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Convexity Properties for Integro-Differential Operators Proposed by Hurwitz-Lerch Zeta Type Functions

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Abstract:

In this paper, new integro-differential operators are introduced that defined by Salagean's differential operator. The major object of the present study is to investigate convexity properties on new geometric subclasses included these new operators.

Keywords: Analytic function, Convex function, Hurwitz–Lerch zeta function, Star-like function.

خصائص التحدب للمؤثرات التكاملية–التفاضلية الذي تم تعريفها بواسطة نوع من الدوال هورويتز ليرش زيتا

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الخلاصة:

في هذا البحث، يتم تقديم مؤثرات تفاضلية–متكاملة جديدة معرفة بواسطة مؤثر سالاجيان التفاضلي، الهدف الرئيسي، هدف هو مناقشة خواص التحدب في الفئات الجزئية الهندسية الجديدة التي تضمنت هؤلاء المؤثرات الجدد.

Introduction:

Let Γ symbolize the class of all functions have the form

$$g(\xi)=\xi+\sum_{m=2}^{\infty}a_m\xi^m,\ (\xi\in\Delta).$$

which is analytic in the open unit disk $\Delta=\{\xi\in C:\xi<1\}$ and $S=\{g\in\Gamma: g$ is univalent in $\Delta\}$.

For $0\le\gamma<1$ and $\xi\in\Delta$, the starlike function of order γ will be the subclass of Γ involving univalent function which is indicated by $S(\gamma)$ and convex function of order γ will be the subclass of Γ involving univalent function which is indicated by $C(\gamma)$, and are defined analytically by

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$Re\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right)>\gamma,$ and $Re\left(\frac{\xi g'(\xi)}{g(\xi)}\right)>\gamma,$ respectively, [1].

The class K and S^* of convex functions and starlike functions, respectively are identical by $K \equiv K(0)$ and $S^* \equiv S^*(0)$. Bharti et al., [2] defined $k-S_p(\alpha)$ to be the class of functions g with $0 \le \alpha < 1$ and $0 \le k < \infty$ that satisfy the condition:

$$Re\left\{\frac{\xi\, g'(\xi)}{g(\xi)}\right\} \ge k\left|\frac{\xi\, g'(\xi)}{g(\xi)}-1\right|+\alpha.$$

Bharti et al., [2] defined $k-UCV(\alpha)$ to be the class of functions g with $0 \le \alpha < 1$ and $0 \le k < \infty$ that satisfy the condition:

$$Re\left\{1+\frac{\xi\, g''(\xi)}{g'(\xi)}\right\} \ge \left|\frac{\xi\, g''(\xi)}{g'(\xi)}\right|+\alpha.$$

On the other hand, the special functions (SFs) are quite advantageous in solving diverse types of differential equations. Those functions have great implementations in other fields of mathematics such as complex analysis, [3]. During the last century, the use of special functions (SFs) has been intensified fruitfully due to their importance in the Geometric Function Theory (GFT). The reason for attracting the authors towards SFs is that the class of hypergeometric functions was employed as a tool for resolving Bieberbach's problem in 1984 by de Branges, [4]. Afterward, numerous significant works on connections between analytic univalent and SFs have been discussed by several complex analysis such as, Jassim [5], Al-Janaby and Ahmad [6], Mahmoud et. al [7], Al-Janaby et. al [8 -10], Atshan et. al [11], Elhaddad and Darus [12], Yan and Liu [13], Oros [14], Ghanim et. al [15], Layth et. al [16] and Mahmood et. al [17].

The familiar $\Phi(\xi,s,v)$ Hurwitz–Lerch zeta function is shown by [18],

$$\Phi(\xi,s,v)=\sum_{m=0}^{\infty}\frac{\xi^m}{(m+v)^s}, \tag{1}$$

such that $(v \in Z^+; Re(s)>1$ where $|\xi|=1,\ s\in C$, and where $|\xi|<1)$.

In [19-21], more clarity can be seen about the exposition of the properties of different generalizations and applications of $\Phi(\xi,s,v)$.

The following extension beta function $B(\hbar_1,\hbar_2;\rho)$ introduced by Chaudhry et al. [22],

$$B(\hbar_1,\hbar_2;\rho)=\int_0^1 t^{\hbar_1-1}(1-t)^{\hbar_2-1}\exp\left(-\frac{\rho}{t(1-t)}\right)dt\,. \tag{2}$$

Furthermore, Choi et al. [23] created the underlying generalization of extended beta functions $B_{\rho,q}(\hbar_1,\hbar_2)$ given by:

$$B_{\rho,q}(\hbar_1,\hbar_2)=\int_0^1 t^{\hbar_1-1}(1-t)^{\hbar_2-1}\exp\left(-\frac{\rho}{t}-\frac{q}{1-t}\right)dt\,, \tag{3}$$

$(Re(q)>0; Re(\rho)>0$ and $Re(\hbar_1)>0; Re(\hbar_2)>0)$.

Motivated by those different fascinating extensions of $\Phi(\xi,s,v)$, researchers have created an extension of the generalized $\Phi(\xi,s,v)$ that includes $B_{p,q}(\hbar_1,\hbar_2;\rho,q)$ the extended beta function.

In [23] a new extension of the generalized Hurwitz-Lerch zeta functions $\Phi_{\delta,\varsigma;y}(\xi,s,v;\rho,q)$ involving $B_{p,q}(\hbar_1,\hbar_2;\rho,q)$ Eq.(3) given by

$$\Phi_{\delta,\varsigma;y}(\xi,s,v;\rho,q)=\sum_{m=0}^{\infty}\frac{B_{\rho,q}(\varsigma+m,\gamma-\varsigma)}{B(\varsigma,\gamma-\varsigma)}\frac{(\delta)_m}{m!}\frac{\xi^m}{(m+v)^s}, \tag{4}$$

such that $(q\ge 0,\rho\ge 0;\varsigma,\delta\in C;\gamma,v\in Z^+;\ Re(s+\gamma-\delta-\varsigma)>1$ where $|\xi|=1, s\in C$ and where $|\xi|<1)$.

For $g\in\Gamma,$ Layth et. al [24] introduced the following linear operator $F_{\delta,\varsigma;y}^{s,v;\rho,q}:\Gamma\to\Gamma$ as following

$$F_{\delta,\varsigma;y}^{s,v;\rho,q}g(\xi)=\Phi_{\delta,\varsigma;y}(\xi,s,v;\rho,q)*g(\xi)$$
$$=\xi+\sum_{m=2}^{\infty}\frac{\Gamma(\delta+m)}{\delta!m!}\frac{B_{\rho,q}(\varsigma+m,y-\varsigma)}{B_{\rho,q}(\varsigma+1,y-\varsigma)}\left(\frac{1+v}{m+v}\right)^s a_m\xi^m, \tag{5}$$

where $\Phi_{\delta,\varsigma;y}(\xi,s,v;\rho,q)$ given by Eq.(4) is the normalized extended Hurwitz-Lerch zeta function in terms of $B_{p,q}(\hbar_1,\hbar_2;\rho,q)$ given by Eq.(3).

Furthermore, for $F_{\delta,\varsigma;y}^{s,v;\rho,q}g(\xi)$ given in Eq.(5) and $g\in\Gamma,$ Layth et. al [24] considered the following Salagean's differential operator

$$\Theta_\lambda^0 g(\xi)=g(\xi),$$
$$\Theta_\lambda^1 g(\xi)=(1-\lambda)g(\xi)+\lambda\xi g'(\xi)=\Theta_\lambda g(\xi),\quad 1\ge\lambda\ge 0,$$
$$=\xi+\sum_{m=2}^{\infty}\frac{\Gamma(\delta+m)}{\delta!m!}\frac{B_{\rho,q}(\varsigma+m,y-\varsigma)}{B_{\rho,q}(\varsigma+1,y-\varsigma)}\left(\frac{1+v}{m+v}\right)^s a_m\xi^m,$$

then

$$\Theta_\lambda^T g(\xi)=\Theta_\lambda\left(\Theta_\lambda^{T-1}g(\xi)\right),\quad T\in N_0.$$
$$=\xi+\sum_{m=2}^{\infty}\left[\frac{\Gamma(\delta+m)}{\delta!m!}\frac{B_{\rho,q}(\varsigma+m,y-\varsigma)}{B_{\rho,q}(\varsigma+1,y-\varsigma)}\left(\frac{1+v}{m+v}\right)^s(1+(m-1)\lambda)\right]^T a_m\xi^m, \tag{6}$$

when

such that $(q\ge 0,\ \rho\ge 0;\ 0\le\lambda\le 1;\ \varsigma,\delta\in C;\ \gamma,v\in Z^+;\ Re(s+\gamma-\delta-\varsigma)>1$ when $|\xi|=1,\ s\in C$ when $|\xi|<1,\ T\in N_0)$.

Let $S_\lambda^\ell(\gamma)$ symbolize the class of function $g(\xi)\in\Gamma$, which satisfies the following condition:

$$Re\left(\frac{z\left(\Theta_\lambda^T g(\xi)\right)'}{\Theta_\lambda^T g(\xi)}\right)>\gamma, \tag{7}$$

for some $0\le\gamma<1$ and $T\in N_0$.

Additionally, in [24], Layth et. al presented the following subclass of uniformly star-like functions.
Let $x-S_{\lambda}^{T}(\gamma)$ symbolize the class of function $g(\xi)\in\Gamma$, such as

$$Re\left(\frac{z\left(\Theta_{\lambda}^{T}g(z)\right)'}{\Theta_{\lambda}^{T}g(z)}-\gamma\right)\geq x\left|\frac{z\left(\Theta_{\lambda}^{T}g(z)\right)'}{\Theta_{\lambda}^{T}g(z)}-1\right|,\quad (0\leq\gamma<1, z\in\Delta). \tag{8}$$

In this work, new generalized subclass of uniformly convex functions is introduced. Let $x-C_{\lambda}^{T}(\gamma)$ symbolize the class of function $g(z)\in\Gamma$, such as

$$Re\left(1+\frac{\xi\left(\Theta_{\lambda}^{T}g(\xi)\right)''}{\left(\Theta_{\lambda}^{T}g(\xi)\right)'}-\gamma\right)\geq x\left|\frac{\xi\left(\Theta_{\lambda}^{T}g(\xi)\right)''}{\left(\Theta_{\lambda}^{T}g(\xi)\right)'}\right|,\quad (0\leq\gamma<1,\xi\in\Delta)\,, \tag{9}$$

where $0\leq\gamma<1,\ x\geq 0,\ T\in N_0,\ \xi\in\Delta.$ Clearly, for $T=0$ the class $x-S_{\lambda}^{T}(\gamma)$ and the class $x-C_{\lambda}^{T}(\gamma)$ coincide respectively, as:

$$Re\left(1+\frac{\xi g''(\xi)}{g'(\xi)}\right)\geq x\left|\frac{\xi g''(\xi)}{g'(\xi)}-1\right|,\quad (\xi\in\Delta), \tag{10}$$

and

$$Re\left(\frac{\xi g'(\xi)}{g(\xi)}\right)\geq x\left|\frac{\xi g'(\xi)}{g(\xi)}-1\right|,\quad (\xi\in\Delta)\,. \tag{11}$$

Let $x-C_{\lambda}^{T}(\upsilon)$ symbolize the class of function $g(\xi)\in\Gamma$, which satisfies the following condition:

$$Re\left(1+\frac{\xi\left(\Theta_{\lambda}^{T}g(\xi)\right)''}{\left(\Theta_{\lambda}^{T}g(\xi)\right)'}\right)\geq x\left|\frac{\xi\left(\Theta_{\lambda}^{T}g(\xi)\right)''}{\left(\Theta_{\lambda}^{T}g(\xi)\right)'}\right|+\upsilon,\ (\xi\in\Delta), \tag{12}$$

for $T=0$ in Eq.(12), here the class studied in Shams et. al [25] in 2004.

Moreover, based on Salagean's operator Eq.(6), the following integral operators can be defined,

$$F_{u_1,\ldots,u_\ell}(\xi)=\int_0^{\xi}\left(\left(\Theta_{\lambda}^{T}g_1(t)\right)'\right)^{u_1}\ldots\left(\left(\Theta_{\lambda}^{T}g_\ell(t)\right)'\right)^{u_\ell}dt, \tag{13}$$

and

$$\psi_{u_1,\ldots,u_\ell}(\xi)=\int_0^{\xi}\left(\frac{\Theta_{\lambda}^{T}g_1(t)}{t}\right)^{u_1}\ldots\left(\frac{\Theta_{\lambda}^{T}g_\ell(t)}{t}\right)^{u_\ell}dt, \tag{14}$$

where $g_j\in\Gamma,\ u_j>0$ and $j=1,2,\ldots,\ell.$
For $T=0,$ there is the operator

$$F_{u_1,\ldots,u_\ell}(\xi)=\int_0^{\xi}\left(\left(g_1(t)\right)'\right)^{u_1}\ldots\left(\left(g_\ell(t)\right)'\right)^{u_\ell}dt,$$

was introduced by Breaz et. al [26].
For $T=0,\ell=1,u=u_1,$ the integral operator can be yield

$$F_u(\xi)=\int_0^{\xi}\left(\left(g(t)\right)'\right)^{u}dt,$$

was studied by Pascu and Pescar [27].

For $T=0$, the operator is give

$$\psi_{u_1,\ldots,u_\ell}(\xi)=\int_0^\xi\left(\frac{g_1(t)}{t}\right)^{u_1}\cdots\left(\frac{g_\ell(t)}{t}\right)^{u_\ell}dt,$$

was posed by Breaz and Breaz [28].

For $T=0, \ell=1, u=u_1,$ the operator can be got

$$\psi_u(\xi)=\int_0^\xi\left(\frac{g(t)}{t}\right)^{u}dt,$$

was considered by Miller et. al [29].

For $T=0, \ell=1$ and $u_1=1,$ the Alexander operator [30] can be obtained by,

$$\psi_u(\xi)=\int_0^\xi\frac{g(t)}{t}dt.$$

Next, for $f_j, g_j\in\Gamma$ and $0<\alpha_j, \beta_j, \; j\in\{1,2,\ldots,\ell\},$ the integro-differential operator was introduced as follows: $\omega_\ell(z):\Gamma^\ell\to\Gamma$, by

$$\omega_\ell(\xi)=\int_0^\xi\prod_{j=1}^{\ell}\left(\frac{\Theta_\lambda^T g_j(t)}{t}\right)^{\alpha_j}\left(\left(\Theta_\lambda^T f_j(t)\right)'\right)^{\beta_j}dt. \tag{15}$$

Remark:

1) For $\alpha_j=0,$ and $g_j=f_j$ operator Eq.(15) reduces to operator Eq.(13).

2) For $m_j=0$ this operator Eq.(15) concides operator given by Eq.(14).

3) Operator $\omega_\ell(\xi)$ Eq.(15) generalizes the integral operators imposed by Breaz et. al. [26], Pascu and Pescar [27], Breaz and Breaz [28], Miller et. al [29], Alexander [30], Frasin [31] and Stanciu and Breaz [32].

Main Results:

Theorem 1. If $g_j\in x_j-S_\lambda^T(\gamma_j)$ with $x_j\geq 0, 0\leq\gamma_j<1$ for all $\sum_{j=1}^{\ell}u_j\leq\frac{1}{2}$ and $j\in\{1,2,\ldots,\ell\},$ then the operator $\psi_{u_1,\ldots,u_\ell}\in C(\sigma)$ where $\sigma=1+\sum_{j=1}^{\ell}u_j(\gamma_j-1).$

Proof.

$$\frac{z\psi''_{u_1,\ldots,u_\ell}(z)}{\psi'_{u_1,\ldots,u_\ell}(z)}=u_1\left(\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)'}{\Theta_\lambda^T g_1(\xi)}-1\right)+\ldots+u_\ell\left(\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)'}{\Theta_\lambda^T g_\ell(\xi)}-1\right)$$

$$Re\left(\frac{\xi\psi''_{u_1,\ldots,u_\ell}(\xi)}{\psi'_{u_1,\ldots,u_\ell}(\xi)}\right)=u_1Re\left(\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)'}{\Theta_\lambda^T g_1(\xi)}-1\right)+\ldots+u_\ell Re\left(\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)'}{\Theta_\lambda^T g_\ell(\xi)}-1\right)$$

$$=u_1Re\left(\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)'}{\Theta_\lambda^T g_1(\xi)}\right)-u_1-u_1\gamma_1+u_1\gamma_1+\ldots+u_\ell Re\left(\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)'}{\Theta_\lambda^T g_\ell(\xi)}\right)-u_\ell-u_\ell\gamma_\ell+u_\ell\gamma_\ell$$

$$= u_1 Re\left(\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)'}{\Theta_\lambda^T g_1(\xi)} - \gamma_1\right) - u_1 + u_1\gamma_1 + \ldots + u_\ell Re\left(\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)'}{\Theta_\lambda^T g_\ell(\xi)} - \gamma_\ell\right) - u_\ell + u_\ell\gamma_\ell$$

Since $g_j \in x_j - S_\lambda^T\left(\gamma_j\right)$ for all $j \in \{1, 2, \ldots, \ell\}$. From Eq.(8), the result is

$$Re\left(\frac{\xi\psi''_{u_1,\ldots,u_\ell}(\xi)}{\psi'_{u_1,\ldots,u_\ell}(\xi)}\right) \geq u_1 x_1 \left|\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)'}{\Theta_\lambda^T g_1(\xi)} - \gamma_1\right| + \ldots + u_1 x_1 \left|\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)'}{\Theta_\lambda^T g_\ell(\xi)} - \gamma_\ell\right| - \sum_{j=1}^{\ell} u_j + \sum_{j=1}^{\ell} u_j\gamma_j$$

$$\geq \sum_{j=1}^{\ell} u_j\left(\gamma_j - 1\right),$$

so,

$$Re\left(\frac{\xi\psi''_{u_1,\ldots,u_\ell}(\xi)}{\psi'_{u_1,\ldots,u_\ell}(\xi)} + 1\right) \geq 1 + \sum_{j=1}^{\ell} u_j\left(\gamma_j - 1\right),$$

Thus, $\psi_{u_1,\ldots,u_\ell} \in C(\sigma)$ where $\sigma = 1 + \sum_{j=1}^{\ell} u_j\left(\gamma_j - 1\right)$, the result is that $0 \leq \sigma < 1$.

Corollary 1. If $g_j \in x_j - S_\lambda^T(\gamma)$ with $x_j \geq 0,\ 0 \leq \gamma < 1$ for all $\sum_{j=1}^{\ell} u_j \leq \frac{1}{2}$ and $j \in \{1, 2, \ldots, \ell\}$, then the operator $\psi_{u_1,\ldots,u_\ell} \in C(\sigma)$ where $\sigma = 1 + (\gamma - 1)\sum_{j=1}^{\ell} u_j$.

Proof. Putting $\ell = 1$ in Theorem 1, we will have the required result.

Corollary 2. If $g_j \in x - S_\lambda^T(\gamma)$ with $u \leq \frac{1}{2}$ and $x \geq 0,\ 0 \leq \gamma < 1$, then the operator $\psi_u \in C(\sigma)$ where

$$\psi_u(\xi) = \int_0^\xi \left(\frac{\left(\Theta_\lambda^T g(t)\right)'}{t}\right)^u dt \text{ and } \sigma = 1 + (\gamma - 1)u$$

Proof. Letting $\ell = 1$ in Theorem 1 . Thus we will have the required result.

Theorem 2 . If $g_j \in x_j - C_\lambda^T\left(\gamma_j\right)$ with $x_j \geq 0,\ 0 \leq \gamma_j < 1$ for all $\sum_{j=1}^{\ell} u_j \leq \frac{1}{2}$ and $j \in \{1, 2, \ldots, \ell\}$, then the operator $F_{u_1,\ldots,u_\ell} \in C(\sigma)$ where $\sigma = 1 + \sum_{j=1}^{\ell} u_j\left(\gamma_j - 1\right)$.

Proof.

$$\frac{\xi F''_{u_1,\ldots,u_\ell}(\xi)}{F'_{u_1,\ldots,u_\ell}(\xi)} = u_1 \frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)''}{\left(\Theta_\lambda^T g_1(\xi)\right)'} + \ldots + u_\ell \frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)''}{\left(\Theta_\lambda^T g_\ell(\xi)\right)'}$$

$$Re\left(\frac{\xi F''_{u_1,\ldots,u_\ell}(\xi)}{F'_{u_1,\ldots,u_\ell}(\xi)}\right) = u_1 Re\left(\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)''}{\left(\Theta_\lambda^T g_1(\xi)\right)'}\right) + \ldots + u_\ell Re\left(\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)''}{\left(\Theta_\lambda^T g_\ell(\xi)\right)'}\right)$$

$$=u_1 Re\left(\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)''}{\left(\Theta_\lambda^T g_1(\xi)\right)'}\right)+u_1-u_1-u_1\gamma_1+u_1\gamma_1+\ldots+u_\ell Re\left(\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)''}{\left(\Theta_\lambda^T g_\ell(\xi)\right)'}\right)+u_\ell-u_\ell-u_\ell\gamma_\ell+u_\ell\gamma_\ell$$

$$=u_1 Re\left(1+\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)''}{\left(\Theta_\lambda^T g_1(\xi)\right)'}-\gamma_1\right)-u_1+u_1\gamma_1+\ldots+u_\ell Re\left(1+\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)''}{\left(\Theta_\lambda^T g_\ell(\xi)\right)'}-\gamma_\ell\right)-u_\ell+u_\ell\gamma_\ell$$

Since $g_j \in x_j - C_\lambda^T\left(\gamma_j\right)$ for all $j\in\{1,2,\ldots,\ell\}$. From Eq.(9), we have

$$Re\left(\frac{\xi F''_{u_1,\ldots,u_\ell}(\xi)}{F'_{u_1,\ldots,u_\ell}(\xi)}\right)\geq u_1x_1\left|\frac{\xi\left(\Theta_\lambda^T g_1(\xi)\right)''}{\left(\Theta_\lambda^T g_1(\xi)\right)'}-1\right|+\ldots+u_1x_1\left|\frac{\xi\left(\Theta_\lambda^T g_\ell(\xi)\right)''}{\left(\Theta_\lambda^T g_\ell(\xi)\right)'}-1\right|-\sum_{j=1}^{\ell}u_j+\sum_{j=1}^{\ell}u_j\gamma_j\geq\sum_{j=1}^{\ell}u_j\left(\gamma_j-1\right)$$

so,

$$Re\left(\frac{\xi F''_{u_1,\ldots,u_\ell}(\xi)}{F'_{u_1,\ldots,u_\ell}(\xi)}+1\right)\geq 1+\sum_{j=1}^{\ell}u_j\left(\gamma_j-1\right),$$

Thus, $F_{u_1,\ldots,u_\ell}\in C\left(\sigma\right)$ where $\sigma=1+\sum_{j=1}^{\ell}u_j\left(\gamma_j-1\right)$, then we obtain $0\leq\sigma<1$.

Corollary 3. If $g_j \in x_j - C_\lambda^T\left(\gamma\right)$ with $x\geq 0,\ 0\leq\gamma<1$ for all $\sum_{j=1}^{\ell}u_j\leq\frac{1}{2}$ and $j\in\{1,2,\ldots,\ell\}$, then the operator $F_{u_1,\ldots,u_\ell}\in C\left(\sigma\right)$ where $\sigma=1+\left(\gamma-1\right)\sum_{j=1}^{\ell}u_j$.

Proof. By putting $\ell=1$ in Theorem 2, the result has been obtained.

Corollary 4. If $g_j \in x - C_\lambda^T\left(\gamma\right)$ with $x\geq 0,\ 0\leq\gamma<1$ and $u\leq\frac{1}{2}$, then the operator $F_u\in C\left(\sigma\right)$ where

$$\sigma=1+\left(\gamma-1\right)u \text{ and } F_u(\xi)=\int_0^{\xi}\left(\left(\Theta_\lambda^T g(t)\right)'\right)^u dt.$$

Proof. Letting $\ell=1$ in Theorem 2. Then we will have the required result.

Theorem 3. Let β_j,α_j be positive real numbers, and $j\in\{1,2,\ldots,\ell\}$. If $g_j\in S_\lambda^\ell\left(\frac{1}{\alpha_j}\right)$ and $f_j\in x_j-C_\lambda^T\left(\upsilon_j\right),\quad x_j\geq 0,\ 0\leq\upsilon_j<1.$ If $\sum_{j=1}^{\ell}\left[\beta_j\left(1-\upsilon_j\right)+\alpha_j\right]-\ell<1$, then $\omega_\ell(\xi)$ given in Eq.(15) is in the class $C(\sigma)$, where

$$\sigma=1+\ell+\sum_{j=1}^{\ell}\left[\beta_j\left(\upsilon_j-1\right)-\alpha_j\right].$$

Proof. Form differentiation of $\omega_\ell(\xi)$ which is given in Eq.(12) and by some calculatation, we have

$$\frac{\xi\omega''_\ell(\xi)}{\omega'_\ell(\xi)}=\sum_{j=1}^{\ell}\left[\alpha_j\left(\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)'}{\Theta_\lambda^T g_j(\xi)}-1\right)+\beta_j\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)''}{\left(\Theta_\lambda^T g_j(\xi)\right)'}\right]$$

$$=\sum_{j=1}^{\ell}\left[\alpha_j \frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)'}{\Theta_\lambda^T g_j(\xi)}-\alpha_j+\beta_j \frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)''}{\left(\Theta_\lambda^T g_j(\xi)\right)'}\right],$$

$$\frac{\xi\,\omega_\ell''(\xi)}{\omega_\ell'(\xi)}+1=\sum_{j=1}^{\ell}\left[\alpha_j \frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)'}{\Theta_\lambda^T g_j(\xi)}-\alpha_j+\beta_j \frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)''}{\left(\Theta_\lambda^T g_j(\xi)\right)'}\right]+1,$$

$$Re\left(\frac{\xi\,\omega_\ell''(\xi)}{\omega_\ell'(\xi)}+1\right)=\sum_{j=1}^{\ell}\left[\alpha_j Re\left(\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)'}{\Theta_\lambda^T g_j(\xi)}\right)-\alpha_j+\beta_j Re\left(\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)''}{\left(\Theta_\lambda^T g_j(\xi)\right)'}+1\right)-\beta_j\right]+1.$$

(16)

Since $g_j \in S_\lambda^\ell\left(\frac{1}{\alpha_j}\right)$, then $Re\left(\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)'}{\Theta_\lambda^T g_j(\xi)}\right)>\frac{1}{\alpha_j}$ and $f_j \in x_j - C_\lambda^T\left(\upsilon_j\right)$, form Eq.(16), there is

$$Re\left(\frac{\xi\,\omega_\ell''(\xi)}{\omega_\ell'(\xi)}+1\right)>\sum_{j=1}^{\ell}\left[1-\alpha_j+\beta_j\left(x_j\left|\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)''}{\left(\Theta_\lambda^T g_j(\xi)\right)'}\right|+\upsilon_j\right)-\beta_j\right]+1$$

$$>1+\ell-\sum_{j=1}^{\ell}\alpha_j+\sum_{j=1}^{\ell}\beta_j x_j\left|\frac{\xi\left(\Theta_\lambda^T g_j(\xi)\right)''}{\left(\Theta_\lambda^T g_j(\xi)\right)'}\right|+\sum_{j=1}^{\ell}\beta_j\left(\upsilon_j-1\right),$$

$$Re\left(\frac{\xi\,\omega_\ell''(\xi)}{\omega_\ell'(\xi)}+1\right)>1+\ell+\sum_{j=1}^{\ell}\alpha_j+\sum_{j=1}^{\ell}\beta_j\left(\upsilon_j-1\right)$$

$$>1+\ell-\sum_{j=1}^{\ell}\left[\beta_j\left(\upsilon_j-1\right)-\alpha_j\right],$$

therefore,

$$\omega_\ell(\xi)\in C(\sigma),\ \sigma=1+\ell+\sum_{j=1}^{\ell}\left[\beta_j\left(\upsilon_j-1\right)-\alpha_j\right].$$

Corollary 5. Let β_j,α_j be positive real numbers, and $j\in\{1,2,\ldots,\ell\}$. If $g_j \in S_\lambda^\ell\left(\frac{1}{\alpha_j}\right)$ and $f_j \in x_j - C_\lambda^T\left(\upsilon_j\right)$, $0\le\upsilon_j<1,\ x_j\ge 0$, if $\beta_j\left(1-\upsilon_j\right)+\alpha_j<2$, then $\omega_\ell(\xi)$ which is given in Eq.(15) will be in the class $C(\sigma)$, where

$\sigma=2+\beta_j\left(1-\upsilon_j\right)+\alpha_j.$

Proof. By setting $T=0$ and $\ell=1$ in Theorem 3, the result has been obtained.

Conclusion:

In the complex domain, Hurwitz–Lerch zeta functions has been importane in geometric function theory. Based on these type special function new integro‑differential operator have been introduced in term Salagean’s differential operator. Moreover, convexity properties on new subclasses involving these considered operators have investigated and presented.

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