zeta}^{\alpha}(fog)(\zeta) \neq f^{\alpha}(\zeta)\big(g(\zeta)\big)g^{\alpha}(\zeta).$$

5. The property of composition of derivatives one function is not satisfied by any fractional derivative formula. Most fractional derivatives do not satisfy in general:

$$D_{\zeta}^{\alpha}(D_{\zeta}^{\beta}\, f(\zeta)) \neq D_{\zeta}^{\alpha+\beta}\big(f(\zeta)\big).$$

6. For all derivatives definitions assumes that the function f is differentiable.

3. Main Results

In this section, the main results of this article have been introduced

3.1 Proposed α-Fractional integral and Derivative of $f(\tau)$

In this section, we have proposed the α- fractional integral and derivative of the function f(τ) and introduced their properties.

Definition 3.1

The new definition of the α-fractional derivative of the real function $f(\tau) : [a,\infty) \to \mathcal{R}.$ is defined as follows:

$$T_{\alpha}\big(f(\tau)\big) = f^{(\alpha)}(\tau) = lim_{\epsilon\to 0}\frac{f(\tau+\epsilon\tau^{\,1-\alpha})-f(\tau-\epsilon\tau^{\,1-\alpha})}{2\epsilon}, \quad (6)$$

Accordingly, in Table 1: the α-fractional derivatives of some basic functions f(τ) of order α are given using the definition 3.1

Table 1: α-Fractional Derivatives of f(τ)

$f(\tau)$	$T_{\alpha}(f)(\tau)$
c	0
1	0
T	$\tau^{1-\alpha}$
τ^2	$2\tau^{2-\alpha}$
τ^n	$n\tau^{n-\alpha}$
$\frac{\tau^{\alpha}}{\alpha}$	1

$e^{c\tau}$	$c\tau^{1-\alpha}e^{c\tau}$
$\frac{1}{\tau}$	$-\frac{\tau^{1-\alpha}}{\tau^2}$
$\frac{1}{\tau^n}$	$\frac{n\tau^{1-\alpha}}{\tau^{n+1}}$
$\frac{1}{\tau^\alpha}$	$\frac{-\alpha\tau^{1-\alpha}}{\tau^{\alpha+1}}$
$\sin(a\tau)$	$a\tau^{1-\alpha}\cos(a\tau)$
$\cos(a\tau)$	$-a\tau^{1-\alpha}\sin(a\tau)$
$\tan(a\tau)$	$a\tau^{1-\alpha}\sec^2(a\tau)$
$\sec(a\tau)$	$a\tau^{1-\alpha}\sec(a\tau)\tan(a\tau)$
$\csc(a\tau)$	$-a\tau^{1-\alpha}\csc(a\tau)\cot(a\tau)$
$\cot(a\tau)$	$-a\tau^{1-\alpha}\csc^2(a\tau)$
$\sinh(a\tau)$	$a\tau^{1-\alpha}\cos h(a\tau)$
$\cosh(a\tau)$	$a\tau^{1-\alpha}\sinh(a\tau)$
$\tanh(a\tau)$	$a\tau^{1-\alpha}\csc^2(a\tau)$
$\mathrm{sech}(a\tau)$	$-a\tau^{1-\alpha}\mathrm{sech}(a\tau)\tanh(a\tau)$
$\mathrm{csch}(a\tau)$	$-a\tau^{1-\alpha}\mathrm{csch}(a\tau)\coth(a\tau)$
$\coth(a\tau)$	$-a\tau^{1-\alpha}\mathrm{csch}^2(a\tau)$
$\ln(a\tau)$	$a\,\tau^{-\alpha}$
log10(aτ)	$\frac{a\,\tau^{-\alpha}}{\ln 10}$

3.1.1 The Comparison of the Proposed Definition with Existing Definitions

To compare the proposed definition of α-fractional derivative in Equation (6) with the existing definitions of the fractional derivatives in Equations (2)-(5), we consider, for example, the basic function $f(\tau) = e^{c\tau}$ then, using the proposed definition α-fractional derivative in Definition 3.1 and the Definitions 2.2-2.5 for Riemann–Liouville derivative, Caputo Derivative, conformable fractional derivative, and a-conformable fractional derivative, we can compare these five definitions of fractional derivative, Firstly, we try to prove the α-fractional derivative of the basic function that is given in Table 1. It means $f^{(\alpha)}(\tau) = c\tau^{1-\alpha}e^{c\tau}$ which is proved as follows:

$$T_\alpha(e^{c\tau}) = \lim_{\epsilon\to 0}\frac{e^{c(\tau+\epsilon\tau^{1-\alpha})} - e^{c(\tau-\tau^{1-\alpha})}}{2\epsilon},$$

$$= e^{c\tau}\lim_{\epsilon\to 0}\frac{e^{c\epsilon\tau^{1-\alpha}} - e^{-c\epsilon\tau^{1-\alpha}}}{2\epsilon} = e^{c\tau}\frac{c\tau^{1-\alpha} + c\tau^{1-\alpha}}{2} = c\tau^{1-\alpha}e^{c\tau}.$$

However, using the Riemann–Liouville derivative definition for $0 < \alpha < 1$, we obtain the following:

$$D_a^\alpha f(\tau) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^\tau \frac{e^{c\tau}dx}{(\tau-x)^\alpha} = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^\tau e^{c\tau}(\tau-x)^{-\alpha}\,dx = c^\alpha e^{c\tau}.$$

While, using the Caputo derivative definition, for $0 < \alpha < 1$, this leads to

$$D_a^\alpha f(\tau) = \frac{1}{\Gamma(1-\alpha)}\int_a^\tau \frac{\frac{d}{dt}e^{c\tau}}{(\tau-x)^\alpha}dx = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^\tau ce^{c\tau}(\tau-x)^{-\alpha}\,dx = c^\alpha e^{c\tau} = c^\alpha e^{c\tau}.$$

On the other hand, using the conformable fractional derivative definition of the function $f(\tau)$, we obtain

$$T_\alpha(e^{c\tau}) = \lim_{\epsilon\to 0}\frac{e^{c(\tau+\epsilon\tau^{1-\alpha})} - e^{c\tau}}{\epsilon},$$
$$= e^{c\tau}\lim_{\epsilon\to 0}\frac{e^{c\epsilon\tau^{1-\alpha}} - 1}{\epsilon} = c\tau^{1-\alpha}e^{c\tau}.$$

Lastly, using a-conformable fractional derivative definition of $f(\tau)$, which leads to the following:

$$D_\alpha^a\big(f(\tau)\big) = f^{(\alpha)}(\tau) = \lim_{\epsilon\to 0}\frac{e^{c(\tau+\epsilon\tau^{1-\alpha}(\tau-a))} - e^{c\tau}}{\epsilon(1-a\tau^{-\alpha})} = c\frac{\tau-a}{\tau^\alpha - a}e^{c\tau}.$$

As well as, by comparing the five definitions of the fractional derivative, we can conclude that the Riemann–Liouville and Caputo's definitions, for example, have the same fractional derivative while the proposed definition and conformable definition also have the same fractional derivative. In contrast, the a-conformable definition is different from all others.

3.1.2 The Properties of Proposed α-Fractional Derivative of f(τ)

To indicate the operator which is known as the α-fractional derivative of order α, let's write $T_\alpha\big(f(\tau)\big)$, when f(τ); g(τ) in the domain of T_1 that satisfy the following conditions for α = 1:

1. $T_1(af+bg)(\tau) = a\,T_1\big(f(\tau)\big)+b\,T_1\big(g(\tau)\big),\ a,b\in\mathcal{R}.$
2. $T_1(\tau^p) = p\tau^{p-1}, p\in\mathcal{R}.$
3. $T_1(fg)(\tau) = f(\tau)\,T_1\big(g(\tau)\big) + g(\tau)\,T_1(f(\tau)).$
4. $T_1\left(\frac{f}{g}(\tau)\right) = \frac{g(\tau)\,T_1\big(f(\tau)\big) - f(\tau)T_1\big(g(\tau)\big)}{g^2(\tau)}.$
5. $T_1\big(f(\tau)\big) = 0$, regarding to $f(\tau)=\lambda,\ \lambda\in\mathcal{R}$, F is constant function.

Proof:

Let $\alpha = 1$, then $T_1\big(f(\tau)\big) = D\big(f(\tau)\big)$, where D is the differential operator of integer order. Hence,

$T_1\big(f(\tau)\big) = D\big(f(\tau)\big) = f'(\tau) = lim_{\epsilon\to 0}\frac{f(\tau+\epsilon)-f(\tau-\epsilon)}{2\epsilon}$,

Parts 1-5 for the differential operator D follow directly from the definition of derivative in the calculus.

Theorem 3.1

If the real function $f : [a,\infty)\to\mathcal{R}$ is **α**-fractional differentiable at the point τ_0 and $\alpha\in(0,1]$, then $f(\tau)$ is a continuous function at τ_0.

Proof

We have to prove that f is continuous at τ_0.

Since $f(\tau_0+\epsilon\tau_0^{1-\alpha}) - f(\tau_0) = f(\tau_0+\epsilon\tau_0^{1-\alpha}) - f(\tau_0) \pm f(\tau_0-\epsilon\tau_0^{1-\alpha})$

Then,

$$\lim_{\epsilon\to 0}(f(\tau_0+\epsilon\tau_0^{1-\alpha}) - f(\tau_0)) =$$
$$\lim_{\epsilon\to 0}\frac{f(\tau_0+\epsilon\tau_0^{1-\alpha}) - f(\tau_0-\epsilon\tau_0^{1-\alpha}) + f(\tau_0-\epsilon\tau_0^{1-\alpha}) - f(\tau_0)}{2\epsilon\tau_0^{1-\alpha}}\lim_{\epsilon\to 0}2\epsilon\tau_0^{1-\alpha}.$$

Suppose $h = \epsilon\tau_0^{1-\alpha}$, then

$$\lim_{h\to 0}(f(\tau_0+h) - f(\tau_0)) = \lim_{h\to 0}\frac{f(\tau_0+h) - f(\tau_0-h) + f(\tau_0-h) - f(\tau_0)}{2h}\lim_{h\to 0}2h,$$
$$= \left(\lim_{h\to 0}\frac{f(\tau_0+h) - f(\tau_0-h)}{2h} - \frac{1}{2}\lim_{\breve{h}\to 0}\frac{f\big(\tau_0+\breve{h}\big) - f(\tau_0)}{\breve{h}}\right)\lim_{h\to 0}2h = 0,$$

Then, $lim_{h\to 0}(f(\tau_0+h)-f(\tau_0))$, which implies to $lim_{\tau\to\tau_0^+} f(\tau_0) = lim_{\tau\to\tau_0^-} f(\tau) = lim_{\tau\to\tau_0} f(\tau_0) = f(\tau_0)$. Hence, $f(\tau)$ is continuous at τ_0.

Theorem 3.2

Consider that the functions $f(\zeta)$ and $g(\zeta)$ are **α**-fractional differentiable for $\alpha \in (0,1]$. Then,

1. $T_\alpha(af \pm bg)(\zeta) = aT_\alpha(f(\zeta)) \pm bT_\alpha(g(\zeta)),\ a,b \in \mathcal{R}$.
2. $T_\alpha(\zeta^q) = q\zeta^{q-\alpha},\ q \in \mathcal{R}$.
3. $T_\alpha(fg)(\zeta) = f(\zeta)\, T_\alpha(g(\zeta)) + g(\zeta)\, T_\alpha(f(\zeta))$.
4. $T_\alpha\left(\frac{f}{g}(\zeta)\right) = \frac{g(\zeta)\, T_\alpha(f(\zeta)) - f(\zeta) T_\alpha(g(\zeta))}{g^2(\zeta)}$.
5. $\mathrm{T}_\alpha(\mathrm{f}(\zeta)) = 0$ for all constant functions $f(\zeta) = \lambda,\ \lambda \in \mathcal{R}$.
6. If $f(\zeta)$ is differentiable with respect to ζ, then, $T_\alpha(f)(\zeta) = \zeta^{\,1-\alpha}\frac{df(\zeta)}{d\zeta}$.
7. If $f(\zeta)$ is (n+1)-differentiable with respect to ζ, then,

$T_\alpha(\zeta)(f(\zeta)) = \zeta^{n+1-\alpha}\ \frac{df(\zeta)}{d\zeta}$ where $\alpha \in [n - 1; n)$.

Proof:

From Definition 3.1, properties 1-7 can be proved directly. For example, we prove only property 3 for its significance.

Now, for fixed ($\tau > 0$),

$$T_\alpha(fg)(\zeta) = (fg)^\alpha(\zeta),$$
$$= \lim_{\epsilon\to 0}\frac{f(\zeta+\epsilon\zeta^{\,1-\alpha})\, g(\zeta+\epsilon\zeta^{\,1-\alpha}) - f(\zeta-\epsilon\zeta^{\,1-\alpha})\, g(\zeta-\epsilon\zeta^{\,1-\alpha})}{2\epsilon}.$$

Then,

$$= \lim_{\epsilon\to 0}\frac{f(\zeta+\epsilon\zeta^{\,1-\alpha})\, g(\zeta+\epsilon\zeta^{\,1-\alpha}) \mp h(\zeta,\epsilon) - f(\zeta-\epsilon\zeta^{\,1-\alpha})\, g(\zeta-\epsilon\zeta^{\,1-\alpha})}{2\epsilon},$$

Where, $h(\zeta,\epsilon) = f(\tau+\epsilon\zeta^{\,1-\alpha})\, g(\zeta-\epsilon\zeta^{\,1-\alpha})$,

Hence,

$$(fg)^\alpha(\zeta) = f(\zeta)\ \lim_{\epsilon\to 0}\frac{g(\zeta+\epsilon\zeta^{\,1-\alpha}) - g(\zeta-\epsilon\zeta^{\,1-\alpha})}{2\epsilon}$$
$$+g(\tau)\lim_{\epsilon\to 0}\frac{f(\zeta+\epsilon\zeta^{\,1-\alpha}) - f(\zeta-\epsilon\zeta^{1-\alpha})}{2\epsilon},$$
$$= f(\zeta)\ T_\alpha(g)(\zeta) + g(\zeta)\ T_\alpha(f)(\zeta),$$

Theorem 3.3

The α-fractional derivative of the real function $f : [a,\infty) \to \mathcal{R}$ in the domain I_α has the following property:

$$T_\alpha(f(\zeta)) = f^{(\alpha)}(\zeta) = \zeta^{1-\alpha} f'(\zeta).$$

Proof:

$$f(\zeta+\epsilon\zeta^{\,1-\alpha}) - f(\zeta-\epsilon\zeta^{\,1-\alpha}) = 2\epsilon\zeta^{\,1-\alpha}f'(\zeta) + f(\zeta^{\,1-\alpha},\epsilon^2,\epsilon^3,\dots)$$

Then,

$$T_\alpha(f(\zeta)) = f^{(\alpha)}(\zeta) = \lim_{\epsilon\to 0}\frac{\epsilon\zeta^{\,1-\alpha}f'(\zeta) + f(\zeta^{\,1-\alpha},\epsilon^2,\epsilon^3,\dots)}{2\epsilon} = \zeta^{1-\alpha} f'(\zeta).$$

Definition 3.2 α-Fractional Integral

The new definition of the α-fractional integral of the real function

$f : [a,\infty) \to \mathcal{R}$ is defined as follows:

$$I_\alpha^a\big(f(\zeta)\big) = I_1^a\big(\tau^{\alpha-1}f(\zeta)\big) = \int_a^\tau \frac{f(\zeta)}{\zeta^{\alpha-1}}d\zeta. \tag{7}$$

where the integral is usual Riemann improper integral and $\alpha \in (0,1)$.

So, $I_{\frac{1}{4}}^0\left(\sqrt[4]{\tau^3}\,cos(2\zeta)\right) = \int_0^\tau cos(2\zeta)\,d\zeta = \frac{1}{2}sin(2\zeta),$

and $I_{\frac{1}{4}}^0\left(\sqrt{\zeta}\,sin\sqrt[4]{\zeta^3}\right) = sin2\sqrt{\zeta}$.

Theorem 3.4

The α −fractional integral of the α -differentiable real function $f\colon [a,\infty) \to \mathcal{R}$ in the domain I_α. has the property which is defined as follows: $T_\alpha(\, I_\alpha^a(f(\zeta))) = f(\zeta)$.

Proof:

Since f is α –differentiable, then, $I_\alpha^a\big(f(\zeta)\big)$ is clearly differentiable. Hence,

$$T_\alpha(\, I_\alpha^a(f))(\zeta) = \zeta^{1-\alpha}\frac{d}{d\tau}I_\alpha^a\big(f(\zeta)\big) = \zeta^{1-\alpha}\frac{d}{d\tau}\int_a^\tau \frac{f(\zeta)}{\tau^{\alpha-1}}d\zeta = f(\zeta).$$

Theorem 3.5

Consider two real continuous functions $f, g : [a,\infty) \to \mathcal{R}$ in the domain I_α^a and β, β_1, β_2 are real parameters. Then, the α-fractional integral has the following properties:

1. $I_\alpha^a(\beta(f))(\zeta) = \beta\, f(\zeta)$.
2. $I_\alpha^a(\beta) = \beta(\zeta^\alpha - a^\alpha)$
3. $I_\alpha^a\big(\beta_1\zeta^{\beta_2}\big) = \beta_1(\zeta^{\beta_2-\alpha} - a^{\beta_2-\alpha})$
4. $I_\alpha^a(\beta_1(f) \pm \beta_2(g))(\zeta) = \beta_1 f(\zeta) \pm \beta_2 g(\zeta)$.
5. $T_\alpha(\, I_\alpha^a\big((\beta_1(f) \pm \beta_2(g)\big)(\zeta) = \beta_1 f(\zeta) \pm \beta_2 g(\zeta)$.

Proof:

We can prove the above five properties of the α-fractional integral by using Definition 3.2 of the α-fractional integral in Equation (7) which depends on the limit then we can prove them by using the properties of the limit of the function.

Table 2: is for α −fractional integrals for f(τ) of order α for some fundamental functions.

$f(\tau)$	$I_\alpha^a(f)(\tau)$
c	$c(\tau^\alpha - a^\alpha)$
1	$\tau^\alpha - a^\alpha$
τ	$\tau^{\alpha+1} - a^{\alpha+1}$
τ^2	$\tau^{\alpha+2} - a^{\alpha+2}$
τ^n	$\tau^{\alpha+n} - a^{\alpha+n}$
τ^α	$\tau^{2\alpha} - a^{2\alpha}$

$\frac{1}{\tau}$	$\tau^{\alpha-1} - a^{\alpha-1}$
$\frac{1}{\tau^2}$	$\tau^{\alpha-2} - a^{\alpha-2}$
$\frac{1}{\tau^n}$	$\tau^{\alpha-n} - a^{\alpha-n}$
$\frac{1}{\tau^\alpha}$	$\ln\tau$
$e^{c\tau}$	$c\tau^{1-\alpha}e^{c\tau}$

Table 2: $\alpha -$ Fractional Integrals for f(τ) of order α

4. Proposed α-Power Series and α-Integration by Parts Methods

The definitions of α-power series and α-integration by parts methods are introduced in this section.

Definition 4.1: The α- integration by Parts
Let $f, g : [a, \infty) \to \mathcal{R}$ be real functions, then

$$I_\alpha^a\left(f(\zeta)\ T_\alpha\big(g(\zeta)\big)\right) = \int_a^\tau \frac{f(\zeta)g'^{(\zeta)}}{\zeta^{\alpha-1}} d\zeta = \frac{f(\zeta)}{\zeta^{\alpha-1}}\ g(\zeta)\Big|_a^\tau - \int_a^\tau g(\zeta)\ D\left(\frac{f(\zeta)}{\zeta^{\alpha-1}}\right) d\zeta. \quad (8)$$

For example, to find the α- integration $I_\alpha^a(\tau\, T_\alpha(e^\tau))$, we can use the rule **α**- integration to obtain

$$I_\alpha^a\big(\tau^\alpha\ T_\alpha(e^\tau)\big) = \frac{\zeta^\alpha}{\zeta^{\alpha-1}}\ e^\zeta\Big|_a^\tau - \int_a^b e^\zeta\ D\left(\frac{\zeta^\alpha}{\zeta^{\alpha-1}}\right) d\zeta = (\tau{-}1)e^\tau + (1-a)e^a.$$

Definition 4.2: α-Power Series

Assume $f : [a, \infty) \to \mathcal{R}$ is an infinitely α-differentiable function, then f can be written as the expansion of a series as follows:

$$f(\zeta) = \sum_{m=0}^{\infty} a_n\, \zeta^{m\alpha}. \quad (9)$$

Then,

$$f^{(\alpha)}(\zeta) = \sum_{m=0}^{\infty} m\alpha\, a_n\, \zeta^{m\alpha-\alpha}, \quad (10)$$

For example, to find the solution of FrDE $y^\alpha(x) + y(x) = 0$ with a given initial condition, we can use the rule of the α-power series using Equations (9)-(10) to obtain $y^\alpha(x) = \sum_{m=0}^{\infty} m\alpha\, a_n\, \zeta^{m\alpha-\alpha}$.
Then,

$$\sum_{m=0}^{\infty}(m\alpha\ \zeta^{-\alpha} + 1)a_n\zeta^{m\alpha} = 0. \quad (11)$$

However, using the given initial condition of the problem and Equation (11), we obtain the coefficients $a_n; n = 0,1,2, \ldots$

4.1 α −Power Series Method

To clarify the implementation of this method, we should follow the following steps for solving FrODE.

4.1.1 Algorithm of α −Power Series Method

1. Substitute the formulas of fractional derivatives in the given FrDE by $f(\zeta), f^{(\alpha)}(\zeta), \ldots$ from Equations (9)-(10).
2. After some algebraic arrangement, by comparing the two sides of the algebraic equation, we obtain the formula of a_n in the $\boldsymbol{\alpha}$ −power series of the required solution to the given FrODE.

5 Implementation

In this section, we have solved three FrODEs according to the proposed definitions of α −fractional derivatives and the α −power series method.

Example 5.1
Consider the following IVP

$$y^{\frac{1}{2}}(\tau) + y(\tau) = \tau^2 + 2\,\tau^{\frac{3}{2}}, \qquad \tau > 0, \tag{12}$$

with the initial condition *y(0) = 0*.
The general solution to the fractional differential equation in Equation (12) is written as $y(\tau) = y_h(\tau) + y_p(\tau)$, where $y_h(\tau)$ is the general solution to the equivalent homogenous equation and $y_p(\tau)$ is the specific solution of the non-homogenous equation. We can check $y_p(\tau) = \tau^2$ as a particular solution of homogenous Equation $y^{\frac{1}{2}}(\tau) + y(\tau) = 0$ and let us looking to $y_h(\tau) = e^{r\sqrt{\tau}}$: Then, the auxiliary equation is $\frac{r}{2}e^{r\sqrt{\tau}} + e^{r\sqrt{\tau}} = 0$. So, $\frac{r}{2} + 1 = 0$ which implies to $y_h(\tau) = e^{-2\sqrt{\tau}}$.

Example 5.2
Consider the following initial value problem

$$y^{\frac{1}{2}}(\tau) = \frac{\tau^{\frac{3}{2}} + \tau^{\frac{1}{2}}y(\tau)}{2\tau + 3y(\tau)}, \qquad \tau > 0, \tag{13}$$

with the initial condition *y(0) = 0*. Using the property 6 in Theorem 3.2, then the FrDE in Equation (13) converts into the following ordinary differential equation

$$y'(\tau) = \frac{\tau + y(\tau)}{2\tau + 3y(\tau)}, \tag{14}$$

It is simple to solve the first-order homogeneous differential equation in Equation (14).

Example 5.3
Consider the IVP

$$f^{(2\alpha)}(\zeta) + f(\zeta) = 0, \qquad \zeta > 0, \qquad 0 < \alpha < 0.5. \tag{15}$$

with the initial conditions $f(0) = 0, f^{(\alpha)}(0) = 1$.

Let $f(\zeta) = \sum_{n=0}^{\infty} a_n\, \zeta^{n\alpha}$, then, $f^{(\alpha)}(\zeta) = \sum_{n=0}^{\infty} n\alpha\, a_n\, \zeta^{n\alpha-\alpha}$,
and $f^{(2\alpha)}(\zeta) = \sum_{n=2}^{\infty} n\alpha^2(n-1)a_n\, \zeta^{n\alpha-2\alpha}$.
Now, put the formulas of $f(\zeta)$, $f^{(2\alpha)}(\zeta)$ in the FrODE in Equation (15) to obtain

$$\sum_{n=2}^{\infty} n\alpha^2(n-1)a_n\, \zeta^{n\alpha-2\alpha} + \sum_{n=0}^{\infty} a_n\, \zeta^{n\alpha} = 0,$$

Put n-2 ≡ n and after comparing two sides of this equation to get

$$a_{n+2} = \frac{-a_n}{\alpha^2(n+1)(n+2)}.$$

Hence, from ICs we get $a_{n+2} = 0$ if n is even number and $a_1 = 1$,
and $a_{2n+1} = \frac{(-1)^n}{\alpha^2(2n+1)!}, n = 1,2,3, \ldots$

5 . Discussion and Conclusion

In this article, the novel definitions of the α-fractional derivative, α-fractional integral, α-power series, and α –rule of integration by parts have been proposed. The properties of these definitions have been introduced. However, the implementations of these definitions in solving FrDEs proved their significance and efficiency of them to be most significant in solving FrDEs.

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