DOI: 10.24996/ijs.2023.64.8.33

ISSN: 0067-2904

On the Stability of Four Dimensional Lotka-Volterra Prey-Predator System

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Received: 9/10/2022 Accepted: 15/2/2023 Published: 30/8/2023

Abstract

The aim of this work is to study a modified version of the four-dimensional Lotka-Volterra model. In this model, all of the four species grow logistically. This model has at most sixteen possible equilibrium points. Five of them always exist without any restriction on the parameters of the model, while the existence of the other points is subject to the fulfillment of some necessary and sufficient conditions. Eight of the points of equilibrium are unstable and the rest are locally asymptotically stable under certain conditions, In addition, a basin of attraction found for each point that can be asymptotically locally stable. Conditions are provided to ensure that all solutions are bounded. Finally, numerical simulations are given to verify and support the obtained theoretical results.

Keywords: Basin of attraction, Equilibrium points, Lyapunov function, Local stability, Prey-Predator.

حول استقرار نظام فريسة–مفترس رباعي للوتكا–فولتيرا

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الخلاصة

الهدف من هذا العمل هو دراسة نموذج معدل للوتكا– فولتيرا رباعي الأبعاد. في هذا النموذج ، تنمو جميع الأنواع الأربعة لوجستيا. يحتوي هذا النموذج على ستة عشر نقطة توازن ممكنة بحد أقصى. يوجد خمسة منها دائمًا دون أي قيود على معلمات النموذج ، بينما يخضع وجود النقاط الأخرى لاستيفاء بعض الشروط الضرورية والكافية. ثمانية من نقاط التوازن غير مستقرة بينما تكون النقاط الأخرى مستقرة محليًا بشكل مقارب في ظل ظروف معينة. بالإضافة إلى ذلك ، تم العثور على حوض جذاب لكل نقطة يمكن أن تكون مستقرة محليًا بشكل مقارب. يتم توفير الشروط للتأكد من أن جميع الحلول محدودة. أخيرًا ، يتم إجراء عمليات محاكاة عددية للتحقق من النتائج النظرية التي تم الحصول عليها ودعمها.

1. INTRODUCTION

The food chain can be described as a transfer of energy from one type of living organism to another, while the energy transferred is the food that ensures the continuity of life is ecological balance. Chemical, physical and biological systems are inherently nonlinear. A large class of

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models that represent predator-prey population dynamics can be described by ordinary differential system or partial differential system. One of the simplest model is the Lotka-Volterra equations, also called predator and prey equations which are two of the first-order nonlinear differential equations. Populations change over time according to the following equations [1]:

$$\dot{x}(t) = x(t)\big(a - by(t)\big), \qquad \dot{y}(t) = -y(t)\big(c - dx(t)\big),$$

where $x(t)$ represents the prey species and $y(t)$ is the density of predator species at time t. All parameters a, b, c and d are nonnegative constants. The parameter a represents the natural growth rate of the prey in the absence of the predator. Parameter b represents the effect of the predator on the prey population. Moreover, if b is the only decreasing factor for the prey population, then prey will be eaten by predators. Parameter c represents the effect of prey on the predator population. Moreover, if c is the only increasing factor for the predator population, then the population growth is proportional to the food available. Parameter d represents the natural death rate of the predator in the absence of prey. In [2], the predator-prey model with at least one predator and two prey has been investigated.

In [3], the following population dynamics of the Lotka-Volterra model consists of three species: two predators and one prey were studied.

$$\dot{x} = ax - xy - xz,\ \dot{y} = -bx + xy, \dot{z} = -cx + x,$$

where $x(t) \geq 0$, represents prey, $y(t) \geq 0$ and $z(t) \geq 0$ represent predators, and a,b,c are positive parameters. The parameters a, b and c are positive and are interpreted as follows: a represents the natural growth rate of the prey in the absence of predators, b represents the natural death rate of the predator y in the absence of prey, c represents the natural death rate of the predator z in the absence of prey. Many authors modified and investigate the classic Lotka-Volterra model. For more details, see [4] and [5].

Global dynamics of 3–dimensional Lotka–Volterra models with two predators competing for a single prey species in a constant and uniform environment has been studied in [6]. It is assumed that the two predator species compete purely exploitatively with no interference between rivals, the growth rate of the prey species is logistic or linear in the absence of predation, respectively and the predator's functional response is linear [6]. In [7], the two prey-one predator system are also discussed and studied. In[8] and [9], the authors investigated the following mathematical model :

$$\dot{x}_1 = x_1\left[g_1\left(1 - \frac{x_1}{k_1}\right) - \frac{\alpha_1 x_2^n y_1}{x_1^n + x_2^n} - \beta y_2\right],$$
$$\dot{x}_2 = x_2\left[g_2\left(1 - \frac{x_2}{k_2}\right) - \frac{\alpha_2 x_1^n y_1}{x_1^n + x_2^n}\right],$$
$$\dot{y}_1 = y_1\left[-\mu_1 + \frac{\delta_1 x_1 x_2^n}{x_1^n + x_2^n} + \frac{\delta_2 x_1^n x_2}{x_1^n + x_2^n}\right],$$
$$\dot{y}_2 = y_2[-\mu_2 + \gamma x_1],$$

where x_i represents the density of the prey in their two divers' habitats; y_i represents the density of the predator. The two species of prey are supposed to grow logistically with a certain growth rate g_i and carrying environmental capacity $k_i$to x_i; α_i represents the rate of predation by the predator y_1, on prey x_i; β represents the rate of predation by the predator y_2 on x_1;μ_i represents the mortality rate of predators y_i such that $i = 1,2$, and δ_1, δ_2 and γ are the corresponding conversion rates. The two functions $\alpha_1 x_2^n y_1 (x_1^n + x_2^n)^{-1}$ and $\alpha_1 x_1^n y_1 (x_1^n +$

$x_2^n)^{-1}$explain the behavior of predator switching y_1. This model includes two prey and two predators. The phenomenon of switching occurs only with one of the two predators, while the two prey species live in two diverse habitats and has the ability of group-defense against one of the two predators.

More details on the modified Lotka-Volterra models and prey-predator models can be found in [10-14].
In [15], a modified for Lotka-Volterra model was proposed and studied, and it represents a food chain consisting of three species. They all grow logistically, which means that the absence or decimation of one species does not cause the death of the others:
$\dot{x} = x(a - \gamma x - by - cz),\ \dot{y} = y(d - \beta y - ez + fx),\ \dot{z} = z(g - \delta z + hy + mx).$

Lions are at the top of the food chain. They do not differentiate between hyenas and deer when they are hungry. Tigers are in the second place; they prey on every animal weaker than them in the wild. In the third-place are hyenas; they live on less powerful animals such as deer or zebra. In our work, a modification of the last mathematical model was introduced such that another predator was added to the model that studied in [14]. The present model has four first-order nonlinear differential equations describing the dynamic behavior for four species, in which all the species grow logistically. The current model is considered more comprehensive than the model studied in [14]. Because of the increase in the number of parameters as well as the number of differential equations, and because of that this model contains sixteen possible equilibrium points, while the number of the possible equilibrium points in the previous model is eight.

In the following section, the modification of the four-dimensional Lotka-Volterra model, such that the four species grow logistically is formulated, and the boundedness of the positive solutions of the model is studied. In Section 3, it is shown that the model has at most sixteen possible equilibrium points. Five of them always exist without any restriction on the parameters of the model, while the existence of the other points is subject to the fulfillment of some necessary and sufficient conditions, moreover the local stability of all possible equilibrium points is discussed, and it is shown that eight of the equilibrium points are unstable while the rest equilibrium points are locally asymptotically stable under certain conditions. In sections 4 a basin of attraction for all equilibrium points of the system (1), which are locally asymptotically stable, will be discussed by finding a suitable Lyapunov function for the mentioned points. In section 5 a numerical simulations are given to verify and support the obtained theoretical results. Finally, a brief conclusion was presented about the findings of this work.

2. THE MATHEMATICAL MODEL

In this section, we modify the Lotka-Volterra model to include four species. The mathematical model is given as follows:

$$\begin{aligned}\dot{x} &= x(\alpha - \beta x - ay - bz - cw),\\ \dot{y} &= y(\gamma - \delta y - dz - ew + fx),\\ \dot{z} &= z(\mu - \varphi z - gw + hy + mx),\\ \dot{w} &= w(\sigma - \varepsilon w + rz + py + qx),\end{aligned} \tag{1}$$

where x, y, z *and* w represent the densities of the species at time t . We assume that the four species grow logistically. The parametersα, γ, μ and σ are positive constants which represent the growth's rates of the species x, y, z, and w, respectively. The nonnegative constantsa, b, and c are the change's rates of x with respect to $y, z,$ w, respectively. While the nonnegative constants and w respectively. The nonnegative d, e and f are the change's rates of y with respect to $z, w,$ and x respectively; $g, h,$ and m are the change's rates of z with respect to w,y, and x respectively; $r, p,$ and q are the change's rates of w with respect to $z, y,$ and x, respectively.

It is clear that the interaction functions of the model (1) are continuous and have continuous partial derivatives on
$R^{+4} = \{(x, y, z, w) \in R4: x(0) \geq 0, y(0) \geq 0, z(0) \geq 0, w(0) \geq 0\}$.

Hence, all these functions are Lipschitizion functions on R^{+4}. Therefore, the solution of the model (1) exists and is unique. Some sufficient conditions are provided in the next theorem which ensures all solutions of system (1) are bounded.

THEOREM 1: Let $r \leq g, p \leq e, q \leq c, h \leq d, m \leq b, and\ f \leq a$. Then all the trajectories of the positive solutions of System (1) are bounded.

Proof:

Consider that $F(t) = x(t) + y(t) + z(t) + w(t)$, then

$$\dot{F}(t) + \tau F(t) = x(\alpha + \tau - \beta x - ay - bz - cw) + y(\gamma + \tau - \delta y - dz - ew + fx)$$
$$+z(\mu + \tau - \varphi z - gw + hy + mx) + w(\sigma + \tau - \varepsilon w + rz + py + qx)$$
$$= xy(f - a) + xz(m - b) + xw(q - c) + yz(h - d) + yw(p - e) + zw(r - g)$$
$$-\beta\left[\left(x - \frac{\alpha + \tau}{\beta}\right)^2 - \left(\frac{\alpha + \tau}{2\beta}\right)^2\right] - \delta\left[\left(y - \frac{\gamma + \tau}{\delta}\right)^2 - \left(\frac{\gamma + \tau}{2\delta}\right)^2\right]$$
$$- \varphi\left[\left(z - \frac{\mu + \tau}{\varphi}\right)^2 - \left(\frac{\mu + \tau}{2\varphi}\right)^2\right] - \sigma\left[\left(w - \frac{\varepsilon + \tau}{\sigma}\right)^2 - \left(\frac{\varepsilon + \tau}{2\sigma}\right)^2\right].$$

It is easy to show that the right hand of the last inequality is less than

$$\Gamma =: \frac{(\alpha + \tau)^2}{4\beta} + \frac{(\gamma + \tau)^2}{4\delta} + \frac{(\mu + \tau)^2}{4\varphi} + \frac{(\varepsilon + \tau)^2}{4\sigma}.$$

So that

$$\dot{F}(t) + \tau F(t) \leq \Gamma.$$

By using Grönwall's inequality, we obtain that

$$F(t) \leq \frac{\Gamma}{\tau} + F(0)e^{-\tau t}$$

Now, when t approaches to infinity, it follows that

$$F(t) \leq \frac{\Gamma}{\tau}.$$

Thus, the proof is completed.

3. THE EQUILIBRIUM POINTS WITH ITS LOCAL STABILITY:

The system (1) contains at most sixteen equilibrium points, some of them exist regardless of the parameter's values. While the other need the fulfillment of some necessary and sufficient conditions to exist. The existence conditions and the local stability analyses of them are given and shown in this section. The possible equilibria of the system(1) are:

1. The equilibrium points $P_0(0,0,0,0), P_1\left(\frac{\alpha}{\beta}, 0,0,0\right)$, $P_2\left(0, \frac{\gamma}{\delta}, 0,0\right)$, $P_3\left(0,0, \frac{\mu}{\varphi}, 0\right)$, and $P_4\left(0,0,0, \frac{\sigma}{\varepsilon}\right)$ always exist.

2. The equilibrium point $P_5(\bar{a}_1, \bar{a}_2, 0,0)$ exists, if and only if $\delta\alpha - a\gamma > 0$,where

$$\bar{a}_1 = \frac{\delta\alpha - a\gamma}{\delta\beta + af}, \bar{a}_2 = \frac{\gamma\beta + \alpha f}{\delta\beta + af}.$$

3. The equilibrium point $P_6(\tilde{a}_1, 0, \tilde{a}_2, 0)$exists if and only if $\alpha\varphi - b\mu > 0$, where

$$\bar{a}_1 = \frac{\alpha\varphi - b\mu}{\beta\varphi + bm}, \bar{a}_2 = \frac{\beta\mu + m\alpha}{\beta\varphi + bm}.$$

4. The equilibrium point $P_7(\breve{a}_1, 0,0, \breve{a}_2)$ exists if and only if $\alpha\varepsilon - c\sigma > 0$, where $\breve{a}_1 = \frac{\alpha\varepsilon - c\sigma}{\beta\varepsilon + cq}, \breve{a}_2 = \frac{\beta\sigma + q\alpha}{\beta\varepsilon + cq}$.

5. The equilibrium point $P_8(0, \breve{a}_1, \breve{a}_2, 0)$, exists if and only if $\gamma\varphi - d\mu > 0$,where
$\breve{a}_1 = \frac{\gamma\varphi - d\mu}{\delta\varphi + dh}, \breve{a}_2 = \frac{\delta\mu + d\gamma}{\delta\varphi + dh}$.

6. The equilibrium point $P_9(0, \tilde{\tilde{a}}_1, 0, \tilde{\tilde{a}}_2)$, exist if and only if $\gamma\varepsilon - e\sigma > 0$, where $\tilde{\tilde{a}}_1 = \frac{\gamma\varepsilon - e\sigma}{\delta\varepsilon + eq}, \tilde{\tilde{a}}_2 = \frac{\delta\sigma + q\gamma}{\delta\varepsilon + eq}$.

7. The equilibrium point $P_{10}(0,0, \hat{a}_1, \hat{a}_2)$, exist if $\mu\varepsilon - g\sigma > 0$, where
$\hat{a}_1 = \frac{\mu\varepsilon - g\sigma}{\varphi\varepsilon + gr}, \hat{a}_2 = \frac{\varphi\sigma + r\mu}{\varphi\varepsilon + gr}$.

8. The equilibrium point $P_{11}(\bar{b}_1, \bar{b}_2, \bar{b}_3, 0)$, where

$$\bar{b}_1 = \frac{|\bar{B}_1|}{|\bar{B}|}, \bar{b}_2 = \frac{|\bar{B}_2|}{|\bar{B}|}, \bar{b}_3 = \frac{|\bar{B}_3|}{|\bar{B}|}, |\bar{B}| = \begin{vmatrix} \beta & a & b \\ -f & \delta & d \\ -m & -h & \varphi \end{vmatrix} |\bar{B}_1| = \begin{vmatrix} \alpha & a & b \\ \gamma & \delta & d \\ \mu & -h & \varphi \end{vmatrix},$$

$$|\bar{B}_2| = \begin{vmatrix} \beta & \alpha & b \\ -f & \gamma & d \\ -q & \mu & \varphi \end{vmatrix}, |\bar{B}_3| = \begin{vmatrix} \beta & a & \alpha \\ -f & \delta & \gamma \\ -m & -h & \mu \end{vmatrix},$$

exists, if and only if $|\bar{B}||\bar{B}_i| > 0, i = 1,2,3$.

9. The equilibrium point $P_{12}(\tilde{b}_1, \tilde{b}_2, 0, \tilde{b}_3)$, where

$$\tilde{b}_1 = \frac{|\tilde{B}_1|}{|\tilde{B}|}, \tilde{b}_2 = \frac{|\tilde{B}_2|}{|\tilde{B}|}, \tilde{b}_3 = \frac{|\tilde{B}_3|}{|\tilde{B}|}, |\tilde{B}| = \begin{vmatrix} \beta & a & c \\ -f & \delta & e \\ -q & -p & \varepsilon \end{vmatrix},$$

$$|\tilde{B}_1| = \begin{vmatrix} \alpha & a & c \\ \gamma & \delta & e \\ \sigma & -p & \varepsilon \end{vmatrix}, |\tilde{B}_2| = \begin{vmatrix} \beta & \alpha & c \\ -f & \gamma & e \\ -q & \sigma & \varepsilon \end{vmatrix}, |\tilde{B}_3| = \begin{vmatrix} \beta & a & \alpha \\ -f & \delta & \gamma \\ -q & -p & \sigma \end{vmatrix},$$

exists, if and only if $|\tilde{B}||\tilde{B}_i| > 0, i = 1,2,3$.

10. The equilibrium point $P_{13}(\breve{b}_1, 0, \breve{b}_2, \breve{b}_3)$, where

$$\breve{b}_1 = \frac{|\breve{B}_1|}{|\breve{B}|}, \breve{b}_2 = \frac{|\breve{B}_2|}{|\breve{B}|}, \breve{b}_3 = \frac{|\breve{B}_3|}{|\breve{B}|}, |\breve{B}| = \begin{vmatrix} \beta & b & c \\ -m & \varphi & g \\ -q & -r & \varepsilon \end{vmatrix}$$

$$|\breve{B}_1| = \begin{vmatrix} \alpha & b & c \\ \mu & \varphi & g \\ \sigma & -r & \varepsilon \end{vmatrix}, |\breve{B}_2| = \begin{vmatrix} \beta & \alpha & c \\ -m & \mu & g \\ -q & \sigma & \varepsilon \end{vmatrix}, |\breve{B}_3| = \begin{vmatrix} \beta & b & \alpha \\ -m & \varphi & \mu \\ -q & -r & \sigma \end{vmatrix},$$

exists, if and only if $|\breve{B}||\breve{B}_i| > 0, i = 1,2,3$.

11. The equilibrium point $P_{14}(0, \hat{b}_1, \hat{b}_2, \hat{b}_3)$, where

$$\hat{b}_1 = \frac{|\hat{B}_1|}{|\hat{B}|}, \hat{b}_2 = \frac{|\hat{B}_2|}{|\hat{B}|}, \hat{b}_3 = \frac{|\hat{B}_3|}{|\hat{B}|}, |\hat{B}| = \begin{vmatrix} \delta & d & e \\ -h & \varphi & g \\ -p & -r & \varepsilon \end{vmatrix}$$

$$|\hat{B}_1| = \begin{vmatrix} \gamma & d & e \\ \mu & \varphi & g \\ \sigma & -r & \varepsilon \end{vmatrix}, |\hat{B}_2| = \begin{vmatrix} \delta & \gamma & e \\ -h & \mu & g \\ -p & \sigma & \varepsilon \end{vmatrix}, |\hat{B}_3| = \begin{vmatrix} \delta & d & \gamma \\ -h & \varphi & \mu \\ -p & -r & \sigma \end{vmatrix}$$

exists, if and only if $|\hat{B}||\hat{B}_i| > 0, i = 1,2,3$.

12. The equilibrium point (The coexistence point) $P_{15}(c_1, c_2, c_3, c_4)$, where

$$c_1 = \frac{|C_1|}{|C|}, c_2 = \frac{|C_2|}{|C|}, c_3 = \frac{|C_3|}{|C|}, c_4 = \frac{|C_4|}{|C|}$$

$$C=\begin{bmatrix}\beta & a & b & c\\ -f & \delta & d & e\\ -m & h & \varphi & g\\ -q & -p & -r & \varepsilon\end{bmatrix}, C_1=\begin{bmatrix}\alpha & a & b & c\\ \gamma & \delta & d & e\\ \mu & h & \varphi & g\\ \sigma & -p & -r & \varepsilon\end{bmatrix}, C_2=\begin{bmatrix}\beta & \alpha & b & c\\ -f & \gamma & d & e\\ -m & \mu & \varphi & g\\ -q & \sigma & -r & \varepsilon\end{bmatrix}$$

$$C_3=\begin{bmatrix}\beta & a & \alpha & c\\ -f & \delta & \gamma & e\\ -m & h & \mu & g\\ -q & -p & \sigma & \varepsilon\end{bmatrix}, C_4=\begin{bmatrix}\beta & a & b & \alpha\\ -f & \delta & d & \gamma\\ -m & h & \varphi & \mu\\ -q & -p & -r & \sigma\end{bmatrix}$$

exist if and only if $|C||C_i|>0, i=1,2,3,4$.

Now**,** we study the local stability of all possible equilibrium points of the system. This will be done by evaluating the Jacobian matrix of system (1) at each equilibrium point. Recall that an equilibrium point x^* of the system (1) is said to be locally asymptotically stable if all eigenvalues of the Jacobian matrix evaluated at x^* has negative real part. If one or more has a positive real part, then x^* is an unstable point. The Jacobian matrix of the system(1) at any point (x,y,z,w) is

$$J_{(x,y,z,w)}=\begin{bmatrix}j_{11} & j_{12} & j_{13} & j_{14}\\ j_{21} & j_{22} & j_{23} & j_{24}\\ j_{31} & j_{32} & j_{33} & j_{34}\\ j_{41} & j_{42} & j_{43} & j_{44}\end{bmatrix},$$

where

$j_{11}=\alpha-2\beta x-ay-bz-cw,\quad j_{12}=-ax,\quad j_{13}=-bx,\quad j_{14}=-cx,$
$j_{21}=fy,\quad j_{22}=\gamma-2\delta y-dz-ew+fx,\quad j_{23}=--dy,\quad j_{24}=-ey,$
$j_{31}=mz,\quad j_{32}=hz,\quad j_{33}=\mu-2\varphi z-gw+hy+mx,\quad j_{34}=-gz,$
$j_{41}=qw,\quad j_{42}=pw,\quad j_{43}=rw,\quad j_{44}=\sigma-2\varepsilon w+rz+py+qx.$

THEOREM 2: Consider the system (1), then the equilibrium points

1. $P_0, P_1, P_2, P_3, P_5, P_6, P_8$ and P_{11} are unstable equilibrium points.

2. P_4 is a locally asymptotically stable point if $\frac{\sigma}{\varepsilon}>max\left\{\frac{\alpha}{c},\frac{\gamma}{e},\frac{\mu}{g}\right\}$,

3. P_7 is a locally asymptotically stable point if $\breve{a}_2>max\left\{\frac{\gamma+f\breve{a}_1}{e},\frac{\mu+m\breve{a}_1}{g}\right\}$,

4. P_9 is a locally asymptotically stable point if $\overset{\frown}{a}_2>max\left\{\frac{\mu+h\overset{\frown}{a}_1}{g},\frac{\alpha-a\overset{\frown}{a}_1}{c}\right\}$,

5. P_{10} is a locally asymptotically stable point if and $\hat{a}_2>max\left\{\frac{\gamma-d\hat{a}_1}{e},\frac{\alpha-b\hat{a}_1}{c}\right\}$.

Proof:

1. It is clear that the eigenvalues of the J at the point $P_0(0,0,0,0)$ are $\lambda_1=\alpha, \lambda_2=\gamma, \lambda_3=\mu$, and $\lambda_4=\sigma$. Therefore, the point $P_0(0,0,0,0)$ is always unstable point .

The Jacobian matrix J at the equilibrium point $P_1\left(\frac{\alpha}{\beta},0,0,0\right)$ is given by

$$J_{P_1}=\begin{bmatrix}-\alpha & \frac{-a\alpha}{\beta} & \frac{-b\alpha}{\beta} & \frac{-c\alpha}{\beta}\\ 0 & \gamma+\frac{f\alpha}{\beta} & 0 & 0\\ 0 & 0 & \mu+\frac{m\alpha}{\beta} & 0\\ 0 & 0 & 0 & \sigma+\frac{q\alpha}{\beta}\end{bmatrix}$$

So that that eigenvalues of J_{P_1} are $\lambda_1 = -\alpha$, $\lambda_2 = \gamma + \frac{f\alpha}{\beta}$, $\lambda_3 = \mu + \frac{m\alpha}{\beta}$ and $\lambda_4 = \sigma + \frac{q\alpha}{\beta} > 0$. Since, some of them are positive, then the point $P_1\left(\frac{\alpha}{\beta}, 0,0,0\right)$ is unstable point.

The Jacobian matrix J of the system (1) at the equilibrium point $P_2\left(0, \frac{\gamma}{\delta}, 0,0\right)$ is

$$J_{P_2} = \begin{bmatrix} \alpha - \frac{a\gamma}{\delta} & 0 & 0 & 0 \\ \frac{f\gamma}{\delta} & -\gamma & -\frac{d\gamma}{\delta} & -\frac{e\gamma}{\delta} \\ 0 & 0 & \mu + \frac{h\gamma}{\delta} & 0 \\ 0 & 0 & 0 & \sigma + \frac{p\gamma}{\delta} \end{bmatrix}$$

Therefore, the eigenvalues of the J_{P_2} are $\lambda_1 = -\gamma, \lambda_2 = \alpha - \frac{a\gamma}{\delta}, \lambda_3 = \mu + \frac{h\gamma}{\delta}$, and $\lambda_4 = \sigma + \frac{p\gamma}{\delta}$.

Since λ_3 and λ_4 are always positive values so that the equilibrium point $P_2\left(0, \frac{\gamma}{\delta}, 0,0\right)$ is unstable point.

For the equilibrium point $P_3\left(0,0, \frac{\mu}{\varphi}, 0\right)$, we can see that the Jacobian matrix of the system (1) at $P_3\left(0,0, \frac{\mu}{\varphi}, 0\right)$ is

$$J_{P_3} = \begin{bmatrix} \alpha - \frac{b\mu}{\rho} - \lambda & 0 & 0 & 0 \\ 0 & \gamma - \frac{d\mu}{\rho} - \lambda & 0 & 0 \\ \frac{m\mu}{\varphi} & \frac{h\mu}{\varphi} & -\mu - \lambda & 0 \\ 0 & 0 & 0 & \sigma + \frac{r\mu}{\varphi} - \lambda \end{bmatrix}.$$

Therefore, the eigenvalues of J_{P_3}, are $\lambda_1 = \alpha - \frac{b\mu}{\varphi}, \lambda_2 = \gamma - \frac{d\mu}{\varphi}, \lambda_3 = -\mu, \lambda_4 = \sigma + \frac{r\mu}{\varphi}$. So that J_{P_3} has one positive eigenvalues, namely $\lambda_4 = \sigma + \frac{r\mu}{\varphi}$. Therefore, the equilibrium point $P_3\left(0,0, \frac{\mu}{\varphi}, 0\right)$ is unstable point.

For the equilibrium point $P_5(\bar{a}_1, \bar{a}_2, 0,0)$, we can see that the Jacobian matrix of the system (1) at $P_5(\bar{a}_1, \bar{a}_2, 0,0)$ is

$$J_{P_5} = \begin{bmatrix} -\beta\bar{a}_1 & -a\bar{a}_1 & -b\bar{a}_1 & -c\bar{a}_1 \\ f\bar{a}_2 & -\delta\bar{a}_2 & -d\bar{a}_2 & -e\bar{a}_2 \\ 0 & 0 & \mu + m\bar{a}_1 + h\bar{a}_2 & 0 \\ 0 & 0 & 0 & \sigma + q\bar{a}_1 + p\bar{a}_2 \end{bmatrix},$$

Therefore, the eigenvalues of the matrix J_{P_5} are

$$\begin{cases} \lambda_1 = \sigma + q\bar{a}_1 + p\bar{a}_2, \quad \lambda_3 = \dfrac{-(\beta\bar{a}_1 + a\bar{a}_2) - \sqrt{(\beta\bar{a}_1 - ay)^2 - 4af(\bar{a}_1\bar{a}_2)^2}}{2} \\ \lambda_2 = \mu + m\bar{a}_1 + h\bar{a}_2, \quad \lambda_4 = \dfrac{-(\beta\bar{a}_1 + a\bar{a}_2) + \sqrt{(\beta\bar{a}_1 - ay)^2 - 4af(\bar{a}_1\bar{a}_2)^2}}{2} \end{cases} \quad (4)$$

It is noticed that the matrix J_{P_5} has two positive eigenvalues, namely $\lambda_1 = \sigma + q\bar{a}_1 + p\bar{a}_2$ and $\lambda_2 = \mu + m\bar{a}_1 + h\bar{a}_2$, so that the equilibrium point $P_5(\bar{a}_1, \bar{a}_2, 0,0)$ is unstable point.

The Jacobian matrix J at equilibrium point $P_6(\tilde{a}_1, 0, \tilde{a}_2, 0)$ is given by

$$J_{P_6}=\begin{bmatrix}-\beta\tilde{a}_1 & -a\tilde{a}_1 & -b\tilde{a}_1 & -c\tilde{a}_1\\ 0 & \gamma-d\tilde{a}_2+f\tilde{a}_1 & 0 & 0\\ m\tilde{a}_2 & h\tilde{a}_2 & -\varphi\tilde{a}_2 & -g\tilde{a}_2\\ 0 & 0 & 0 & \sigma+r\tilde{a}_2+q\tilde{a}_1\end{bmatrix}$$

Therefore, the eigenvalues of the matrix J_{P_6} are

$$\begin{cases}\lambda_1=\sigma+r\tilde{a}_2+q\tilde{a}_1, & \lambda_3=\dfrac{-(\beta\tilde{a}_1+\varphi\tilde{a}_2)-\sqrt{(\beta\tilde{a}_1-\varphi\tilde{a}_2)^2-4mb\tilde{a}_1\tilde{a}_2}}{2},\\ \lambda_2=\gamma-d\tilde{a}_2+f\tilde{a}_1, & \lambda_4=\dfrac{-(\beta\tilde{a}_1+\varphi\tilde{a}_2)+\sqrt{(\beta\tilde{a}_1-\varphi\tilde{a}_2)^2-4mb\tilde{a}_1\tilde{a}_2}}{2},\end{cases}$$

It is notice that the matrix J_{P_6} has one positive eigenvalue, namely $\lambda_1=\sigma+r\tilde{a}_2+q\tilde{a}_1$, so that the equilibrium point $P_6(\tilde{a}_1,0,\tilde{a}_2,0)$ is unstable point.
The Jacobi's matrix J at the equilibrium point $P_8(0,\breve{a}_1,\breve{a}_2,0)$,is given by

$$J_{P_8}=\begin{bmatrix}\alpha-a\breve{a}_1-b\breve{a}_2 & 0 & 0 & 0\\ f\breve{a}_1 & -\delta\breve{a}_1 & -d\breve{a}_1 & -e\breve{a}_1\\ m\breve{a}_2 & p\breve{a}_2 & -\varphi\breve{a}_2 & -g\breve{a}_2\\ 0 & 0 & 0 & \sigma+r\breve{a}_2+p\breve{a}_1\end{bmatrix}.$$

Therefore, the eigenvalues of J_{P_8} are

$$\begin{cases}\lambda_1=\alpha-a\breve{a}_1-b\breve{a}_2, & \lambda_3=\dfrac{-(\delta\breve{a}_1+\varphi\breve{a}_2)-\sqrt{(\delta\breve{a}_1-\varphi\breve{a}_2)^2-4dp\breve{a}_1\breve{a}_2}}{2},\\ \lambda_2=\sigma+r\breve{a}_2+p\breve{a}_1, & \lambda_4=\dfrac{-(\delta\breve{a}_1+\varphi\breve{a}_2)+\sqrt{(\delta\breve{a}_1-\varphi\breve{a}_2)^2-4dp\breve{a}_1\breve{a}_2}}{2},\end{cases} \quad (10.4)$$

Note that $\lambda_2=\sigma+r\breve{a}_2+p\breve{a}_1$ is positive eigenvalue which means that the equilibrium point $P_8(0,\breve{a}_1,\breve{a}_2,0)$ is unstable.
The Jacobi's matrix J at the point $P_{11}(\bar{b}_1,\bar{b}_2,\bar{b}_3,0)$ is given by

$$J_{P_{11}}=\begin{bmatrix}-\beta\bar{b}_1 & -a\bar{b}_1 & -b\bar{b}_1 & -c\bar{b}_1\\ f\bar{b}_2 & -\delta\bar{b}_2 & -d\bar{b}_2 & -e\bar{b}_2\\ m\bar{b}_3 & h\bar{b}_3 & -\varphi\bar{b}_3 & -g\bar{b}_3\\ 0 & 0 & 0 & \sigma+r\bar{b}_3+p\bar{b}_2+q\bar{b}_1\end{bmatrix},$$

Therefore, the characteristic equation of $J_{P_{11}}$ is

$$\left(\lambda_1-\sigma-r\bar{b}_3-p\bar{b}_2-q\bar{b}_1\right)(\lambda^3-\mathrm{tr}\bar{\mathrm{H}}\lambda^2+\textstyle\sum_{i=1}^{3}|\bar{\mathrm{H}}_{ii}|\,\lambda-|\bar{\mathrm{H}}|)=0,$$

where

$$\bar{\mathrm{H}}=\begin{bmatrix}-\beta\bar{b}_1 & -a\bar{b}_1 & -b\bar{b}_1\\ f\bar{b}_2 & -\delta\bar{b}_2 & -d\bar{b}_2\\ m\bar{b}_3 & h\bar{b}_3 & -\varphi\bar{b}_3\end{bmatrix}$$

$\bar{\mathrm{H}}_{ii}, i=1,2,3$ are the diagonal minors of the matrix $\bar{\mathrm{H}}$. It is easy to see, that one of its eigenvalues, namely $\lambda_1=\sigma+r\bar{b}_3+p\bar{b}_2+q\bar{b}_1$ is positive. Thus, the equilibrium point $P_{11}(\bar{b}_1,\bar{b}_2,\bar{b}_3,0)$ is unstable.

2. The Jacobian matrix of system (1) at $P_4\left(0,0,0,\frac{\sigma}{\varepsilon}\right)$ is

$$J_{P_4} = \begin{bmatrix} \alpha - \frac{c\sigma}{\varepsilon} & 0 & 0 & 0 \\ 0 & \gamma - \frac{e\sigma}{\varepsilon} & 0 & 0 \\ 0 & 0 & \mu - \frac{g\sigma}{\varepsilon} & 0 \\ \frac{q\sigma}{\varepsilon} & \frac{p\sigma}{\varepsilon} & \frac{r\sigma}{\varepsilon} & -\sigma \end{bmatrix}.$$

The eigenvalues of J_{P_4} are $\lambda_1 = \alpha - \frac{c\sigma}{\varepsilon}, \lambda_2 = \gamma - \frac{e\sigma}{\varepsilon}, \lambda_3 = \mu - \frac{g\sigma}{\varepsilon}, \lambda_4 = -\sigma$. So that the equilibrium point $P_4\left(0,0,0,\frac{\sigma}{\varepsilon}\right)$ is locally asymptotically stable if and only if

$\frac{\sigma}{\varepsilon} > max\left\{\frac{\alpha}{c}, \frac{\gamma}{e}, \frac{\mu}{g}\right\}$.

3. The Jacobian matrix of system (1) at the equilibrium point $P_7(\breve{a}_1, 0,0, \breve{a}_2)$ is given by

$$J_{P_7} = \begin{bmatrix} -\beta\breve{a}_1 & -a\breve{a}_1 & -b\breve{a}_1 & -c\breve{a}_1 \\ 0 & \gamma + f\breve{a}_1 - e\breve{a}_2 & 0 & 0 \\ 0 & 0 & \mu + m\breve{a}_1 - g\breve{a}_2 & 0 \\ q\breve{a}_2 & p\breve{a}_2 & r\breve{a}_2 & -\varepsilon\breve{a}_2 \end{bmatrix},$$

Therefore, the eigenvalues of J_{P_7} are

$$\begin{cases} \lambda_1 = \gamma + f\breve{a}_1 - e\breve{a}_2, \quad \lambda_3 = \dfrac{-(\beta\breve{a}_1 + \varepsilon\breve{a}_2) - \sqrt{(\beta\breve{a}_1 - \varepsilon\breve{a}_2)^2 - 4cq\breve{a}_1\breve{a}_2}}{2}, \\ \lambda_2 = \mu + m\breve{a}_1 - g\breve{a}_2, \quad \lambda_4 = \dfrac{-(\beta\breve{a}_1 + \varepsilon\breve{a}_2) + \sqrt{(\beta\breve{a}_1 - \varepsilon\breve{a}_2)^2 - 4cq\breve{a}_1\breve{a}_2}}{2}. \end{cases}$$

the point is locally asymptotically stable if and only if

$\gamma + f\breve{a}_1 < e\breve{a}_2$ and $\mu + m\breve{a}_1 < g\breve{a}_2$ or

$\breve{a}_2 > max\left\{\frac{\gamma + f\breve{a}_1}{e}, \frac{\mu + m\breve{a}_1}{g}\right\}$,

4. The Jacobi's matrix of the system (1) at the equilibrium point $P_9\left(0, \breve{\breve{a}}_1, 0, \breve{\breve{a}}_2\right)$ is as follows:

$$J_{P_9} = \begin{bmatrix} \alpha - a\breve{\breve{a}}_1 - c\breve{\breve{a}}_2 & 0 & 0 & 0 \\ f\breve{\breve{a}}_1 & -\delta\breve{\breve{a}}_1 & -d\breve{\breve{a}}_1 & -e\breve{\breve{a}}_1 \\ 0 & 0 & \mu + h\breve{\breve{a}}_1 - g\breve{\breve{a}}_2 & 0 \\ q\breve{\breve{a}}_2 & p\breve{\breve{a}}_2 & r\breve{\breve{a}}_2 & -\varepsilon\breve{\breve{a}}_2 \end{bmatrix},$$

Therefore, the eigenvalues of J_{P_7} are

$$\begin{cases} \lambda_1 = \alpha - a\breve{\breve{a}}_1 - c\breve{\breve{a}}_2, \quad \lambda_3 = \dfrac{-\left(\delta\breve{\breve{a}}_1 + \varepsilon\breve{\breve{a}}_2\right) - \sqrt{\left(\delta\breve{\breve{a}}_1 - \varepsilon\breve{\breve{a}}_2\right)^2 - 4dp\breve{\breve{a}}_1\breve{\breve{a}}_2}}{2}, \\ \lambda_2 = \mu + h\breve{\breve{a}}_1 - g\breve{\breve{a}}_2, \quad \lambda_4 = \dfrac{-\left(\delta\breve{\breve{a}}_1 + \varepsilon\breve{\breve{a}}_2\right) + \sqrt{\left(\delta\breve{\breve{a}}_1 - \varepsilon\breve{\breve{a}}_2\right)^2 - 4dp\breve{\breve{a}}_1\breve{\breve{a}}_2}}{2}. \end{cases}$$

so that for the point $P_9\left(0, \breve{\breve{a}}_1, 0, \breve{\breve{a}}_2\right)$is to be locally asymptotically stable, the following must be achieved

$\alpha < a\breve{\breve{a}}_1 + c\breve{\breve{a}}_2$ and $\mu + h\breve{\breve{a}}_1 < g\breve{\breve{a}}_2$, or $\breve{\breve{a}}_2 > max\left\{\frac{\mu + h\breve{\breve{a}}_1}{g}, \frac{\alpha - a\breve{\breve{a}}_1}{c}\right\}$.

5. The Jacobian matrix of the system (1) at the equilibrium point $P_{10}(0,0,\hat{a}_1,\hat{a}_2)$ is given

$$J_{P_{10}}=\begin{bmatrix}\alpha-b\hat{a}_1-c\hat{a}_2 & 0 & 0 & 0\\ 0 & \gamma-d\hat{a}_1-e\hat{a}_2 & 0 & 0\\ m\hat{a}_1 & h\hat{a}_1 & -\varphi\hat{a}_1 & -g\hat{a}_1\\ q\hat{a}_2 & p\hat{a}_2 & r\hat{a}_2 & -\varepsilon\hat{a}_2\end{bmatrix}$$

Therefore, the eigenvalues of $J_{P_{10}}$ are $\begin{cases}\lambda_1=\alpha-b\hat{a}_1-c\hat{a}_2, & \lambda_3=\frac{-(\varphi\hat{a}_1+\varepsilon\hat{a}_2)-\sqrt{(\varphi\hat{a}_1-\varepsilon\hat{a}_2)^2-4rg\hat{a}_1\hat{a}_2}}{2},\\ \lambda_2=\gamma-d\hat{a}_1-e\hat{a}_2, & \lambda_4=\frac{-(\varphi\hat{a}_1+\varepsilon\hat{a}_2)+\sqrt{(\varphi\hat{a}_1-\varepsilon\hat{a}_2)^2-4rg\hat{a}_1\hat{a}_2}}{2}.\end{cases}$

so that the point $P_{10}(0,0,\hat{a}_1,\hat{a}_2)$ is to be locally asymptotically stable if
$\alpha<b\hat{a}_1+c\hat{a}_2$ and, $\gamma<d\hat{a}_1+e\hat{a}_2$, or $\hat{a}_2>max\left\{\frac{\gamma-d\hat{a}_1}{e},\frac{\alpha-b\hat{a}_1}{c}\right\}$.
Thus, the proof is completed.

THEOREM 3: Consider the system (1), then we have the following:

1. the equilibrium point P_{12} is locally asymptotically stable point if the following conditions hold

$$\begin{cases}\mu+h\tilde{b}_2+m\tilde{b}_1<g\tilde{b}_3\\ det\tilde{\text{H}}<0,\\ \sum_{i=1}^{3}\left|\tilde{\text{H}}_{ii}\right|\text{tr}\tilde{\text{H}}<det\tilde{\text{H}},\end{cases}\quad \text{where, } \tilde{\text{H}}=\begin{bmatrix}-\beta\tilde{b}_1 & -a\tilde{b}_1 & -c\tilde{b}_1\\ f\tilde{b}_2 & -\delta\tilde{b}_2 & -e\tilde{b}_2\\ q\tilde{b}_3 & p\tilde{b}_3 & -\varepsilon\tilde{b}_3\end{bmatrix}.$$

2. the equilibrium point P_{13} is locally asymptotically stable point provided the conditions

$$\begin{cases}\gamma+f\breve{b}_1<\delta\breve{b}_2+e\breve{b}_3,\\ det\breve{\text{H}}<0,\\ \sum_{i=1}^{3}\left|\breve{\text{H}}_{ii}\right|\text{tr}\breve{\text{H}}<det\breve{\text{H}},\end{cases}\quad \text{where, } \breve{\text{H}}=\begin{bmatrix}-\beta\breve{b}_1 & -b\breve{b}_1 & -c\breve{b}_1\\ m\breve{b}_2 & -\varphi\breve{b}_2 & -g\breve{b}_2\\ q\breve{b}_3 & r\breve{b}_3 & -\varepsilon\breve{b}_3\end{bmatrix}$$

3. the equilibrium point P_{14} is locally asymptotically stable point provided the conditions

$$\begin{cases}\alpha<a\hat{b}_1+b\hat{b}_2+c\hat{b}_3,\\ \text{det}\hat{\text{H}}<0,\\ \sum_{i=1}^{3}\left|\hat{\text{H}}_{ii}\right|\text{tr}\hat{\text{H}}<det\hat{\text{H}},\end{cases}\quad \text{where, } \hat{\text{H}}=\begin{bmatrix}-\delta\hat{b}_1 & -d\hat{b}_1 & -e\hat{b}_1\\ h\hat{b}_2 & -\varphi\hat{b}_2 & -g\hat{b}_2\\ p\hat{b}_3 & r\hat{b}_3 & -\varepsilon\hat{b}_3\end{bmatrix}.$$

4. the equilibrium point P_{15} is locally asymptotically stable point provided the conditions

$$\begin{cases}\Delta_i>0, i=1,2,3,4,\\ \Delta_1\Delta_2-\Delta_3>0\\ \Delta_3(\Delta_1\Delta_2-\Delta_3)-\Delta_4\Delta_1^2>0.\end{cases}\quad \text{where,}$$

$\Delta_1=-trJ_{15}$
$\Delta_2=(\beta\delta+af)p_1p_2+(\beta\varphi+mb)p_1p_3+(\beta\varepsilon+cq)p_1p_4$
$\quad+(\delta\varphi+hd)p_2p_3+(\delta\varepsilon+pe)p_2p_4+(\varphi\varepsilon+gr)p_3p_4,$
$\Delta_3=-\sum_{i=1}^{4}J_{P_{15}ii}$, $\Delta_4=detJ_{15}$, and

$$J_{P_{15}}=\begin{bmatrix}-\beta p_1 & -ap_1 & -bp_1 & -cp_1\\ fp_2 & -\delta p_2 & -dp_2 & -ep_2\\ mp_3 & hp_3 & -\varphi p_3 & -gp_3\\ qp_4 & pp_4 & rp_4 & -\varepsilon p_4\end{bmatrix}.$$

Proof:

1. The Jacobian matrix of the system (1) at the equilibrium point $P_{12}(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)$ Is given as follows

$$J_{P_{12}}=\begin{bmatrix}-\beta\tilde{b}_1 & -a\tilde{b}_1 & -b\tilde{b}_1 & -c\tilde{b}_1\\ f\tilde{b}_2 & -\delta\tilde{b}_2 & -d\tilde{b}_2 & -e\tilde{b}_2\\ 0 & 0 & \mu-g\tilde{b}_3+h\tilde{b}_2+m\tilde{b}_1 & 0\\ q\tilde{b}_3 & p\tilde{b}_3 & r\tilde{b}_3 & -\varepsilon\tilde{b}_3\end{bmatrix},$$

Therefore, the characteristic equation of the $J_{P_{12}}$ is

$$\left[\lambda-\mu-h\tilde{b}_2-m\tilde{b}_1+g\tilde{b}_3\right]\left[\lambda^3-\mathrm{tr}\tilde{\mathrm{H}}\lambda^2+\textstyle\sum_{i=1}^{3}\left|\tilde{\mathrm{H}}_{ii}\right|\lambda-det\tilde{\mathrm{H}}\right]=0,$$

Where, $\tilde{\mathrm{H}}_{ii}, i=1,2,3$ are the diagonal minors of the following matrix

$$\tilde{\mathrm{H}}=\begin{bmatrix}-\beta\tilde{b}_1 & -a\tilde{b}_1 & -c\tilde{b}_1\\ f\tilde{b}_2 & -\delta\tilde{b}_2 & -e\tilde{b}_2\\ q\tilde{b}_3 & p\tilde{b}_3 & -\varepsilon\tilde{b}_3\end{bmatrix},$$

According to the Routh-Hurwitz principle, the equilibrium point $P_{12}(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)$ is to be locally asymptotically stable if:

$$\begin{cases}\mu+h\tilde{b}_2+m\tilde{b}_1<g\tilde{b}_3,\\ \mathrm{tr}\tilde{\mathrm{H}}<0,\\ det\tilde{\mathrm{H}}<0,\\ \sum_{i=1}^{3}\left|\tilde{\mathrm{H}}_{ii}\right|\mathrm{tr}\tilde{\mathrm{H}}<det\tilde{\mathrm{H}}.\end{cases}$$

Note that, the trace of the matrix $J_{P_{12}}$ is $(-\beta\tilde{b}_1-\delta\tilde{b}_2-\varepsilon\tilde{b}_3)$, which is always negative. So that the previous conditions can be written as follows:

$$\begin{cases}\mu+h\tilde{b}_2+m\tilde{b}_1<g\tilde{b}_3\\ det\tilde{\mathrm{H}}<0,\\ \sum_{i=1}^{3}\left|\tilde{\mathrm{H}}_{ii}\right|\mathrm{tr}\tilde{\mathrm{H}}<det\tilde{\mathrm{H}}.\end{cases}$$

2. The Jacobian matrix of the system (1) at the equilibrium point $P_{13}(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)$is given by

$$J_{P_{13}}=\begin{bmatrix}-\beta\breve{b}_1 & -a\breve{b}_1 & -b\breve{b}_1 & -c\breve{b}_1\\ 0 & \gamma-\delta\breve{b}_2-e\breve{b}_3+f\breve{b}_1 & 0 & 0\\ m\breve{b}_2 & h\breve{b}_2 & -\varphi\breve{b}_2 & -g\breve{b}_2\\ q\breve{b}_3 & p\breve{b}_3 & r\breve{b}_3 & -\varepsilon\breve{b}_3\end{bmatrix}$$

Therefore, the characteristic equation of the $J_{P_{13}}$ *is*

$$\left[\lambda_1-\gamma-f\breve{b}_1+\delta\breve{b}_2+e\breve{b}_3\right]\left[\lambda^3-\mathrm{tr}\breve{\mathrm{H}}\lambda^2+\textstyle\sum_{i=1}^{3}\left|\breve{\mathrm{H}}_{ii}\right|\lambda-\mathrm{det}\breve{\mathrm{H}}\right]=0,$$

where,$\breve{\mathrm{H}}_{ii}, i=1,2,3$ are the diagonal minors of the following matrix:

$$\breve{\mathrm{H}}=\begin{bmatrix}-\beta\breve{b}_1 & -b\breve{b}_1 & -c\breve{b}_1\\ m\breve{b}_2 & -\varphi\breve{b}_2 & -g\breve{b}_2\\ q\breve{b}_3 & r\breve{b}_3 & -\varepsilon\breve{b}_3\end{bmatrix}.$$

According to the Routh-Hurwitz principle, the equilibrium point $P_{13}(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)$is to be locally asymptotically stable if:

$$\begin{cases} \gamma + f\breve{b}_1 < \delta\breve{b}_2 + e\breve{b}_3, \\ \mathrm{tr}\breve{\mathrm{H}} < 0, \\ det\breve{\mathrm{H}} < 0, \\ \sum_{i=1}^{3} \left|\breve{\mathrm{H}}_{ii}\right| \mathrm{tr}\breve{\mathrm{H}} < det\breve{\mathrm{H}}. \end{cases}$$

Note that, the trace of the matrix $J_{P_{13}}$ is $(-\beta\tilde{b}_1 - \varphi\breve{b}_2 - \varepsilon\tilde{b}_3)$, which is always negative. So that the previous conditions can be written as

$$\begin{cases} \gamma + f\breve{b}_1 < \delta\breve{b}_2 + e\breve{b}_3, \\ det\breve{\mathrm{H}} < 0, \\ \sum_{i=1}^{3} \left|\breve{\mathrm{H}}_{ii}\right| \mathrm{tr}\breve{\mathrm{H}} < det\breve{\mathrm{H}}. \end{cases}$$

3. The Jacobian matrix of the system (1) at the equilibrium point $P_{14}\left(0, \hat{b}_1, \hat{b}_2, \hat{b}_3\right)$is given by

$$J_{P_{14}} = \begin{bmatrix} \alpha - a\hat{b}_1 - b\hat{b}_2 - c\hat{b}_3 & 0 & 0 & 0 \\ f\hat{b}_1 & -\delta\hat{b}_1 & -d\hat{b}_1 & -e\hat{b}_1 \\ m\hat{b}_2 & h\hat{b}_2 & -\varphi\hat{b}_2 & -g\hat{b}_2 \\ q\hat{b}_3 & p\hat{b}_3 & r\hat{b}_3 & -\varepsilon\hat{b}_3 \end{bmatrix}$$

Therefore, the characteristic equation of the $J_{P_{14}}$ *is*

$\left(\lambda_1 - \alpha + a\hat{b}_1 + b\hat{b}_2 + c\hat{b}_3\right)\left(\lambda^3 - \lambda^2 \mathrm{tr}\hat{\mathrm{H}} + \sum_{i=1}^{3}\left|\hat{\mathrm{H}}_{ii}\right| \lambda - \left|\hat{\mathrm{H}}\right|\right) = 0,$

where,$\hat{\mathrm{H}}_{ii}, i = 1,2,3$ are the diagonal minors of the following matrix

$$\hat{\mathrm{H}} = \begin{bmatrix} -\delta\hat{b}_1 & -d\hat{b}_1 & -e\hat{b}_1 \\ h\hat{b}_2 & -\varphi\hat{b}_2 & -g\hat{b}_2 \\ p\hat{b}_3 & r\hat{b}_3 & -\varepsilon\hat{b}_3 \end{bmatrix}$$

According to the Routh-Hurwitz principle, the equilibrium point $P_{14}\left(0, \hat{b}_1, \hat{b}_2, \hat{b}_3\right)$ is to be locally asymptotically stable if the following the conditions hold :

$$\begin{cases} \alpha < a\hat{b}_1 + b\hat{b}_2 + c\hat{b}_3, \\ \mathrm{tr}\hat{\mathrm{H}} < 0, \\ \mathrm{det}\hat{\mathrm{H}} < 0, \\ \sum_{i=1}^{3} \left|\hat{\mathrm{H}}_{ii}\right| \mathrm{tr}\hat{\mathrm{H}} < det\hat{\mathrm{H}}. \end{cases}$$

Note that, the trace of the matrix $J_{P_{14}}$ is $(-\delta\tilde{b}_1 - \varphi\breve{b}_2 - \varepsilon\tilde{b}_3)$, which is always negative, it follows that the previous conditions can be written as

$$\begin{cases} \alpha < a\hat{b}_1 + b\hat{b}_2 + c\hat{b}_3,, \\ \mathrm{det}\hat{\mathrm{H}} < 0, \\ \sum_{i=1}^{3} \left|\hat{\mathrm{H}}_{ii}\right| \mathrm{tr}\hat{\mathrm{H}} < det\hat{\mathrm{H}}. \end{cases}$$

4. The Jacobian matrix of the system (1) at the point $P_{15}(c_1, c_2, c_3, c_4)$

is given by $J_{P_{15}} = \begin{bmatrix} -\beta p_1 & -ap_1 & -bp_1 & -cp_1 \\ fp_2 & -\delta p_2 & -dp_2 & -ep_2 \\ mp_3 & hp_3 & -\varphi p_3 & -gp_3 \\ qp_4 & pp_4 & rp_4 & -\varepsilon p_4 \end{bmatrix}.$

Therefore, the characteristic equation of the $J_{P_{15}}$ *is*

$\lambda^4 + \Delta_1\lambda^3 + \Delta_2\lambda^2 + \Delta_3\lambda + \Delta_4 = 0,$

where,

$\Delta_1 = -trJ_{P_{15}}$

$\Delta_2 = (\beta\delta + af)p_1p_2 + (\beta\varphi + mb)p_1p_3 + (\beta\varepsilon + cq)p_1p_4$

$+(\delta\varphi + hd)p_2p_3 + (\delta\varepsilon + pe)p_2p_4 + (\varphi\varepsilon + gr)p_3p_4$,
$\Delta_3 = -\sum_{i=1}^{4}|J_{P_{15}ii}|$, and
$\Delta_4 = detJ_{P_{15}}$.
$J_{P_{15}ii}, i = 1,2,3,4$ are the diagonal minors of $J_{P_{15}}$.

According to the Routh-Hurwitz principle, the equilibrium point $P_{15}(c_1, c_2, c_3, c_4)$ *is to* be locally asymptotically stable if:

$$\begin{cases} \Delta_i > 0, i = 1,2,3,4, \\ \Delta_1\Delta_2 - \Delta_3 > 0 \\ \Delta_3(\Delta_1\Delta_2 - \Delta_3) - \Delta_4\Delta_1^2 > 0. \end{cases}$$

Thus, the proof is completed.

THEOREM 4:
1. If P_4 is locally asymptotically stable, then the equilibrium points P_7, P_9 and P_{10} can not be existed.
2. If one of the equilibrium points P_7, P_9 and P_{10} does not exist, then the P_4 is not stable.

Proof:
1. Let P_4 be a locally asymptotically stable**,** then according to Theorem 2**,** the following condition must be satisfied**:**
$\frac{\sigma}{\varepsilon} > max\left\{\frac{\alpha}{c}, \frac{\gamma}{e}, \frac{\mu}{g}\right\}$. Therefore, we have $\alpha\varepsilon - c\sigma < 0$, $\gamma\varepsilon - e\sigma < 0$ and $\mu\varepsilon - g\sigma < 0$**.** Hence, the conditions of the existence of the equilibrium points P_7, P_9 and P_{10} can not be satisfied so that these points are not existed.
2. Now, if one of the equilibrium points $P_7, P_9,$ or P_{10} exists. Therefore, $\alpha\varepsilon - c\sigma > 0$, $\gamma\varepsilon - e\sigma > 0$ or $\mu\varepsilon - g\sigma > 0$, that is $\frac{\sigma}{\varepsilon} < \frac{\alpha}{c}$, $\frac{\sigma}{\varepsilon} < \frac{\gamma}{e}$ *or* $\frac{\sigma}{\varepsilon} < \frac{\mu}{g}$. So that P_4 is locally not stable. Thus, the proof is completed.

4. BASIN OF ATTRACTION

In this section, a basin of attraction for all equilibrium points of the system (1), which are locally asymptotically stable**,** will be discussed by finding a suitable Lyapunov function for the mentioned points.

THEOREM 5: If $P_4\left(0,0,0,\frac{\sigma}{\varepsilon}\right)$ is locally asymptotically stable and $f \leq a, m < b$, $h < d$, then the following region

$$Q_4 = \left\{(x, y, z, w): 0 \leq x, 0 \leq y, 0 \leq z, max\left\{\frac{\alpha}{c}, \frac{\gamma}{e}, \frac{\mu}{g}\right\} \leq w \leq \frac{\sigma}{\varepsilon}\right\},$$

is a basin of attraction for the equilibrium points P_4 $\frac{\sigma}{\varepsilon} > max\left\{\frac{\alpha}{c}, \frac{\gamma}{e}, \frac{\mu}{g}\right\}$.

Proof: Consider the following real valued function

$$V_4 = \left(x + y + z + \frac{\left(w - \frac{\sigma}{\varepsilon}\right)^2}{2}\right).$$

It is clear that $V_4(x, y, z, w) > 0$, for all $(x, y, z, w)\epsilon\mathcal{R}_+^4\backslash\left\{\left(0,0,0,\frac{\sigma}{\varepsilon}\right)\right\}$, and is zero at $\left(0,0,0,\frac{\sigma}{\varepsilon}\right)$. The function $V_4(x, y, z, w)$ is differentiable with respect to time t and its derivative is given by
$\dot{V}_4 = \dot{x} + \dot{y} + \dot{z} + \left(w - \frac{\sigma}{\varepsilon}\right)\dot{w}$.

It is easy to notes that $\dot{V}_4\left(0,0,0,\frac{\sigma}{\varepsilon}\right) = 0$.

To prove that $\dot{V}_4$ is negative in $Q_4\backslash\left\{\left(0,0,0,\frac{\sigma}{\varepsilon}\right)\right\}$, it is sufficient to prove that $\dot{x} + \dot{y} + \dot{z}$, and $\left(w - \frac{\sigma}{\varepsilon}\right)\dot{w}$ are negative in $Q_4\backslash\left\{\left(0,0,0,\frac{\sigma}{\varepsilon}\right)\right\}$, and this is what we will do as follows

$\dot{x}+\dot{y}+\dot{z}=\alpha x-\beta x^2-axy-bxz-cxw+\gamma y-\delta y^2-dyz-eyw+fyx+\mu z-\varphi z^2$
$-gzw+hyz+mxz=$
$=(\alpha-cw)x+y(\gamma-ew)+z(\mu-gw)+(f-a)yx+(m-b)xz+(h-d)yz$
$-\beta x^2-\delta y^2-\varphi z^2\leq-\beta x^2-\delta y^2-\varphi z^2<0,\ \forall\ (x,y,z,w)\epsilon Q_4\backslash\left\{\left(0,0,0,\frac{\sigma}{\varepsilon}\right)\right\},$
$\left(w-\frac{\sigma}{\varepsilon}\right)\dot{w}=\left(w-\frac{\sigma}{\varepsilon}\right)w(\sigma-\varepsilon w+rz+py+qx)\leq$
$\leq\left(w-\frac{\sigma}{\varepsilon}\right)w(rz+py+qx)<0,\ \forall\ (x,y,z,w)\epsilon Q_4\backslash\left\{\left(0,0,0,\frac{\sigma}{\varepsilon}\right)\right\}.$

Hence for all $(x,y,z,w)\epsilon Q_4\backslash\left\{\left(0,0,0,\frac{\sigma}{\varepsilon}\right)\right\}$, we have that $\dot{V_4}<0$, and $\dot{V_4}\left(0,0,0,\frac{\sigma}{\varepsilon}\right)=0$. So that V_4 is a Lyapunov function . Therefore Q_4 is a basin of attraction for the equilibrium points P_4. Thus, the proof of Theorem5 has been completed.

THEOREM 6: If $P_7(\breve{\alpha}_1,0,0,\breve{\alpha}_2)$ is locally asymptotically stable and, $h<d,\varepsilon\breve{\alpha}_2<\sigma,$ then the following region
$Q_7=\left\{(x,y,z,w):\breve{\alpha}_1\leq x\leq min\left\{\frac{\alpha e-\gamma c}{fc},\frac{\alpha g-\mu c}{mc}\right\},0\leq y,0\leq z,\frac{\alpha}{c}<w\leq\breve{\alpha}_2\right\},$
is a basin of attraction for the equilibrium points P_7.

Proof: Consider the following real valued function
$V_7=\frac{(x-\breve{\alpha}_1)^2}{2}+y+z+\frac{(w-\breve{\alpha}_2)^2}{2}.$

It is clear that $V_7(x,y,z,w)>0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(\breve{\alpha}_1,0,0,\breve{\alpha}_2)\}$, and is zero at $(\breve{\alpha}_1,0,0,\breve{\alpha}_2)$. The function $V_7(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by
$\dot{V_7}=(x-\breve{\alpha}_1)\dot{x}+\dot{y}+\dot{z}+(w-\breve{\alpha}_2)\dot{w}.$
It is easy to notes that $\dot{V_7}=0$, at $(x,y,z,w)=(\breve{\alpha}_1,0,0,\breve{\alpha}_2)$.
To prove that $\dot{V_7}$ is negative in $Q_7\backslash\{(0,\breve{\breve{\alpha}}_1,0,\breve{\breve{\alpha}}_2)\}$, it is sufficient to prove that $(x-\breve{\alpha}_1)\dot{x}$, $(w-\breve{\alpha}_2)$ and $\dot{y}+\dot{z}w$ are negative in $Q_7\backslash\{(0,\breve{\breve{\alpha}}_1,0,\breve{\breve{\alpha}}_2)\}$, and this is what we will do as follows
$(x-\breve{\alpha}_1)\dot{x}=(x-\breve{\alpha}_1)x(\alpha-\beta x-ay-bz-cw)\leq(x-\breve{\alpha}_1)x\big((\alpha-cw)-ay-bz-cw\big)<0,\ \forall\ (x,y,z,w)\epsilon Q_7\backslash\{(\breve{\alpha}_1,0,0,\breve{\alpha}_2)\}.$
$(w-\breve{\alpha}_2)\dot{w}=(w-\breve{\alpha}_2)\,w(\sigma-\varepsilon w+rz+py+qx)<(w-\breve{\alpha}_2)\,w\big((\sigma-\varepsilon\breve{\alpha}_2)+rz+py+qx\big)0,\ \forall\ (x,y,z,w)\epsilon Q_7\backslash\{(\breve{\alpha}_1,0,0,\breve{\alpha}_2)\}.$
We will now prove that $\dot{y}+\dot{z}$ is negative $Q_7\backslash\{(\breve{\alpha}_1,0,0,\breve{\alpha}_2)\}$.
$\dot{y}+\dot{z}=\gamma y-\delta y^2-dyz-eyw+fyx+\mu z-\varphi z^2-gzw+hyz+mxz=$
$=-\delta y^2-\varphi z^2+yz(h-d)+y(\gamma-ew+fx)+z(\mu-gw+mx)$
$\leq=-\delta y^2-\varphi z^2+y\left(\gamma-e\frac{\alpha}{c}+fx\right)+z(\mu-g\frac{\alpha}{c}+mx)<0$
$\leq=-\delta y^2-\varphi z^2+y\left(\gamma-e\frac{\alpha}{c}+fx\right)+z(\mu-g\frac{\alpha}{c}+mx)<0,\qquad\forall(x,y,z,w)\epsilon Q_7\backslash\{(\breve{\alpha}_1,0,0,\breve{\alpha}_2)\}$.

Hence $\dot{V_7}<0$ for all $(x,y,z,w)\ \epsilon\ Q_7\backslash\{(\breve{\alpha}_1,0,0,\breve{\alpha}_2)\}$, and $\dot{V_7}(\breve{\alpha}_1,0,0,\breve{\alpha}_2)=0$. So that V_7 is a Lyapunov function. Therefore Q_7 is a basin of attraction for the equilibrium points P_7. Thus, the proof of Theorem6 is done.

THEOREM 7: If $P_9(0,\breve{\breve{\alpha}}_1,0,\breve{\breve{\alpha}}_2)$ is locally asymptotically stable and, $m\leq b$, $\varepsilon\breve{\breve{\alpha}}_2<\sigma$, then the following region
$Q_9=\left\{(x,y,z,w):0\leq x<\frac{\alpha e-\delta c}{fc},0\leq y<\frac{g\alpha-\mu c}{hc},0\leq z,\frac{\alpha}{c}<w\leq\breve{\breve{\alpha}}_2\right\},$

is a basin of attraction for the equilibrium points P_9.

Proof: Consider the following real valued function

$$V_9 = x + \frac{\left(y - \widetilde{\overline{a}}_1\right)^2}{2} + z + \frac{\left(w - \widetilde{\overline{a}}_2\right)^2}{2}.$$

It is clear that $V_9(x,y,z,w) > 0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}$, and is zero at $(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)$. The function $V_9(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by

$$\dot{V}_9 = \dot{x} + \left(y - \widetilde{\overline{a}}_1\right)\dot{y} + \dot{z} + \left(w - \widetilde{\overline{a}}_2\right)\dot{w}.$$

It is easy to notice that $\dot{V}_7 = 0$, at $(x,y,z,w) = (0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)$.

To prove that V_9 is negative in $Q_7\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}$, it is sufficient to prove that $\dot{x} + \dot{z}$, $\left(y - \widetilde{\overline{a}}_1\right)\dot{y}$, and $\left(w - \widetilde{\overline{a}}_2\right)\dot{w}$ are negative in $Q_9\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}$, and this is done as follows

$$\dot{x} + \dot{z} = x(\alpha - \beta x - ay - bz - cw) + z(\mu - \varphi z - gw + hy + mx) \leq$$
$$\leq (m-b)xz - (\beta x^2 + \varphi z^2) + x\left(\alpha - ay - c\frac{\alpha}{c}\right) + z\left(\mu - g\frac{\alpha}{c} + hy\right) \leq$$
$$\leq -(\beta x^2 + \varphi z^2) < 0, \ \forall\ (x,y,z,w)\epsilon Q_9\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}.$$

$$\dot{y} = y(\gamma - \delta y - dz - ew + fx) \leq -\delta y^2 - dyz + y\left(\gamma - e\frac{\alpha}{c} + fx\right) < -\delta y^2 < 0,$$
$\forall (x,y,z,w)\epsilon Q_9\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}$.

$$\left(w - \widetilde{\overline{a}}_2\right)\dot{w} = \left(w - \widetilde{\overline{a}}_2\right)w(\sigma - \varepsilon w + rz + py + qx) < \left(w - \widetilde{\overline{a}}_2\right)w\left(\sigma - \varepsilon\,\widetilde{\overline{a}}_2\right), < 0 \quad , \quad \forall$$
$(x,y,z,w)\epsilon Q_9\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}$.

Hence $\dot{V}_9 < 0$ for all $(x,y,z,w)\epsilon Q_9\backslash\{(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)\}$, and $\dot{V}_9(0,\widetilde{\overline{a}}_1,\dot{0},\widetilde{\overline{a}}_2) = 0$. So that V_9 is a Lyapunov function. Therefore Q_9 is a basin of attraction for the equilibrium points $P_9(0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)$. Thus, the proof of Theorem7 is done.

THEOREM 8 : If $P_{10}(0,0,\hat{a}_1,\hat{a}_2)$ is locally asymptotically stable and $f \leq a$, $\varepsilon\hat{a}_2 < \sigma$, and $\mu > \varphi\hat{a}_1 + g\hat{a}_2$, then the following region

$$Q_{10} = \left\{(x,y,z,w): 0 \leq x, 0 \leq y, z \leq \hat{a}_1, \max\{\tfrac{\alpha}{c},\tfrac{\gamma}{e}\} \leq w \leq \hat{a}_2\right\},$$

is a basin of attraction for the equilibrium points P_{10}.

Proof: Consider the following real valued function

$$V_{10} = x + y + \frac{(z - \hat{a}_1)^2}{2} + \frac{(w - \hat{a}_2)^2}{2}.$$

It is clear that $V_{10}(x,y,z,w) > 0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(0,0,\hat{a}_1,\hat{a}_2)\}$, and is zero at $(0,0,\hat{a}_1,\hat{a}_2)$. The function $V_{10}(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by

$$\dot{V}_{10} = \dot{x} + \dot{y} + (z - \hat{a}_1)\dot{z} + (w - \hat{a}_2)\dot{w}.$$

It is easy to notice that $\dot{V}_{10} = 0$, at $(x,y,z,w) = (0,\widetilde{\overline{a}}_1,0,\widetilde{\overline{a}}_2)$. To prove that $\dot{V}_{10}$ is negative in $Q_{10}\backslash\{(0,0,\hat{a}_1,\hat{a}_2)\}$, it is sufficient to prove that $\dot{x} + \dot{y}, (z - \hat{a}_1)\dot{z}$ and $(w - \hat{a}_2)\dot{w}$ are negative in $Q_{10}\backslash\{(0,0,\hat{a}_1,\hat{a}_2)\}$, and this is done as follows

$$\dot{x} + \dot{y} = x(\alpha - \beta x - ay - bz - cw) + y(\gamma - \delta y - dz - ew + fx) =$$
$$= x(\alpha - cw) + y(\gamma - ew) + (f - a)xy + x(-\beta x - bz) + y(-\delta y - dz) < 0 \quad , \quad \forall$$
$\forall (x,y,z,w)\epsilon Q_{10}\backslash\{(0,0,\hat{a}_1,\hat{a}_2)\}$.

$(z - \hat{a}_1)\dot{z} = (z - \hat{a}_1)z(\mu - \varphi z - gw + hy + mx) < (z - \hat{a}_1)z(\mu - \varphi\hat{a}_1 - g\hat{a}_2)(z - \hat{a}_1)z(hy + mx) < 0, \forall\ (x,y,z,w)\epsilon Q_{10}\backslash\{(0,0,\hat{a}_1,\hat{a}_2)\}$.

$(w - \hat{a}_2)\dot{w} = (w - \hat{a}_2)\,w(\sigma - \varepsilon w + rz + py + qx) <$

$< (w-\hat{a}_2)\, w(\sigma-\varepsilon\hat{a}_2)+(w-\hat{a}_2)\, w(rz+py+qx), < 0, \qquad \forall (x,y,z,w)\epsilon Q_{10} \backslash\{(0,0,\hat{a}_1,\hat{a}_2)\}$.

Hence $\dot{V}_{10} < 0$, for all $(x,y,z,w)\epsilon Q_{10}\backslash\{(\breve{a}_1,0,0,\breve{a}_2)\}$, and $\dot{V}_{10}(0,0,\hat{a}_1,\hat{a}_2)=0$. So that V_{10} is a Lyapunov function. Therefore Q_{10} is a basin of attraction for the equilibrium points $P_{10}(0,0,\hat{a}_1,\hat{a}_2)$. . Thus, the proof of Theorem 8 is completed.

THEOREM 9: If $P_{12}(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)$ is locally asymptotically stable, $\alpha<\beta\breve{b}_1$ $ehf\,\tilde{b}_2<\delta\tilde{b}_2(g-me)+e(m\gamma-f\mu)$, $ef\tilde{b}_3>\delta\tilde{b}_2$, $\varepsilon\tilde{b}_3\le\sigma$, then the following region

$$Q_{12}=\left\{(x,y,z,w):\tilde{b}_1\le x\le\frac{\delta\tilde{b}_2-\gamma}{f},\tilde{b}_2\le y\le\frac{\delta\tilde{b}_2(g-me)+e(m\gamma-f\mu)}{efh},0\le z,\frac{\delta\tilde{b}_2}{ef}\le w\le\tilde{b}_3\right\},$$

is a basin of attraction for the equilibrium points P_{12}.

Proof: Consider the following real valued function

$$V_{12}=\frac{\left(x-\tilde{b}_1\right)^2}{2}+\frac{\left(y-\tilde{b}_2\right)^2}{2}+z+\frac{\left(w-\tilde{b}_3\right)^2}{2}.$$

It is clear that $V_{12}(x,y,z,w)>0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)\}$, and it is zero at $(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)$. The function $V_{12}(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by

$$\dot{V}_{12}=(x-\tilde{b}_1)\dot{x}+(y-\tilde{b}_2)\dot{y}+\dot{z}+(w-\tilde{b}_3)\dot{w}$$

It is easy to notice that $\dot{V}_{12}=0$, at $(x,y,z,w)=(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)$. To prove that $\dot{V}_{12}$ is negative in $Q_{12}\backslash\{(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)\}$, it is sufficient to prove that $(x-\tilde{b}_1)\dot{x},0,(y-\tilde{b}_2)\dot{y}$.

$\dot{z}$, amd $(w-\tilde{b}_3)\dot{w}$ are negatives in $Q_{12}\backslash\{(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)\}$, and this is done as follows

$$\begin{aligned}(x-\tilde{b}_1)\dot{x} &= (x-\tilde{b}_1)x(\alpha-\beta x-ay-bz-cw)\\ &\le (x-\tilde{b}_1)x\left((\alpha-\beta\tilde{b}_1)-ay-bz-cw\right)\le(x-\tilde{b}_1)x(-ay-bz-cw)\\ &<0.\end{aligned}$$

$$(y-\tilde{b}_2)\dot{y}=y(y-\tilde{b}_2)(\gamma-\delta y-dz-ew+fx)\le(y-\tilde{b}_2)y\left(\gamma-\delta\tilde{b}_2-dz-\frac{\delta\tilde{b}_2}{f}+f\frac{\delta\tilde{b}_2-\gamma}{f}\right)\le y(y-\tilde{b}_2)\left(-dz-\frac{\delta\tilde{b}_2}{f}\right)<0.$$

$$\begin{aligned}\dot{z} &= z(\mu-\varphi z-gw+hy+mx)\le z\left(\mu-\varphi z-g\frac{\delta\tilde{b}_2}{ef}+hy+m\frac{\delta\tilde{b}_2-\gamma}{f}\right)\\ &\le-\varphi z^2+z\left(hy-\frac{\delta\tilde{b}_2(g-me)+e(m\gamma-f\mu)}{efh}\right)\le-\varphi z^2<0.\end{aligned}$$

$$\begin{aligned}(w-\tilde{b}_3)w(\sigma-\varepsilon w+rz+py+qx) &\le (w-\tilde{b}_3)w\left((\sigma-\varepsilon\tilde{b}_3)+rz+py+qx\right)\le\\ &\le(w-\tilde{b}_3)w(rz+py+qx)<0.\end{aligned}$$

Hence $\dot{V}_{12}<0$, for all $(x,y,z,w)\epsilon Q_{12}\backslash\{(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)\}$, and $\dot{V}_{12}(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)=0$. So that V_{12} is a Lyapunov function. Therefore Q_{12} is a basin of attraction for the equilibrium points $P_{12}(\tilde{b}_1,\tilde{b}_2,0,\tilde{b}_3)$. Thus, the proof of Theorem 9 is completed.

THEOREM 10: If $P_{13}(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)$ is locally asymptotically stable, $\beta\breve{b}_1<\alpha$ $\varphi\breve{b}_2+g\breve{b}_3\le\mu+m\breve{b}_1$, and $\varepsilon\breve{b}_3\le\sigma$, then the following region

$$Q_{13}=\left\{(x,y,z,w):\breve{b}_1\le x\le\frac{\alpha}{\beta},0\le y,z\le\breve{b}_2,\frac{f\alpha+\gamma\beta}{e\beta}\le w\le\breve{b}_3\right\}$$

is a basin of attraction for the equilibrium points P_{13}.

Proof: Consider the following real valued function

$V_{13} = \frac{(x-\breve{b}_1)^2}{2} + y + \frac{(z-\breve{b}_2)^2}{2} + \frac{(w-\breve{b}_3)^2}{2}.$

It is clear that $V_{13}(x,y,z,w) > 0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)\}$, and is zero at $(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)$. The function $V_{13}(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by

$\dot{V}_{13} = \dot{x}(x-\breve{b}_1) + \dot{y} + (z-\breve{b}_2)\dot{z} + (w-\breve{b}_3)\dot{w}.$

It is easy to notice that $\dot{V}_{13} = 0$, at $(x,y,z,w) = (\breve{b}_1,0,\breve{b}_2,\breve{b}_3)$.

To prove that $\dot{V}_{13}$ is negative $Q_{13}\backslash\{(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)\}$

From the conditions of Q_{13}, we have

$(w-\breve{b}_1) > 0, (z-\breve{b}_2) < 0$, and $(w-\breve{b}_3) < 0$.

So, in order to prove that $\dot{V}_{13}$ is negative, it is sufficient to prove that

$\dot{x} < 0, \dot{y} < 0, \dot{z} > 0$ and $\dot{w} > 0$, and this is done as follows

$\dot{x} = x(\alpha - \beta x - ay - bz - cw) \le x\left((\alpha - \beta\breve{b}_1) - ay - bz - cw\right) \le x(-ay - bz - cw) < 0.$

$\dot{y} = y(\gamma - \delta y - dz - ew + fx) \le y\left(\gamma - e\frac{f\alpha+\gamma\beta}{e\beta} + f\frac{\alpha}{\beta}\right) + y(-\delta y - dz) = y(-\delta y - dz) < 0.$

$\dot{z} = z(\mu - \varphi z - gw + hy + mx) \ge z(\mu + m\hat{b}_1 - \varphi\hat{b}_2 - g\hat{b}_3 + hy) \ge z(hy) > 0.$

$\dot{w} = w(\sigma - \varepsilon w + rz + py + qx) \ge w\left((\sigma - \varepsilon\hat{b}_3) + rz + py + qx\right) \ge w(rz + py + qx) > 0.$

Hence $\dot{V}_{13} < 0$, for all $(x,y,z,w)\epsilon Q_{14}\backslash\{(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)\}$, and $\dot{V}_{13}(\breve{b}_1,0,\dot{\breve{b}}_2,\breve{b}_3) = 0$. So that V_{13} is a Lyapunov function. Therefore Q_{13} is a basin of attraction for the equilibrium points $P_{13}(\breve{b}_1,0,\breve{b}_2,\breve{b}_3)$. Thus, the proof of Theorem 10 is done.

THEOREM 11: If $P_{14}(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)$ is locally asymptotically stable, $\alpha \le a\hat{b}_1$, $\varphi\hat{b}_2 + g\hat{b}_3 \le \mu + h\hat{b}_1$ and $\varepsilon\hat{b}_3 \le \sigma$, then the following region

$Q_{14} = \left\{(x,y,z,w): 0 \le x < \frac{\delta\hat{b}_1}{f}, \hat{b}_1 \le y, \frac{\gamma}{d} < z \le \hat{b}_2, w \le \hat{b}_3\right\}$

is a basin of attraction for the equilibrium points P_{14}.

Proof: Consider the following real valued function

$V_{14} = x + \frac{(y-\hat{b}_1)^2}{2} + \frac{(z-\hat{b}_2)^2}{2} + \frac{(w-\hat{b}_3)^2}{2}.$

It is clear that $V(x,y,z,w) > 0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)\}$, and is zero at $(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)$. The function $V(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by

$\dot{V}_{14} = \dot{x} + (y-\hat{b}_1)\dot{y} + (y-\hat{b}_2)\dot{z} + (y-\hat{b}_3)\dot{w}$

It is easy to notice that $\dot{V}_{14} = 0$, at $(x,y,z,w) = (0,\hat{b}_1,\hat{b}_2,\hat{b}_3)$.

To prove that $\dot{V}_{14}$ is negative $Q_{14}\backslash\{(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)\}$, it is sufficient to show that $\dot{x}, (y-\hat{b}_1)\dot{y}, (y-\hat{b}_2)\dot{z}$, and $(y-\hat{b}_3)\dot{w}$ are negative, and this is done as follows

$\dot{x} = x(\alpha - \beta x - ay - bz - cw) \le x\left((\alpha - a\hat{b}_1) - \beta x - bz - cw\right) \le x(-\beta x - bz - cw) < 0.$

$$
\begin{aligned}
(y-\hat{b}_1)\dot{y} &= (y-\hat{b}_1)y(\gamma-\delta y-dz-ew+fx)\\
&\le (y-\hat{b}_1)y\left(\gamma-\delta\hat{b}_1-d\frac{\gamma}{d}-ew+fx\right)\\
&\le (y-\hat{b}_1)y\left(-\delta\hat{b}_1-d\frac{\gamma}{d}-ew+fx\right)\\
&\le -(y-\hat{b}_1)eyw+(y-\hat{b}_1)y(fx-\delta\hat{b}_1) < -eyw < 0\\
(z-\hat{b}_2)\dot{z} &= (z-\hat{b}_2)z(\mu-\varphi z-gw+hy+mx)\\
&\le (z-\hat{b}_2)z(\mu-\varphi\hat{b}_2+h\hat{b}_1-g\hat{b}_3+mx) \le (z-\hat{b}_2)mxz < 0\\
(w-\hat{b}_3)\dot{w} &= (w-\hat{b}_3)w(\sigma-\varepsilon w+rz+py+qx)\\
&\le (w-\hat{b}_3)w\left((\sigma-\varepsilon\hat{b}_3)+rz+py+qx\right) \le (w-\hat{b}_3)w(+rz+py+qx)\\
&< 0.
\end{aligned}
$$

Hence $\dot{V}_{14} < 0$, for all $(x,y,z,w)\epsilon Q_{14}\backslash\{(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)\}$, and $\dot{V}_{14}(0,\hat{b}_1,\dot{\hat{b}}_2,\hat{b}_3) = 0$. So that V_{14} is a Lyapunov function. Therefore Q_{14} is a basin of attraction for the equilibrium points $P_{14}(0,\hat{b}_1,\hat{b}_2,\hat{b}_3)$. Thus, the proof of Theorem11 is done.

THEOREM 12: If $P_{15}(c_1,c_2,c_3,c_4)$ is locally asymptotically stable, $\beta c_1 \ge \alpha$, $\varphi c_3 + gc_4 \le \mu + hc_2$ and $\varepsilon c_4 \le \sigma$, then the following region

$Q_{15} = \left\{(x,y,z,w): c_1 \le x \le \frac{\delta c_2}{f}, c_2 \le y, \frac{\gamma}{d} \le z \le c_3, w \le c_4\right\}$

is a basin of attraction for the equilibrium points P_{15}.

Proof: Consider the following real valued function

$$V_{15} = \frac{(x-c_1)^2}{2} + \frac{(y-c_2)^2}{2} + \frac{(z-c_3)^2}{2} + \frac{(w-c_4)^2}{2}.$$

It is clear that $V_{15}(x,y,z,w) > 0$, for all $(x,y,z,w)\epsilon\mathcal{R}_+^4\backslash\{(c_1,c_2,c_3,c_4)\}$, and is zero at (c_1,c_2,c_3,c_4). The function $V(x,y,z,w)$ is differentiable with respect to time t and its derivative is given by

$\dot{V} = (x-c_1)\dot{x} + (y-c_2)\dot{y} + (z-c_3)\dot{z} + (w-c_4)\dot{w}$

It is easy to notice that $\dot{V}_{15} = 0$, at $(x,y,z,w) = (c_1,c_2,c_3,c_4)$

To prove that $\dot{V}_{14}$ is negative $Q_{15}\backslash\{(c_1,c_2,c_3,c_4)\}$, , it is sufficient to show that $(x-c_1)\dot{x}$, $(y-c_2)\dot{y}$, $(z-c_3)\dot{z}$, and $(w-c_4)\dot{w}$, are negatives in the region $Q_{15}\backslash\{(c_1,c_2,c_3,c_4)\}$, and this is done as follows

$$
\begin{aligned}
(x-c_1)\dot{x} &= (x-c_1)x(\alpha-\beta x-ay-bz-cw) \le (x-c_1)x(-ay-bz-cw) < 0.\\
(y-c_2)\dot{y} &= (y-c_2)y(\gamma-\delta y-dz-ew+fx) \le (y-c_2)y(\gamma-dz+fx-\delta c_2-ew)\\
&\le -exw(y-c_2) < 0.\\
(z-c_3)\dot{z} &= (z-c_3)z(\mu-\varphi z-gw+hy+mx)\\
&\le (z-c_3)z(\mu-\varphi c_3-gc_4+hc_2+mx) \le mxz(z-c_3) < 0\\
(w-c_4)\dot{w} &= (w-c_4)w(\sigma-\varepsilon w+rz+py+qx) \le (w-c_4)w(+rz+py+qx) < 0
\end{aligned}
$$

Hence $\dot{V}_{15} < 0$, for all $(x,y,z,w)\epsilon Q\backslash\{(c_1,c_2,c_3,c_4)\}$, and $\dot{V}(c_1,c_2,c_3,c_4) = 0$. So that V_{15} is a Lyapunov function. Therefore Q_{15} is a basin of attraction for the equilibrium points $P_{15}(c_1,c_2,c_3,c_4)$. Thus, the proof of Theorem12 is done.

5. NUMERICAL SIMULATIONS AND DISCUSSIONS :

In this section, a numerical example of the system (1) will be given for the following set of parameters.

$\alpha = 0.34, \beta = 0.2, a = 0.4, b = 0.3, c = 0.1, \gamma = 0.18, \delta = 0.4, d = 0.3, e = 0.$
$f = 0.2, \mu = 0.09, \varphi = 0.5, g = 0.4, h = 0.3, m = 0.6, \sigma = 0.2, \varepsilon = 0.4, r = 0.1,$
$p = 0.2, q = 0.1.$

The purpose of the presenting this example is to verify and confirm the theoretical results that have been obtained in Section 3, it was demonstrated in Theorem 2 that the equilibrium points P_0, P_1, P_2, P_3, $P_5(0.4,0.65,0,0)$, $P_6(0.5107,0,0.729,0)$, $P_8(0,02172,0.3103,0)$ and $P_{11}(0.4,0.107,0.724,0)$ are unstable points. So we will find the rest of the points and determine the local stability for each of them. Easy calculations show that the points $P_4(0,0,0,0.5)$, $P_7(0.2889,0,0,0.8222)$, $P_9(0,0.359,0,0.5765)$, $P_{12}(0.4,0.4444,0,0.822)$ and, $P_{13}(0.705,0,0.404,0.777)$ are not stable because the conditions given in 2., 3., 4., of Theorem 2 and the conditions given in 1., 2., 3., of Theorem 3 are not met, for each equilibrium point respectively, see Figures 1-5. While the two equilibrium points $P_{10}(0,0,\hat{a}_1,\hat{a}_2)$, and $P_{14}\left(0,\hat{b}_1,\hat{b}_2,\hat{b}_3\right)$ do not exist, because the conditions given in 7., and 12. of Section 4 are not satisfied, for each point respectively. The last equilibrium point of the system (1), is $P_{15}(0.4,0.3,0.2,0.8)$. This point is locally asymptotically stable due to the fulfillment of the conditions 4., of Theorem 3 that has been mentioned in the Section 4, that is:

$$\begin{cases} \Delta_1 = 0.62,\ \Delta_2 = 0.1774, \quad \Delta_3 = 0.026,\ \Delta_{14} = 1.4274x10^{-5} \\ \Delta_1\Delta_2 - \Delta_3 = 0.0840 > 0, \\ \Delta_3(\Delta_1\Delta_2 - \Delta_3) - \Delta_4\Delta_1^2 = 0.0022 > 0, \end{cases}$$

see Figures 1-6.

Now, if $\alpha = \gamma = 0.04$, but the rest of the parameters keep their values as given. The equilibrium points P_5, P_6, P_7, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14} and P_{15} do not exists because the conditions given in 4., 6., 7., 9., 10., 11. and 14. As for equilibrium point P_4, it is locally asymptotically stable, see Figure 7.

The system (1) with the parameters shown in section 6 has 14 equilibrium points, 13 of which are unstable and only one point, which is the coexistence point P_{15}, is locally asymptotically stable. The point $P_{10}(0,0,\hat{a}_1,\hat{a}_2)$, and $P_{14}\left(0,\hat{b}_1,\hat{b}_2,\hat{b}_3\right)$ do not exist. According to Theorem 4, if P_7, P_9 and P_{10} do not exist then P_4 is locally asymptotically stable. Which follows that if both α and γ are smaller than 0.05 and μ is are smaller than 0.2, then P_4 is locally asymptotically stable, which annihilates the equilibrium points P_7, P_9 and P_{10}. Furthermore, eight equilibrium points were annihilated due to the change in these two parameters. This is due to the influence of each parameter on the dynamic behaviors of the model.

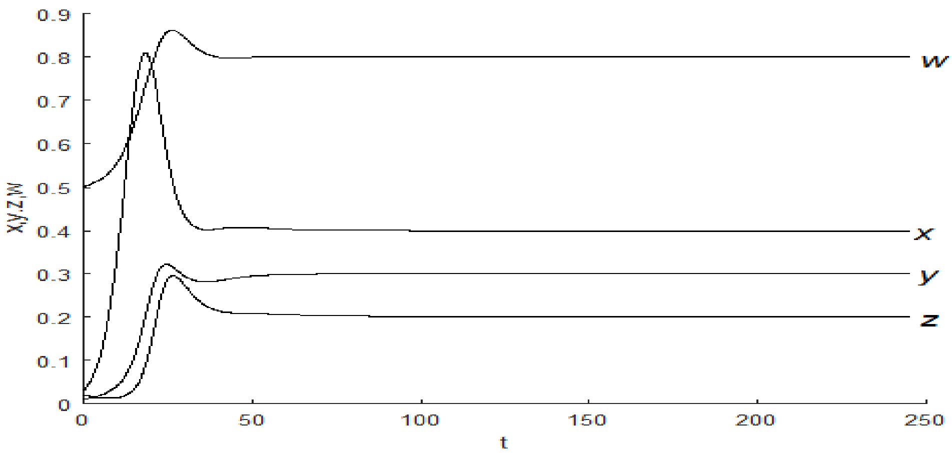

Figure 1: P_4 is unstable point. The trajectory with the parameter values is given above, and for the initial point (0.03,0.01 ,0.02,0.5), located close P_4 and it diverges from P_4 and, converges to P_{15}.

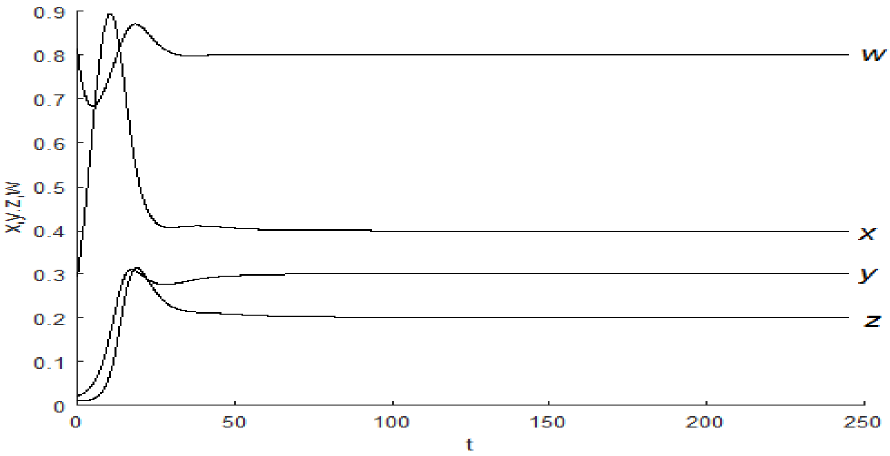

Figure 2: $P_7$is unstable point. The trajectory with the parameter values is given above, and for the initial point $(0.28,0.02,0.01,0.82)$, located closeP_7 and it diverges from P_7 and, converges to P_{15}..

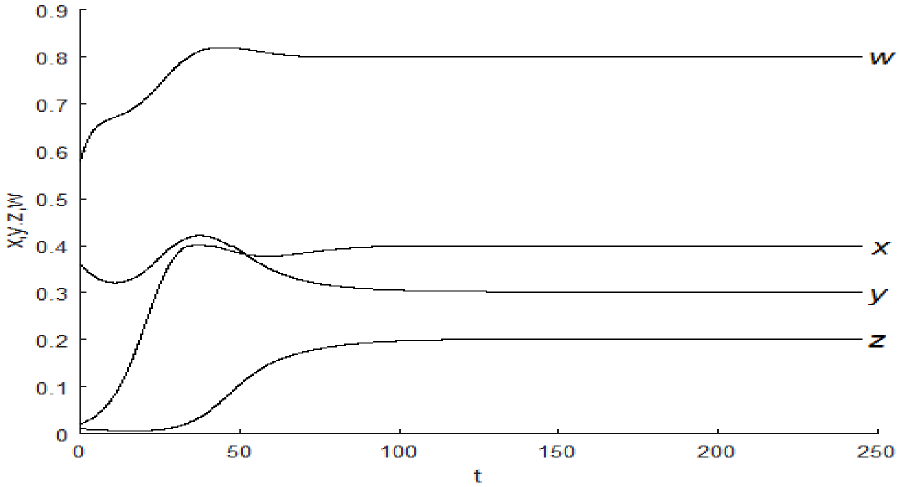

Figure 3: $P_9$is unstable point. The trajectory with the parameter values is given above, and for the initial point $(0.02,0.358,0.01,0.577)$, located close P_9 and it diverges from P_9 and, converges to P_{15}.

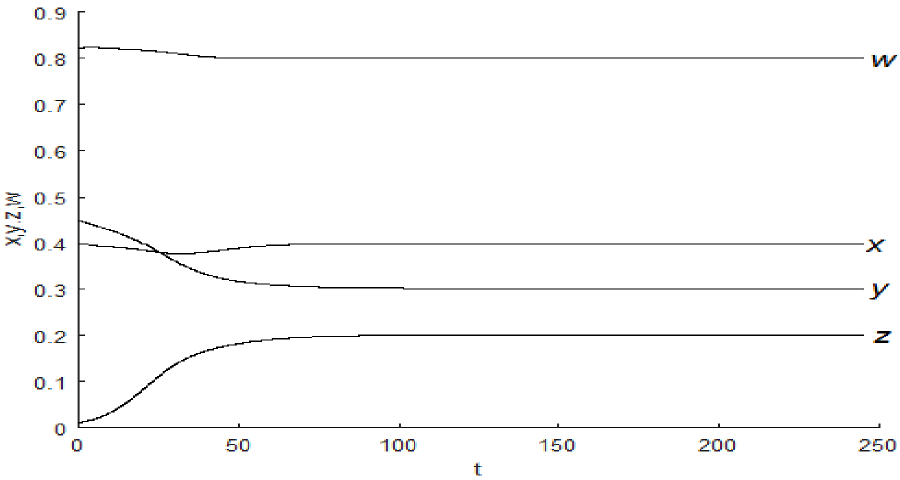

Figure 4: $P_{12}$is unstable point. The trajectory with the parameter values is given above, and for the initial point $(0.4,0.45,0.1,0.82)$, located close P_{12} and it diverge from P_{12} and, converges to P_{15}..

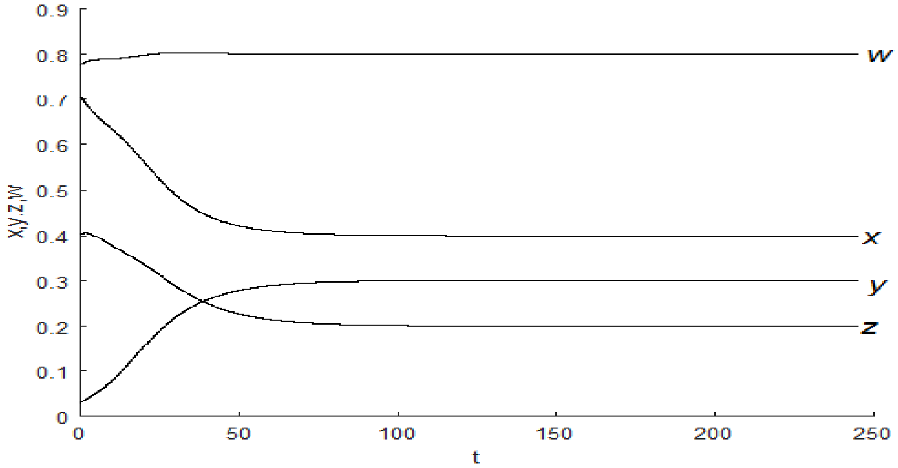

Figure 5: $P_{13}$is unstable point. The trajectory with the parameter values is given above, and for the initial point (0.705,0.03 ,0.403,0.777), located close P_{13} , diverge from P_{13} and, converges to P_{15}..

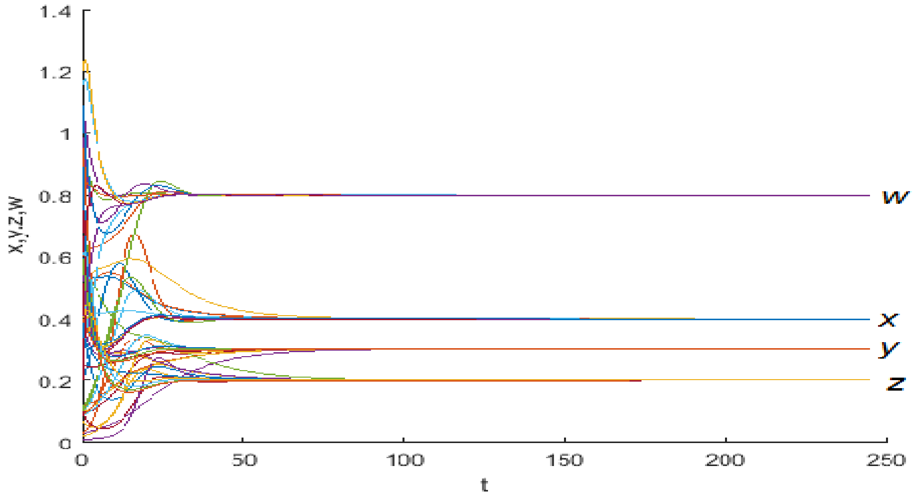

Figure 6: $P_{15}$is locally stable point. The trajectory with the parameter values given above and, for different initial points, converges to P_{15}.

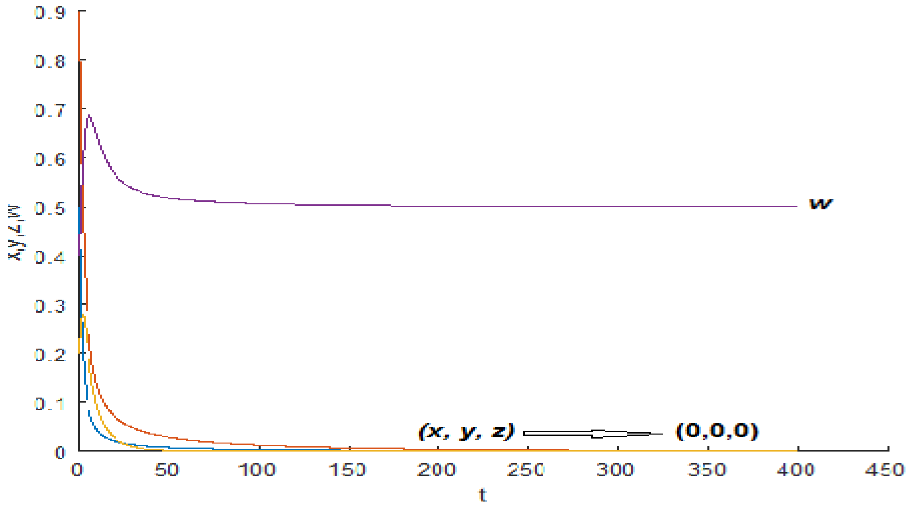

Figure 7: $P_4$is locally stable point. The trajectory when $\alpha = \gamma = 0.04$ and the rest parameter keep their values as given above, and the initial point (0.5,0.6,0.3,0.1), converges to $P_4(0,0,0,0.5)$.

6. CONCLOSIONS

In this paper, a modified model of the Lotka-Volterra model was presented such that the proposed model is a complete food chain consisting of four species. The model has sixteen possible equilibrium points; five of them always exist, whatever the values of the model parameters. The number of unstable equilibrium points is eight, while the rest are locally asymptotically stable if they meet the conditions specified in this paper. For each of the equilibrium points that can be locally asymptotically stable, a basin of attraction was found using the Lyapunov function. In a numerical example, it is found that the number of equilibrium points for the system (1) was fourteen, all of which were unstable, except the coexistence point, which was locally asymptotically stable and, two points, do not exist in the mentioned examples due to not meeting some specific conditions. Changing two value parameters eliminates eleven equilibrium points.

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