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Extended Idempotent Divisor Graph of Commutative Rings

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Abstract

Associate graph Л(R) is said to be idempotent divisor graph with vertices set V(Л(R)) = R*, if any two non- zero elements $a_1$and a_2 are adjacent if and only if $a_1.a_2$ = e, where e is an idempotent element not equal 1. In this work we study and introduce the extended idempotent divisor graph that is for any two non-zero elements $a_1$and a_2 adjacent if $a_1^{t_1}.\ a_2^{t_2}$=e , where $t_1, t_2 \in Z$ and e an idempotent element not equal one, and we give some results for properties such as diameter and the girth of this graph. Also, we investigated rings isomorphic to direct product two finite local rings.

Keywords: extended zero divisor graph, idempotent divisor graph, degree of vertices, size of a graph, reduced ring.

توسيع بيانات العناصر المتحايدة للحلقات الابدالية

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الخلاصة

البيان البسيط Л(R) و الذي يسمى بيان قاسم العناصر المتحايدة الذي مجوعة رؤوسه تقع في V(Л(R)) = R* وان كل رأسين غير صفريين مختلفين a_1 و a_2 يكونان متجاوران اذا و فقط اذا كان $a_1.a_2 = e$ حيث ان e عنصر متحايد لا يساوي 1.في هذا العمل ندرس توسيع بيان قاسم العناصر المتحايدة بحيث ان اي رأسين غير صفريين a_1 و a_2 يكونان متجاوران اذا وجد عددين صحيحين t_1, t_2 بحيث ان $a_1^{t_1}.\ a_2^{t_2}$=e وان e عنصر متحايد لا يساوي 1.و وجدنا بعض الخصائص مثل القطر و الدارة لهذا البيان .كذلك وجدنا البيان لبعض انواع من الحلقات مثل الحلقات المتشاكلة مع حاصل ضرب حلقتين محليتين .

1. Introduction

In this paper, we assume that R a finite commutative ring with identity $1 \neq 0$, we used Id(R) (U(R), respectively) the set of all idempotent (identity respectively) elements in a ring R and Z(R) the set of all zero divisor. In 1988 Beck give relationship between two branches in mathematics ring and graph theory, when studied the coloring of commutative ring [1]. In 1999 Anderson modify this definition as zero divisor graph denoted by Γ(R) and the vertices of this graph equal $Z(R) - \{0\} = Z(R)^*$ and the elements $a_1.a_2$ in V(Γ(R)) , a_1, a_2 adjacent whenever $a_1.a_2$=0 in R [2]. Later many authors gave different definition see for example [3]–

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[10]. In 2016 Bennis, Mikram and Taraza introduced the extended zero-divisor graph of R, denoted by $\overline{\Gamma(R)}$ and two distinct vertices a_1 and a_2 are adjacent if and only if, there are positive integers t_1 and t_2 such that $a_1^{t_1}.a_2^{t_2} = 0$ and $a_1^{t_1}, a_2^{t_2} \neq 0$ [10]. Recently, Mohammad and Shuker introduced an idempotent divisor graph with vertices set in $R^* = R - \{0\}$, and two non- zero distinct vertices a_1 and a_2 are adjacent if and only if $a_1.a_2 = e$, for some non-unit idempotent element e in a commutative ring R with identity $1 \neq 0$, this graph denoted by Л(R)[11]. Clearly $\Gamma(R) \subset$ Л(R) when R non-local ring and $\Gamma(R) =$ Л(R) when R local ring. In this paper we give extended idempotent divisor graph $\overline{Л(R)}$ with $V(\overline{Л(R)}) = R^*$ and two elements $a_1, a_2 \in V(\overline{Л(R)})$ adjacent if and only if there are two positive integers t_1 and t_2 such that $a_1^{t_1}.\ a_2^{t_2} = e$, where e is an idempotent element and $e = e^2 \neq 1$ with $a_1^{t_1}, a_2^{t_2} \neq 0$. This paper contained three sections. In section two we gave the definition and some properties and examples of extended idempotent divisor graph for reduce ring, in section three we gave the diameter and the girth for some rings as well as the order and the size for these rings and the degree of their vertices.

In a graph theory the diameter of a graph G is the greatest distance between any two vertices of a graph G denoted by $diam(G)$, and the girth is the length of a shortest cycle contained in a graph G denoted as $gr(G)$. Also, the simple graph G is a graph without loops neither multiple-edge connected graph G is the graph that has a path between every pair of vertices. Moreover, m is the size of a graph G and which is the number of the edges, n denote as the order of a graph G and it represents the number of vertices. The center of a graph is the set of all vertices of minimum eccentricity, where $\omega(G)$ the clique number G is a greats complete sub-graph of a graph G. The chromatic number of a graph G denoted by $\chi(G)$ and defined as a minimal number of colours needed to colour the vertices in such a way that no two adjacent vertices have the same colour. Furthermore, the complete graph K is a graph in which every two distinct vertices are adjacent. For more details see for example [12], [13].

In a ring theory ring R is a local if it has only one maximal ideal and F_q is a fild of order q. A reduced ring is a ring has no non-zero nilpotent elements. The idempotent element it is an element such that: $a^2 = a$. Boolean ring is a ring in which every element is idempotent. An element x in a ring R is called nilpotent if there exists a smallest positive integer t such that $x^t = 0$. $v(x)$ denotes the order of nilpotency of x, the degree of nilpotency of a ring R defined to be the supremum of the orders of nilpotency of its nilpotent elements and it is denoted by $v(R)$. It well known that any finite non-local ring can be written as a direct product of a local ring, while every reduced ring can be written as a direct product of a field. For more details see [14].

2. Definition and properties of extended idempotent divisor graph of reduced ring

In this part we introduced extended idempotent divisor graph and give example, as well as we investigate when R a reduced ring.

Definition 2.1:

The extended idempotent divisor graph of a ring R is the simple graph denoted by $\overline{Л(R)}$ with $V(\overline{Л(R)}) = R^*$such that two distinct vertices a_1, a_2 are adjacent if there exists two positive integers t_1 and t_2 such that $a_1^{t_1}.\ a_2^{t_2} = e$, where e is an idempotent element such that $e \neq 1$ and $a_1^{t_1}.\ a_2^{t_2} \neq 0$. Where $V(\overline{Л(R)}) = R^*$ is the set of all non-zero vertices of $\overline{Л(R)}$.

Example 1: Let $R = Z_{12}$, then
$I(Z_{12}) = \{0,1,4,9\}$ then the idempotent divisor graph and the extended idempotent divisor graph show in Fig. 2.1

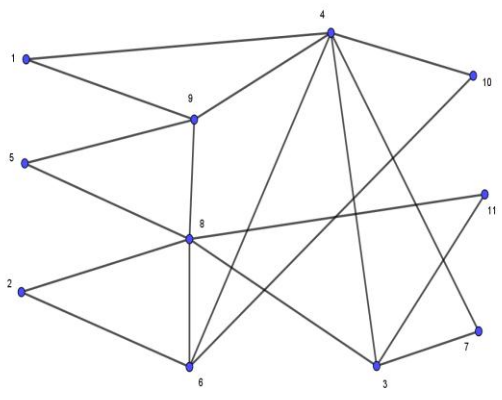


Figure: 2.1 $Л(Z_{12})$

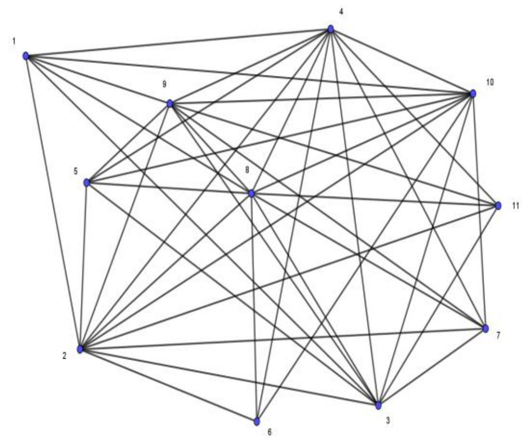


Figure: 2.2 $\overline{Л(Z_{12})}$

Remarks 2.2:
If R finite commutative ring with unit, then
1- $R = U(R) \cup Z(R)$,
2- Every element $a \in R$ is either nilpotent or there exists appositive integer t such that a^t is an idempotent.
3- If $u \in U(R)$, then $u^l = 1$, for some $l \in \mathbb{Z}^+$.

Theorem 2.3:
For every reduced ring R, $\text{diam}(\overline{Л(R)}) \leq 2$.
Proof:
Since R a finite reduced ring then R non-local and $V(\overline{Л(R)}) = U(R) \cup Z(R)^*$. Now, for any $x, y \in \overline{Л(R)}$,since R finite ring, then there are three cases:

Case1: If $x, y \in U(R)$ then, there are $l_1, l_2 \in \mathbb{Z}^+$ such that $x^{l_1} = y^{l_2} = 1$. So, for any non-identity idempotent element $e \in R$, we have $x^{l_1} . e = y^{l_2} . e = e$. Therefore, $\text{diam}(x, y) = 2$ in this case.

Case2: If $x, y \in Z(R)^*$, since R reduced ring then R does not contains a nilpotent element, so that $x^{l_1} = e_1$ and $y^{l_2} = e_2$ ((where e_1, e_2 non-unit idempotent elements and $l_1, l_2 \in \mathbb{Z}^+$). Which implies that $x^{l_1} . y^{l_2} = e_1 . e_2$. Hence $\text{diam}(\overline{Л(R)}) = 1$ in this case.
Case 3: If $y \in Z(R)^*$, $x \in U(R)$ and. It's obviously by Remarks 2.2 since $x \in U(R)$ then $x^{l_1} = 1$. Also since $y \in Z(R)^*$ and R reduced then $y^{l_2} = e \notin \{0,1\}$, then we have $x^{l_1} . y^{l_2} = 1. e = e$ and we get $\text{diam}\ (\overline{Л(R)}) = 1$ in this case. So, that for all cases $\text{diam}(x, y) \leq 2$. ■

Theorem 2.4:
If R be a reduced ring, then $\omega(\overline{Л(R)}) = \chi(\overline{Л(R)}) = |Z(R)^*| + 1$.

Proof:
Since, R finite reduced ring, then $R \cong F_1 \times F_2 \times \dots \dots \times F_n$, where F_i are fields for all $i = 1,2, \dots, n$.
First, we have to prove every two elements $z_1, z_2 \in Z(R)^*$ are adjacent in $\overline{Л(R)}$. Since $z_1, z_2 \in Z(R)$, then $z_1$and z_2 have a unique representation as $z_1 = (a_1, a_2, \dots \dots a_n)$ and $z_2 = (b_1, b_2, \dots \dots b_n)$ where $a_i, b_i \in F_i$ and there is at least one element $a_i \in F_i$(respctively, $b_j \in F_j$) such that $a_i = 0_i$($b_j = 0_j$resp.), $i, j \in \{1,2, .., n\}$, Which implies that there are $t_1, t_2 \in \mathbb{Z}$ such that $a_i^{t_1} = \begin{cases} 0_i & \text{if } a_i = 0 \\ 1_i & \text{if } a_i \neq 0 \end{cases}$, $b_j^{t_2} = \begin{cases} 0_j & \text{if } b_j = 0 \\ 1_j & \text{if } b_j \neq 0 \end{cases}$

Hence z_1, z_2 are idempotent elements not equal to 1 and $z_1 . z_2$ is an idempotent element not equal 1. Consequently, $K_{|Z(R)^*|}$ is a complete sub-graph induced by $Z(R)^*$ as well as $1 \in R$ adjacent with every non-zero divisor in $\overline{Л(R)}$. Hence $\overline{Л(R)}$ has a complete sub-graph $K_{|Z(R)^*|+1}$. Finally, if $u \in U(R) - \{1\}$, then u non-adjacent with 1. Therefore $K_{|Z(R)^*|+1}$ is a largest complete sub graph and $\omega(\overline{Л(R)}) = \chi(\overline{Л(R)}) = |Z(R)^*| + 1$. ■

Proposition2.5:
For a reduced ring R, $\deg(v)_{v \in \overline{Л(R)}} = \begin{Bmatrix} |R^*| - 1 & \text{if } v \in Z(R)^* \\ |Z(R)^*| & \text{if } v \in U(R) \end{Bmatrix}$. Moreover $\overline{Л(R)} = |U(R)|K_1 + K_{|Z(R)^*|}$.

Proof:
If $v \in Z(R)^*$, then by the same way as the proof of the Theorem 2.4. We get adjacent with every element $\overline{Л(R)}$. So that $\deg_{v \in Z(R)^*}(v) = |R^*| - 1$.
If $v \in U(R)$, then v adjacent with every element in $Z(R)^*$, and non-adjacent with any element in U(R). Therefore, $\deg_{v \in U(R)}(v) = |Z(R)^*|$. ■

Corollary 2.6:
Let R be a reduced ring, then the center of R: Cent(R)=Z(R^*).

3. More properties in $\overline{Л(R)}$.

It well-known $Л(R)$ is a connected simple graph and the diameter less than or equal three as well as a girth equal three or ∞, Thus $\overline{Л(R)}$ is also. More than $Л(R) \subseteq \overline{Л(R)}$, so we start this part give a necessary and sufficient conditions to be $Л(R) = \overline{Л(R)}$.

Definition 3.1: A ring R is called Boolean ring if every element in R is an idempotent element[14].

Theorem 3.2:
Let R be a ring, then $Л(R) = \overline{Л(R)}$ if and only if R Boolean ring or local ring with $Z(R)^2 = 0$.

Proof:
Clearly if R Boolean ring or local ring with $Z(R)^2 = 0$, then $Л(R)$ is complete graph so that $Л(R) = \overline{Л(R)}$.

Conversely, let $Л(R) = \overline{Л(R)}$, if R local, then $Л(R) = \Gamma(R)$, so by [10] we get $Z(R)^2 = 0$.

If R non-local. Since R finite ring then $R \cong R_1 \times R_2 \times \dots \dots \times R_n$, where $n \geq 2$ and R_i local ring for each $i \in \{1,2,..n\}$. We claim that $R_i \cong Z_2$.
If not, then there exists $a_i \in R_i - \{0_i, 1_i\}$ for some $i = 1,2, \dots, n$. without loss generality let $i = 1$. We note that an element $(a_1, 1_2, 1_3, \dots \dots.., 1_n)$ is non-adjacent with an element $(1_1, 0_2, 1_3, \dots \dots.., 1_n)$ in $Л(R)$. On the other hands, since R_1 a finite local ring, then there exists $t \in \mathbb{Z}$ such that:

$$a_1^t = \begin{cases} 0_1 & \text{if } a_1 \in Z(R_1) \\ 1_1 & \text{if } a_1 \in U(R_1) \end{cases}.$$

Then we have

$$(a_1, 1_2, 1_3, .., 1_n)^t . (1_1, 0_2, 1_3, \dots, 1_n) = \begin{cases} (0_1, 0_2, 1_3, \dots, 1_n), \text{if } a_1 \in Z(R_1) \\ (1_1, 0_2, 1_3, \dots, 1_n), \text{if } a_1 \in U(R_1) \end{cases}$$

is a non-unit idempotent element in R. Therefore, $Л(R) = \overline{Л(R)}$ which is a contradiction. So $R \cong Z_2 \times Z_2 \times \dots \dots \times Z_2$. Then we have R Boolean ring or local ring with $Z(R)^2 = 0$. ■

Now, we classify rings by diameter of a graph $\overline{Л(R)}$.

Theorem 3.3:
Let R be a finite commutative ring, then:
1- diam $(\overline{Л(R)}) = 0$ if and only if $R \cong Z_4$ or $Z_2[x]/(x^2)$.
2- diam $(\overline{Л(R)}) = 1$ if and only if R local with $x^{v(x)-1}.y^{v(y)-1} = 0$, for all $x, y \in Z(R)^*$ or R Boolean ring.
3- diam $(\overline{Л(R)}) = 3$ if and only if $R \cong R_1 \times R_2 \times \dots \dots .. R_n$, where R_i local rings which are not fields.
4- otherwise diam $(\overline{Л(R)}) = 2$.

Proof:
1- It's clear.
2- If R local with $x^{v(x)-1}.y^{v(y)-1} = 0$, then by [11] we have $Л(R) = \Gamma(R)$, whence by leads [11] to diam $(\overline{Л(R)}) = 1$. Also, If R non- local Boolean ring, then by [11], R is a complete graph, consequently diam$(Л(R))$ = diam $(\overline{Л(R)}) = 1$.

Conversely, let R be a ring with diam $(\overline{Л(R)}) = 1$. If R local whence by [10], $x^{v(x)-1}.y^{v(y)-1} = 0$ for all $x, y \in Z(R)^*$. If *R* non-local, since diam $(\overline{Л(R)}) = 1$ then for any two elements $u_1, u_2 \in U(R)$ are adjacent in $(\overline{Л(R)})$ this means, there exist positive integers l_1 and l_2 and non-unit idempotent element e such that $(u_1)^{l_1}.(u_2)^{l_2} = e$. But $u_1, u_2 \in U(R)$, so that $e = 1$. Which is a contradiction, hence R has only one unit element, there for $R = Z(R) \cup U(R) = Z(R) \cup \{1\}$ and we get $R \cong z_2 \times z_2 \times \dots \dots .. z_2$ (n-times). Consequently, R is a Boolean ring.
3- Let R_i local ring not field, for all $i \in \{1,2, \dots n\}$, it is enough to prove there are two vertices x and y in $\overline{Л(R)}$ such that $d(x, y) = 3$. Since R_i local not field, then by [15] there exists $z_i \in Z(R_i)$ for all i such that, $z_i^2 = 0$ and $z_i.x_i = 0$, where $x_i \in Z(R_i)$. Let $z = (z_1, z_2, \dots \dots, z_n) \in R$, then $z^2 = 0$. We note that z adjacent with an element $x = (x_1, x_2, \dots \dots x_n) \in R$ if and only if $x_i \in Z(R_i)$ [because if $x_i \in U(R_i)$, then $z_i.x_i \notin \{0_i, 1_i\}$]. Also $1 = (1_1, 1_2, \dots \dots 1_n)$ adjacent with *y* where $y = (y_1, y_2, \dots \dots y_n)$ if and only if $y_i \in U(R_i) \cup \{0_i\}$ and $y \neq 0$. So that $1 - y - x - z$, hence $d(1, z) = 3$. Therefore, diam $\left(\overline{Л(R)}\right) = 3$.

Conversely, let $\mathrm{diam}(\overline{Л(R)}) = 3$, since R finite ring, then $R \cong R_1 \times R_2 \times \ldots \times R_n$,where R_i are local ring for all $i \in \{1, 2, .., n\}$. If R_i is a field for some i , say R_1 is a field then there are three cases:

Case1: If $x, y \in Z(R)^*$ and $xy \neq 0$, then there exist three sub cases:
Sub-case a: If there exists positive integers $t_1$and t_2 such that $x^{t_1+1}.y^{t_2+1} = 0$. Since $x^{t_1}.xy = 0$, then x adjacent to xy also since $xy.y^{t_2} = 0$, then y adjacent to xy. So $x - xy - y$ and $d(x, y) = 2$.
Sub-case b: If there is $t \in \mathbb{Z}$ such that $x^t = 0$ and for each $l \in \mathbb{Z}$, , $y^l \neq 0$ then y is an idempotent element in R. As long as $x, y \in Z(R)^*$, then there are $a, b \in Z(R)^* - \{x, y\}$ such that $ax = by = 0$. If $bx = 0$, then $x - b - y$ and $d(x, y) \leq 2$. if $bx \neq 0$, since $x^{n-1}.(bx) = 0$ then $x - xb$. Which implies that $x - xb - y$ and $d(x, y) \leq 2$.
Sub-case c: If $x^{l_1} = e_1$ and $y^{l_2} = e_2$, where $l_1, l_2 \in \mathbb{Z}$ and $e_1, e_2 \in Id(R) - \{0,1\}$, then x adjacent to y and $d(x, y) = 1$.
Case 2: if $x, y \in U(R)$, then by Remark 2.2 we have $x^{l_1} = y^{l_2} = 1$, for some $l_1, l_2 \in \mathbb{Z}$. So $x - e - y$ and $d(x, y) \leq 2$
Case 3: if $x \in U(R)$, and $y \in Z(R)^*$. Then $x = (x_1, x_2, \ldots x_n)$, where $x_i \in U(R_i)$ and $y = (y_1, y_2, \ldots y_n)$, $y_i \in R_i$ and, there exists $y_j \in Z(R_j)$. So $x^l = 1$, $y^t = e$ for some $l, t \in \mathbb{Z}$ and $e \in Id(R) - \{1\}$. If $e \neq 0$, then $x - y$ and $d(x, y) = 1$. On other side if $e = 0$, since R_1 is a field, then $y_1 = 0$ and hence $y = (0, y_2, y_3, \ldots y_n)$. Therefore $x^l.(1_1, 0_2, \ldots, 0_n) = (1_1, 1_2, \ldots, 1_n).(1_1, 0_2, \ldots, 0_n) = (1_1, 0_2, \ldots, 0_n)$ is an idempotent in R and $(1_1, 0_2, \ldots, 0_n).y = (1_1, 0_2, \ldots, 0_n).(0, y_2, y_3, \ldots y_n) = (0_1, 0_2, \ldots, 0_n)$. Whence $d(x, y) = 2$. So that for all cases $d(x, y) \leq 2$ which is contradicts the assumption then every R_i not fields.

Corollary 3.4:
Let $R \cong F \times R_1 \times R_2 \times \ldots \times R_n$ where F is afield, and every R_i local ring, then the center of R is: $cent(R) = \{(1,0,0, \ldots .0)\}$.

Proposition 3.5:
For any finite ring R, $gr(\overline{Л(R)}) = 3$ except for the cases $R \cong Z_4, Z_2[x]/(x^2), Z_9$ or $Z_3[x]/(x^2)$, then $gr(\overline{Л(R)}) = \infty$.
Proof:
Clearly if $R \cong Z_8$ or $Z_2[x]/(x^3)$, then $gr(\overline{Л(R)}) = 3$. Also, If $R \cong Z_4, Z_2[x]/(x^2)$, Z_9 or $Z_3[x]/(x^2)$, then By Theorem 3.1 we have $Л(R) = \overline{Л(R)}$. Which implies that $gr(\pi(R)) = gr(\overline{Л(R)}) = \infty$. Otherwise, by [11] we have $gr(\overline{Л(R)}) = 3$. ■
The next result we gave an extended for any ring $R \cong F_q \times R'$, where F_q is field of order q and R' a local ring.

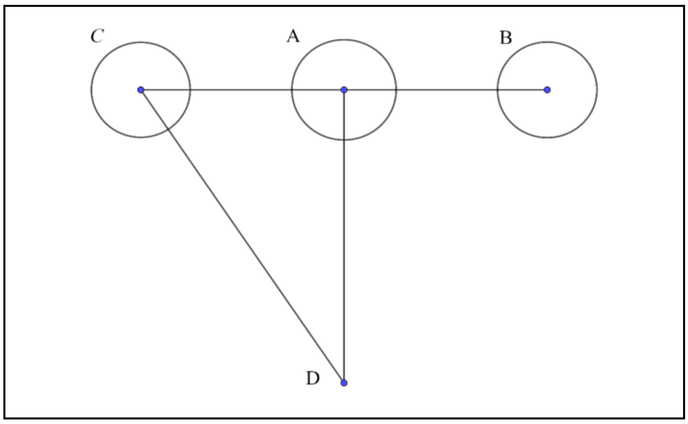
Theorem 3.6:
Let $R \cong F_q \times R'$, where F_q is a field of order q, and R' local ring with $a^{v(a)-1}.b^{v(b)-1} = 0$, for any $a, b \in Z(R')^*$, then

$$\deg(v)_{v \in \overline{Л(R)}} = \begin{cases} v = (r, s) := |R^*| - 1 \\ v = (0, s) := |F^*||Z(R')| + |Z(R')^*| - 1 \\ v = (0, u) := |F^*||Z(R')| + |U(R')| + |F^*||U(R')| - 1 \\ v = (r, u) := |F^*||Z(R')| + |U(R')| \end{cases}$$

where $r \in F^*$, $s \in Z(R')^*$, $u \in U(R')$, and the *order* of $\overline{Л(R)}$:
$n = q \times |R'| - 1$, and the *size* of $\overline{Л(R)}$:

$$m = \frac{1}{2}\Big[(|\mathrm{F}| - 1)(\,|\mathrm{Z(R')}|(|\mathrm{R}| - 2) + |\mathrm{Z(R')}||\mathrm{F}||\mathrm{U(R')}|) + (|\mathrm{Z(R')}| - 1)(|\mathrm{Z(R')}||\mathrm{F}| - 2) + \left|(\mathrm{U(R')})^2\right|(2|\mathrm{F}| - 1) - |\mathrm{U(R')}|\Big].$$

Proof:
The *order* of $\overline{Л(\mathrm{R})}$: $n = R^* = q \times |R'| - 1$.
Now, let $v = (r, z) \in R^*$, be any element in R^*; where $\mathrm{r} \in \mathrm{F_q}$, $z \in \mathrm{R'}$, then we can distinguish the vertices into disjoint subsets:
$A = \{(r, s): r \in F^*, s \in Z(R')\}$, $B = \{(0, s): s \in Z(R')^*\}$,
$C = \{(0, u): u \in U(R')\}$ and $D = \{(r, u): r \in F_q^*, u \in U(R')\}$. Clearly
$|\mathrm{A}| = |\mathrm{F^*}||\mathrm{Z(R')}|$, $|\mathrm{B}| = |\mathrm{Z(R')^*}|$, $|\mathrm{C}| = |\mathrm{U(R')}|$, $|\mathrm{D}| = |\mathrm{F^*}||\mathrm{U(R')}|$
Firstly: if $v = (r, s) \in A$, since by Remark 2.2, there exists a positive integer t such that :$(r, s)^t = (1,0)$, then (r, s) adjacent with every other vertices in the graph $\overline{Л(\mathrm{R})}$, so $\deg(\mathrm{v})_{\mathrm{v}\in\mathrm{A}} = |\mathrm{R^*}| - 1$.
Secondly: if $v = (0, s) \in B$, since for any $s_1, s_2 \in Z(R')^*$, $s_1^{v(s_1)-1}.s_2^{v(s_2)-1} = 0$,then v adjacent with every element in A and B , so $\deg(\mathrm{v})_{\mathrm{v}\in\mathrm{B}} = |\mathrm{A}| + |\mathrm{B}| - 1 = |\mathrm{F^*}||\mathrm{Z(R')}| + |\mathrm{Z(R')^*}| - 1$.
Thirdly: if $v = (0, u) \in C$, again by Remark 2.2, there exists a positive integer l such that :$(0, u)^l = (0,1)$, then we have v adjacent with every element in A, C and D, so $\deg(\mathrm{v})_{\mathrm{v}\in\mathrm{C}} = |\mathrm{A}| + |\mathrm{C}| + |\mathrm{D}| - 1 = |\mathrm{F^*}||\mathrm{Z(R')}| + |\mathrm{U(R')}| + |\mathrm{F^*}||\mathrm{U(R')}| - 1$.
Finally: If $v = (\mathrm{r}, \mathrm{u}) \in \mathrm{D}$, since v not adjacent for any element in D, so that: $\deg(\mathrm{v})_{\mathrm{v}\in\mathrm{D}} = |\mathrm{A}| + |\mathrm{C}| = |\mathrm{F^*}||\mathrm{Z(R')}| + |\mathrm{U(R')}|$
Now, to find the *size* of the graph $\overline{Л(\mathrm{R})}$:since
$m = \frac{1}{2}[\sum_{\mathrm{v}\in\mathrm{A}} \deg(\mathrm{v}) + \sum_{\mathrm{v}\in\mathrm{B}} \deg(\mathrm{v}) + \sum_{\mathrm{v}\in\mathrm{C}} \deg(\mathrm{v}) + \sum_{\mathrm{v}\in\mathrm{D}} \deg(\mathrm{v})]$
$= \frac{1}{2}[(|\mathrm{R^*}| - 1)\,|\mathrm{F^*}||\mathrm{Z(R')}| + (|\mathrm{F^*}||\mathrm{Z(R')}| + |\mathrm{Z(R')^*}| - 1)|\mathrm{Z(R')^*}| + (|\mathrm{F^*}||\mathrm{Z(R')}| + |\mathrm{U(R')}| + |\mathrm{F^*}||\mathrm{U(R')}| - 1)\,|\mathrm{U(R')}| + (|\mathrm{F^*}||\mathrm{Z(R')}| + |\mathrm{U(R')}|)\,|\mathrm{F^*}||\mathrm{U(R')}|]$
Now put: $|\mathrm{R^*}| = |\mathrm{R}| - 1$, $|\mathrm{F^*}| = |\mathrm{F}| - 1$, $|\mathrm{Z(R')^*}| = |\mathrm{Z(R')}| - 1$, and simplify the size of $\overline{Л(\mathrm{R})}$, we get
$m = \frac{1}{2}[(|\mathrm{F}| - 1)(\,|\mathrm{Z(R')}|(|\mathrm{R}| - 2) + |\mathrm{Z(R')}||\mathrm{F}||\mathrm{U(R')}|) + (|\mathrm{Z(R')}| - 1)(|\mathrm{Z(R')}||\mathrm{F}| - 2) + \left|(\mathrm{U(R')})^2\right|(2|\mathrm{F}| - 1) - |\mathrm{U(R')}|]$.



Corollary 3.7:
If $\mathrm{R} \cong \mathrm{F_q} \times \mathrm{R'}$, where $\mathrm{F_q}$ is a field of order q , and $\mathrm{R'}$ local ring with $\mathrm{a}^{\mathrm{v(a)}-1}.\mathrm{b}^{\mathrm{v(b)}-1} = 0$, for any $a, b \in Z(R')$, Then $\mathrm{Cent(R)} = \{(r, s): r \in F^*, s \in Z(R')\}$.

Theorem 3.8:

Let $R \cong R_1' \times R_2'$,where R_1', R_2' two rings such that $a_i^{v(a_i)-1}.b_i^{v(b_i)-1} = 0$, for any $a_i, b_i \in Z(R_i')$, $i \in \{1,2\}$ Then:

$$\deg(v)_{v\in\overline{\text{Л}(R)}} = \begin{cases} v = (s_1,0) := |Z(R_1')^*||R_2'| + |(R_2')^*| - 1 \\ v = (0,s_2) := |Z(R_2')^*||R_1'| + |(R_1')^*| - 1 \\ v = (u_1,0) := |Z(R_2')^*|(1 + |U(R_1')|) + |U(R_2')||R_1'| + |U(R_1')| - 1 \\ v = (0,u_2) := |Z(R_1')^*|(1 + |U(R_2')|) + |U(R_1')||R_2'| + |U(R_2')| - 1 \\ v = (u_1,s_2) := |Z(R_2')^*|(1 + |U(R_1')|) + |U(R_2')||R_1'| + |U(R_1')| - 1 \\ v = (s_1,u_2) := |Z(R_1')^*|(1 + |U(R_2')|) + |U(R_1')||R_2'| + |U(R_2')| - 1 \\ v = (s_1,s_2) := |Z(R_1')^*| + |Z(R_2')^*| + |Z(R_1')^*||Z(R_2')^*| - 1 \\ v = (u_1,u_2) := |U(R_1')| + |U(R_2')| + |U(R_1')||Z(R_2')^*| + |U(R_2')||Z(R_1')^*| \end{cases}$$

where $s_i \in Z(R_i')^*$, $u_i \in U(R_i')$ and $i \in \{1,2\}$.

Moreover, the order of $\overline{\text{Л}(R)}$ is $n = |R_1| \times |R_2| - 1$, and

The *size* of $(\overline{\text{Л}(R)})$; $m = \frac{1}{2}[(|Z(R_1')|-1)[\ |R_1'||(U(R_2'))^2| + (|Z(R_1')| + |U(R_1')||U(R_2')|)\ |R_2'| + |U(R_2')|\ (|Z(R_1')|-1)-2] + (|Z(R_2')|-1)[\ |R_2'|\ |(U(R_1'))^2| + (|Z(R_2')| + |U(R_1')||U(R_2')|)\ |R_1'| + |U(R_1')|\ (|Z(R_2')|-1)-2] + |U(R_1')|\ |U(R_2')|\ (|R_1'| + |R_2'| + |U(R_1')| + |U(R_2')|) + |U(R_1')|\ (|U(R_1')|-1) + |U(R_2')|\ (|U(R_2')|-1) + (|Z(R_1')| + |Z(R_2')|)\ (2-|Z(R_1')|\ |Z(R_2')|)-1$

Proof:

The order of $\overline{\text{Л}(R)}$;$n = R^* = |R_1| \times |R_2| - 1$, to find degree of any element $v = (r_1, r_2) \in R^*$,where $r_1 \in R_1'$,$r_2 \in R_2'$. We divide the vertices into disjoint subsets:

$A = \{(s_1,0): s_1 \in Z(R_1')^*\}$, $B = \{(0,s_2): s_2 \in Z(R_2')^*\}$, $C = \{(u_1,0): u_1 \in U(R_1')\}$,
$D = \{(0,u_2): u_2 \in U(R_2')\}$,
$E = \{(u_1,s_2): u_1 \in U(R_1'), s_2 \in Z(R_2')^*\}$,
$F = \{(s_1,u_2): s_1 \in Z(R_1')^*, u_2 \in U(R_2')\}$, $G = \{(s_1,s_2): s_1 \in Z(R_1')^*, s_2 \in Z(R_2')^*\}$,$H = \{(u_1,u_2): u_1 \in U(R_1'), u_2 \in U(R_2')\}$, Then by Remark 2.2
$u_1^{l_1}, u_2^{l_2} = 1$, $s_1^{v(s_1)}, s_2^{v(s_2)} = 0$, where l_1, l_2 are positive integers
It is obvious that, $|A| = |Z(R_1')^*|$, $|B| = |Z(R_2')^*|$, $|C| = |U(R_1')|$,
$|D| = |U(R_2')|$,$|E| = |U(R_1')||Z(R_2')^*|$,$|F| = |Z(R_1')^*||U(R_2')|$,
$|G| = |Z(R_1')^*||Z(R_2')^*|$,$|H| = |U(R_1')||U(R_2')|$. So in the same way of Theorem 3.5 we have:
every element in A adjacent with every element in B, D, G, F and A
$\deg(v)_{v\in A} = |Z(R_1')^*||R_2'| + |(R_2')^*| - 1$
every element in B adjacent with every element in A, C, G, E and B
$\deg(v)_{v\in B} = |Z(R_2')^*||R_1'| + |(R_1')^*| - 1$
every element in C adjacent with every element in B, D, E, F, H and C
$\deg(v)_{v\in C} = |Z(R_2')^*|(1 + |U(R_1')|) + |U(R_2')||R_1'| + |U(R_1')| - 1$
every element in D adjacent with every element in A, C, E, F, H and D
$\deg(v)_{v\in D} = |Z(R_1')^*|(1 + |U(R_2')|) + |U(R_1')||R_2'| + |U(R_2')| - 1$
every element in E adjacent with every element in B, C, D, F, H and E
$\deg(v)_{v\in E} = |Z(R_2')^*|(1 + |U(R_1')|) + |U(R_2')||R_1'| + |U(R_1')| - 1$
every element in F adjacent with every element in A, C, D, E, H and F
$\deg(v)_{v\in F} = |Z(R_1')^*|(1 + |U(R_2')|) + |U(R_1')||R_2'| + |U(R_2')| - 1$
every element in G adjacent with every element in A, B and G
$\deg(v)_{v\in G} = |Z(R_1')^*| + |Z(R_2')^*| + |Z(R_1')^*||Z(R_2')^*| - 1$
every element in H adjacent with every element in C, D, E and F
$\deg(v)_{v\in H} = |U(R_1')| + |U(R_2')| + |U(R_1')||Z(R_2')^*| + |U(R_2')||Z(R_1')^*|$
To find the *size* of $\overline{\text{Л}(R)}$

$m = size(\overline{Л(R)}) = \frac{1}{2}\sum_{v\in\overline{Л(R)}} \deg(v)$, then

$m = \frac{1}{2}[\sum_{v\in A}\deg(v) + \sum_{v\in B}\deg(v) + \sum_{v\in C}\deg(v) + \sum_{v\in D}\deg(v) + \sum_{v\in E}\deg(v) + \sum_{v\in F}\deg(v) + \sum_{v\in G}\deg(v) + \sum_{v\in H}\deg(v)]$.

$$\begin{aligned} m = \frac{1}{2}[&(|Z(R_1')^*||R_2'| + |(R_2')^*| - 1)|Z(R_1')^*| + (|Z(R_2')^*||R_1'| + |(R_1')^*| - 1)|Z(R_2')^*| \\ &+ (|Z(R_2')^*|(1 + |U(R_1')|) + |U(R_2')||R_1'| + |U(R_1')| - 1)|U(R_1')| \\ &+ (|Z(R_1')^*|(1 + |U(R_2')|) + |U(R_1')||R_2'| + |U(R_2')| - 1)|U(R_2')| \\ &+ (|Z(R_2')^*|(1 + |U(R_1')|) + |U(R_2')||R_1'| + |U(R_1')| - 1)|U(R_1')||Z(R_2')^*| \\ &+ (|Z(R_1')^*|(1 + |U(R_2')|) + |U(R_1')||R_2'| + |U(R_2')| - 1)|U(R_2')||Z(R_1')^*| \\ &+ (|Z(R_1')^*| + |Z(R_2')^*| + |Z(R_1')^*||Z(R_2')^*| - 1)|Z(R_1')^*||Z(R_2')^*| \\ &+ (|U(R_1')| + |U(R_2')| + |U(R_1')||Z(R_2')^*| \\ &+ |U(R_2')||Z(R_1')^*|)|U(R_1')||U(R_2')|. \end{aligned}$$

If we put $|Z(R_1')^*| = |Z(R_1')| - 1$, $|Z(R_2')^*| = |Z(R_2')| - 1$, $|(R_1')^*| = |R_1'| - 1$ and $|(R_2')^*| = |R_2'| - 1$, and simplify the size we get:

$m = \frac{1}{2}[(|Z(R_1')|$-1)[$|R_1'||(U(R_2'))^2|$ +($|Z(R_1')|$+$|U(R_1')||U(R_2')|$) $|R_2'|$ +$|U(R_2')|$ ($|Z(R_1')|$-1)-2]+ ($|Z(R_2')|$-1)[$|R_2'|$ $|(U(R_1'))^2|$+($|Z(R_2')|$ +$|U(R_1')||U(R_2')|$) $|R_1'|$ +$|U(R_1')|$ ($|Z(R_2')|$-1)-2]+ $|U(R_1')|$ $|U(R_2')|$ ($|R_1'|$+$|R_2'|$ +$|U(R_1')|$ +$|U(R_2')|$)+ $|U(R_1')|$ ($|U(R_1')|$-1)+ $|U(R_2')|$ ($|U(R_2')|$-1)+ ($|Z(R_1')|$+$|Z(R_2')|$) (2-$|Z(R_1')|$ $|Z(R_2')|$)-1. ■

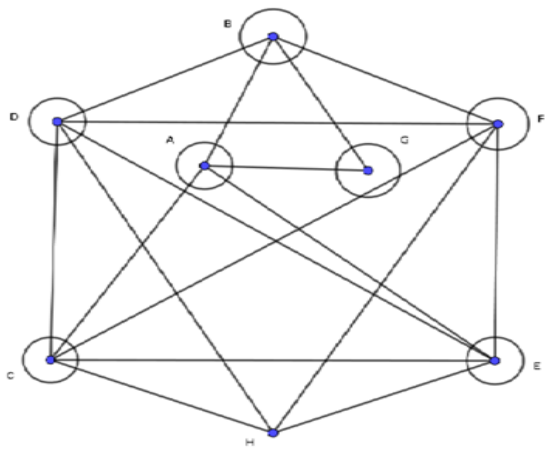


Figure 3.2 $\overline{Л(R_1' \times R'_2)}$

Corollary 3.9:

Let $R \cong R_1' \times R_2'$,where R_1', R_2' two rings, such that $a_i^{v(a_i)-1}.b_i^{v(b_i)-1} = 0$, for any $a_i, b_i \in Z(R_i')$, $i \in \{1,2\}$. Then $cent(R) = A \cup B \cup C \cup D \cup E \cup F$, where A, B, C, D, E and F defined in Theorem 3.9.

4. Conclusions

In Theorem 2.3 and 2.4, for every reduced ring R, we have: (i) $diam((\overline{Л(R)})) \le 2$. (ii) $\omega((\overline{Л(R)})) = \chi((\overline{Л(R)})) = |Z(R)^*| + 1$. iii- $Cent((\overline{Л(R)})) = Z(R)^*$. Moreover, Proposition 2.5, we show $\overline{Л(R)} = |U(R)|K_1 + K_{|Z(R)^*|}$.

In Theorem 3.2, we show an extended idempotent divisor graph equal idempotent divisor graph if and only if R Boolean ring or local ring and every element has index 2.

In Theorem 3.3, we classify an extended idempotent divisor graph by using a diameter of it. Finally, we give some properties when R a direct product two local rings see Theorem 3.8.

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