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Approximate Solution of Linear Fuzzy Random Ordinary Differential Equations Using Laplace Variational Iteration Method

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Abstract

In this article, the Laplace transformation method in connection with the variational iteration method will be used to solve approximately fuzzy random ordinary differential equations. After that, the sequence of approximated closed form iterated solutions is derived based on the general Lagrange multiplier evaluated using the well-known convolution theorem of the Laplace transformation method. In addition, two examples are given and solved to illustrate the reliability, efficiency and applicability of the proposed method, they are simulated using computer programs with two different generations of stochastic processes, namely the Wiener process or Brownian motion, which are 1000 and 10000, respectively.

Keywords: Variational iteration method, Laplace transformation, Fuzzy random differential equations, Random differential equations.

الحل التقريبي للمعادلات التفاضلية الخطية الضبابية العشوائية الاعتيادية باستخدام طريقة لابلاس للتكرار المتغير

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الخلاصة

في هذه البحث، سنقدم طريقة تحويل لابلاس مع طريقة التكرار المتغير لايجاد الحل التقريبي للمعادلات التفاضلية الخطية الاعتيادية العشوائية الضبابية. حيث تم اشتقاق متتابعة دوال الحلول التقريبية استناداً الى ايجاد مضروب لاكرانج والذي تم حسابه عن طريق مبرهنة الالتفاف المعروفة في تحويلات لابلاس. إضافة إلى ذلك، تم تقديم مثالين وحلهما لتوضيح موثوقية وإمكانية تطبيق الطريقة المقترحة. بالإضافة إلى ذلك، تم تقديم مثالين وحلهما لبيان مدى موثوقية وكفاءة وإمكانية تطبيق الطريقة المقترحة، والتي تمت محاكاتها باستخدام برامج حاسوبية مع توليد مجموعتين مختلفين من المتغيرات التصادفية والتي تسمى متغيرات وينر التصادفية أو حركة براونين، وهما 1000 و 10000، على التوالي.

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1. Introduction

The subject of fuzzy differential equations (FDEs) is generally one of the most important topics in mathematics. It is particularly important in many fields including medicine, engineering and technology, …etc. On the other hand, the theory of stochastic processes considers random processes that satisfy certain particular properties as well as the co-cycle property. Therefore, deterministic fuzzy random differential equations were developed as a result of research into dynamical systems, and they have various applications in the modelling of classical problems in control theory, physics, biology, engineering economics, and finance in which random disturbances are the only source of uncertainty for such type of problems. Thus, the stochastic analysis approaches must be employed to address these circumstances. Also, it is still in most real-world problems a second source of uncertainty encountered due to the ambiguous nature or imprecise, fuzzy, etc. For such work, one must investigate fuzzy random differential equations that are concerned with fuzzy random differential equations with classical Brownian motion. The fuzzy random ordinary differential equations (FRODEs) are defined as a fuzzy ordinary differential equation (RODE) involving random variables as a stochastic process where fuzziness appears when triangular fuzzy numbers are included in the differential equation.

In this paper, the FRODEs will be solved using the Laplace variational iteration method (LVIM) which is a hybrid method that combines the usual Laplace transformation method and the variational iteration method (VIM). Such equations are so difficult to solve analytically, however, numerical and approximate methods seem to be necessary and reliable to give resolvable accurate results.

2. Literature Review and Problem Statement

Fuzzy random ordinary differential equations are used in real-world systems such that the phenomena are related to randomness and fuzziness as two kinds of uncertainties. These problems are encountered in economics and finance. There are several work projects and articles on the fuzzy stochastic differential equations and each one is different from the others in the approach.

In the literature, there is a large number of studies that have emerged in recent years that are concerned with differential equations containing either random variables or fuzzy sets (see [1], [4-6], [8-11], [16-19]). In 2008, Abbasbandy et al. [1] proposed a solution for general dual fuzzy linear systems, in which the existence of a minimal solution of the general dual fuzzy linear equation systems is investigated. Cortés et al. in 2011 [8] provided the numerical solution of random differential equations by means of random improved Euler's method. Khudair et al. in 2011 [16] used the Adomian decomposition method and the VIM to solve certain types of second order RODEs. In 2013, Guo et al. [11] presented the approximate solutions of the second-order linear differential equations with fuzzy boundary conditions, in which the indeterminate fuzzy coefficients approach converts the fuzzy linear boundary value problem into a crisp function system of linear equations. Behzadi in 2014 used the VIM to solve the second-order fuzzy Abel-Volterra integro-differential equations, [4]. The Runge-Kutta method was used by Nouri and Ranjbar in 2015 [18] to find a numerical solution to the initial value problems of RODEs. Chakraverty et al. presented in 2016 a new direction in the use of basic concepts of fuzzy differential equations, solutions and the applications of FDEs for engineers and scientists, [6]. Ghazanfari et al. [10] used the VIM to solve FDEs. The mean and variance of the approximate solutions of the second-order RODEs using the homotopy analysis method are proposed by Khudair et al. in 2016 [17]. Tchier et al. in 2017 [20] studied a family of RDEs with boundary conditions using a random fixed-point theorem. In 2019 Abdulsahib et al.

presented a modified approach based on the VIM and numerical integration methods to solve the initial value problems of the n-th order RODEs, [3]. In 2021, Phu and Lupulescu provided the statement and the proof of the existence and uniqueness theorem of a solution for the class of fuzzy fractional functional differential equations, [19]. Fadhel et al. in 2021 [6] used the LVIM to solve RODEs. In 2022, Bica and Satmari [5] presented the numerical method to solve fuzzy Volterra integral equations based on the fuzzy Bernstein spline interpolation procedure.

The parameters, variables, and initial conditions of a mathematical model governed by ordinary differential equations (ODEs) are assumed to be precisely prescribed. In fact, there may be imprecise, ambiguous, random processes or insufficient information available regarding the variables and parameters that appeared in the differential equation. All of these are caused by accuracies in measurements, observations, or experimental data; the application of varied operating conditions; or maintenance-induced errors due to imprecise or randomness. To avoid uncertainties or lack of precision, a fuzzy environment in parameters, variables, and initial conditions can be used instead of accurate (fixed) ones by converting conventional differential equations into FRODEs. Because of the complexity of fuzzy logic and stochastic calculus, it can be difficult to find accurate solutions to such problems in real-world applications. So that the employment of reliable and efficient numerical and/or approximate methods in the solution of FRODEs is needed

The statement of the considered problem of this paper is to use the LVIM to find the approximate solution of the following n-th order FRODE:

$$\tilde{x}^{(n)}(t,\omega) = f(t, \tilde{x}(t,\omega), \tilde{x}'(t,\omega), \ldots, \tilde{x}^{(n-1)}(t,\omega)),\ t \geq 0, \tag{1}$$

with initial conditions that are given as triangular fuzzy numbers:

$\tilde{x}^{(i)}(0,\omega) = \tilde{x}_0^i$, i = 0,1,…,$n$–1,

where ω is a random process that is taken to be of Brownian motion, [20].

3. Preliminary and Fundamental Concepts

In this section, some basic concepts that are necessary for this work will be introduced. These concepts include basic definitions of fuzzy set theory, stochastic calculus and some basic properties of the Laplace transform.

Definition 1, [14]. A triangular fuzzy number is a mapping $u: \mathbb{R} \longrightarrow [0,1]$, which satisfies:

1. u is upper semi-continuous.
2. There exist real numbers a, b and c, such that $a \leq b \leq c$ and

i. $u(x) = 0$ outside some interval $[a,c]$,

ii. $u(x)$ is monotonic increasing on $[a,b]$,

iii. $u(x)$ is monotonic decreasing on $[b,c]$,

iv. $u(x) = 1$ if $x = b$.

v.

Definition 2, [15]. The set of all elements that belong to the fuzzy set $\tilde{u}$ at least to the degree α is called the α-level set, which is defined by:

$u_\alpha = \{x \in X : \mu_{\tilde{u}}(x) \geq \alpha\}$,

where $\mu_{\tilde{u}}$ refers to the membership function related to the fuzzy set $\tilde{u}$.

Fuzzy numbers may be characterized or parameterized in terms of α-level sets.

Definition 3, [14]. A fuzzy number $\tilde{u}$ is in parametric form as an interval $[\underline{u}(\alpha), \overline{u}(\alpha)]$, $0 \leq \alpha \leq 1$, which satisfies the following requirements:

1. $\underline{u}(\alpha)$ is a bounded monotonic increasing right continuous function,
2. $\overline{u}(\alpha)$ is a bounded monotonic decreasing left continuous function,

3. $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, for all $0 \leq \alpha \leq 1$.

Remark 1. For an arbitrary two fuzzy numbers $\tilde{u} = [\underline{u}(\alpha), \overline{u}(\alpha)]$ and $\tilde{v} = [\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in [0,1]$ the following algebraic operations may be fulfilled:

1. If k is any real number, then:

$$k\tilde{u} = \begin{cases} (k\underline{u}(\alpha), k\overline{u}(\alpha)), \text{if } k \geq 0 \\ (k\overline{u}(\alpha), k\underline{u}\alpha)), \text{if } k < 0 \end{cases}$$

2. $\tilde{u} \mp \tilde{v} = (\underline{u}(\alpha) \mp \underline{v}(\alpha), \overline{u}(\alpha) \mp \bar{v}(\alpha))$.
3. $\tilde{u}\tilde{v}$ = (min s, max s), where $s = \{\underline{u}\,\underline{v}, \underline{u}\,\overline{v}, \overline{u}\,\underline{v}, \overline{u}\,\overline{v}\}$.

Some preliminary concepts of the stochastic process are also given for completeness purpose, which are encountered in stochastic calculus. These concepts start with the following basic definition:

Definition 4, [3,9]. A stochastic process is a family of random variables $x(t, \omega)$ (or briefly $X_t(\omega)$ or X_t) of independent variables t and parameter ω. Let $t \in [t_0,T] \subset [0,\infty)$, $\omega \in \Omega$ on a common probability space $(\Omega,\mathbf{F},P)$, where Ω is a sample space which is the set of all possible outcomes of random increment, $\mathbf{F}$ is the class of all subset of Ω and P is a probability measure whose domain is Ω and the codomain is the interval [0,1], which is assumed to be real values and P is measurable as a function of ω for each fixed t. The parameter t is the time and $X_t(.)$ represents a random variable on the above probability space Ω, while $X.(\omega)$ is called a sample path or trajectory of the stochastic process.

Definition 5, [3,9]. A stochastic process W_t for all $t \in [0,\infty)$ is called a Wiener process or Brownian motion if:

1. $P(\{\omega \in \Omega \mid W_0(\omega) = 0\}) = 1$.
2. For $0 < t_0 < t_1 <\ldots< t_n$, the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$,…, $W_{t_n} - W_{t_{n-1}}$ are independent.
3. For an arbitrary t and $h>0$, $W_{t+h}-W_t$ has the normal distribution with mean 0 and variance h.

4. Solution of FRODEs Using Laplace VIM

In order to be familiar with Eq.(1), the α-level sets associated with fuzzy sets will be considered, for all $\alpha \in [0,1]$. Hence, the general form of the n-th order linear FRODE with constant coefficients related to Eq.(1) will take the form:

$$\tilde{x}^{(n)}(t,\omega,\alpha) = f(t, \tilde{x}(t,\omega,\alpha), \tilde{x}'(t,\omega,\alpha), \ldots, \tilde{x}^{(n-1)}(t,\omega,\alpha)),\ t \geq 0,\ \alpha \in [0,1], \quad (2)$$

with initial conditions that are given as fuzzy numbers:

$$\tilde{x}^{(i)}(0,\omega,\alpha) = (G_i(t,\omega,\alpha), H_i(t,\omega,\alpha)),\ i = 0,1,\ldots,n-1.$$

Let $\tilde{x} = [\underline{x}, \bar{x}]$, then Eq.(2) may be written in terms of its lower and upper solutions related to the α-levels depending on their coefficients [7]. The linear FRODE with constant coefficients in terms of α-levels is given by:

$$\tilde{x}^{(n)}(t,\omega,\alpha) + c_{n-1}\tilde{x}^{(n-1)}(t,\omega,\alpha) + \ldots + c_1\tilde{x}'(t,\omega,\alpha) + c_0\tilde{x}(t,\omega,\alpha) = g(t,\omega),\ t \geq 0, \quad (3)$$

with initial conditions are given as a triangular fuzzy numbers:

$$\tilde{x}(0,\omega_0,\alpha) = \tilde{b}_0, \tilde{x}'(0,\omega_0,\alpha) = \tilde{b}_1, \ldots, \tilde{x}^{(n-1)}(0,\omega_0,\alpha) = \tilde{b}_{n-1},$$

where c_i are real constants for all $i = 0, 1, \ldots, n-1$, g is a given function and $\tilde{b}_i$ are triangular fuzzy numbers for all $i = 0, 1, \ldots, n-1$, and $\tilde{x}$ is the solution to be determined as a fuzzy function.

To demonstrate the concept of the VIM [2,12-15], consider the following general FRODE in operator form:

$$L[\tilde{x}(t,\omega,\alpha)] + N[\tilde{x}(t,\omega,\alpha)] = g(t,\omega),\ t \in [t_0,T], \quad (4)$$

where L is a linear differential operator, N is a nonlinear differential operator and g is a given function that contains random variable ω.

If the α-level sets are considered, then the correction functional of Eq.(4) for the lower and upper fuzzy solution $\tilde{x}$ will read for all $m = 0, 1, \ldots$. Three cases are considered to solve the fuzzy differential equation, these cases may be summarized as follows:

Case 1: All the coefficients $c_{n-1}, c_{n-2}, \ldots, c_0$ are positive and hence Eq.(4) will be decomposed into the following non-fuzzy or crisp ordinary differential equations:

$$\underline{x}_{m+1}(t,\omega,\alpha) = \underline{x}_m(t,\omega,\alpha) + \int_0^t \underline{\lambda}(\xi,t)\Big[\frac{d^n}{d\xi^n}\underline{x}_m(\xi,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(\xi,\omega,\alpha)+\ldots+c_1\underline{x}'_m(\xi,\omega,\alpha) + c_0\underline{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\Big]d\xi, \tag{5}$$

$$\bar{x}_{m+1}(t,\omega,\alpha) = \bar{x}_m(t,\omega,\alpha) + \int_0^t \bar{\lambda}(\xi,t)\Big[\frac{d^n}{d\xi^n}\bar{x}_m(\xi,\omega,\alpha) + c_{n-1}\bar{x}_m^{(n-1)}(\xi,\omega,\alpha)+\ldots+c_1\bar{x}'_m(\xi,\omega,\alpha) + c_0\bar{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\Big]d\xi. \tag{6}$$

Case 2: The coefficients $c_{n-1}, c_{n-2}, \ldots, c_{n-m}$ are positive and $c_{n-m-1}, c_{n-m-2}, \ldots, c_1, c_0$ are negative, then the lower and upper fuzzy solution of $\tilde{x}$ related to equation (4) are written as:

$$\underline{x}_{m+1}(t,\omega,\alpha) = \underline{x}_m(t,\omega,\alpha) + \int_0^t \underline{\lambda}(\xi,t)\Big[\frac{d^n}{d\xi^n}\underline{x}_m(\xi,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(\xi,\omega,\alpha)+\ldots+c_{n-m}\underline{x}_m^{(n-m)}(\xi,\omega,\alpha) + c_{n-m-1}\bar{x}_m^{(n-m-1)}(\xi,\omega,\alpha) + c_{n-m-2}\bar{x}_m^{(n-m-2)}(\xi,\omega,\alpha)+\ldots+c_1\bar{x}'_m(\xi,\omega,\alpha) + c_0\bar{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\Big]d\xi, \tag{7}$$

$$\bar{x}_{m+1}(t,\omega,\alpha) = \bar{x}_m(t,\omega,\alpha) + \int_0^t \bar{\lambda}(\xi,t)\Big[\frac{d^n}{d\xi^n}\bar{x}_m(\xi,\omega,\alpha) + c_{n-1}\bar{x}_m^{(n-1)}(\xi,\omega,\alpha)+\ldots+c_{n-m}\bar{x}_m^{(n-m)}(\xi,\omega,\alpha) + c_{n-m-1}\underline{x}_m^{(n-m-1)}(\xi,\omega,\alpha) + c_{n-m-2}\underline{x}_m^{(n-m-2)}(\xi,\omega,\alpha)+\ldots+c_1\underline{x}'_m(\xi,\omega,\alpha) + c_0\underline{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\Big]d\xi. \tag{8}$$

Case 3: All the coefficients $c_{n-1}, c_{n-2}, \ldots, c_0$ are negative, then Eq. (4) will be decomposed into the following upper and lower cases solutions:

$$\underline{x}_{m+1}(t,\omega,\alpha) = \underline{x}_m(t,\omega,\alpha) + \int_0^t \underline{\lambda}(\xi,t)\Big[\frac{d^n}{d\xi^n}\underline{x}_m(\xi,\omega,\alpha) + c_{n-1}\bar{x}_m^{(n-1)}(\xi,\omega,\alpha)+\ldots+c_1\bar{x}'_m(\xi,\omega,\alpha) + c_0\bar{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\Big]d\xi \tag{9}$$

$$\bar{x}_{m+1}(t,\omega,\alpha) = \bar{x}_m(t,\omega,\alpha) + \int_0^t \bar{\lambda}(\xi,t)\Big[\frac{d^n}{d\xi^n}\bar{x}_m(\xi,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(\xi,\omega,\alpha)+\ldots+c_1\underline{x}'_m(\xi,\omega,\alpha) + c_0\underline{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\Big]d\xi. \tag{10}$$

The proposed approach starts by taking the Laplace transform of Eq.(4), yields to:

$$\mathcal{L}\{L(\tilde{x}(t,\omega,\alpha))\} + \mathcal{L}\{N(\tilde{x}(t,\omega,\alpha))\} = \mathcal{L}\{\mathrm{g}(t,\omega)\} \tag{11}$$

where the Laplace transformation of the differential operator of the correction functional (11) using the VIM will give:

$$\mathcal{L}\{\tilde{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\tilde{x}_m(t,\omega,\alpha)\} + \mathcal{L}\int_0^t \lambda(\xi,t)\big\{L\big(\tilde{x}_m(\xi,\omega,\alpha)\big) + N\big(\tilde{x}_m(\xi,\omega,\alpha)\big) - \mathrm{g}(\xi,\omega)\big\}\,d\xi,$$ for all $m = 0,1,\ldots$

The correction functional depends on the fuzzy differential equations which mean it depends on the coefficients to be either positive or negative so one of the following cases should be considered:

Case 1: If the all coefficients $c_0, c_1, \ldots, c_{n-1}$ are positive, this yields to:

$\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}(\xi,t)\left[\frac{d^n}{d\xi^n}\underline{x}_m(\xi,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(\xi,\omega,\alpha) + \ldots + c_1\underline{x}'_m(\xi,\omega,\alpha) + c_0\underline{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\right]d\xi,\right.$

$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \bar{\lambda}(\xi,t)\left[\frac{d^n}{d\xi^n}\bar{x}_m(\xi,\omega,\alpha) + c_{n-1}\bar{x}_m^{(n-1)}(\xi,\omega,\alpha) + \cdots + c_1\bar{x}'_m(\xi,\omega,\alpha) + c_0\bar{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\right]d\xi.\right.$

Therefore, upon using the convolution theorem for the lower-case solution with respect to t, this implies that:

$$\begin{aligned}\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} &= \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\{\lambda(t-s) * \{L(\underline{x}_m(t,\omega,\alpha)) + N(\underline{x}_m(t,\omega,\alpha)) - \mathrm{g}(t,\omega)\}\} \\ &= \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\{\lambda(s,t)\{L(\underline{x}(t,\omega,\alpha)) + N(\underline{x}_m(t,\omega,\alpha)) - g(t,\omega)\}\} \\ &= \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\underline{\lambda}(s,t) * \left[\frac{d^n}{dt^n}\underline{x}_m(t,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(t,\omega,\alpha) + \ldots + c_1\underline{x}'_m(t,\omega,\alpha) + c_0\underline{x}_m(t,\omega,\alpha) - \mathrm{g}(t,\omega)\right],\right.\end{aligned} \tag{12}$$

where * refers to the usual convolution operation between two functions.

Also, for the upper-case solution, we have:

$$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\bar{\lambda}(s,t) * \left[\frac{d^n}{dt^n}\bar{x}_m(t,\omega,\alpha) + c_{n-1}\bar{x}_m^{(n-1)}(t,\omega,\alpha) + \cdots + c_1\bar{x}'_m(t,\omega,\alpha) + c_0\bar{x}_m(t,\omega,\alpha) - \mathrm{g}(t,\omega)\right]\right\}, \tag{13}$$

then Eqs.(12) and (13) become as follows:

$$\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \underline{\Lambda}(s)\left\{s^n\mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} - c_{n-1}\underline{x}_m^{(n-1)}(0,\omega,\alpha) + \ldots + c_1\underline{x}'_m(0,\omega,\alpha) + c_0\underline{x}_m(0,\omega,\alpha)\right\} + \mathcal{L}\{N(\underline{x}_m(t;\omega)) - \mathrm{g}(t;\omega)\} \tag{14}$$

$$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}(\bar{x}_m(t,\omega,\alpha)) + \bar{\Lambda}(s)\left\{s^n\mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} - c_{n-1}\bar{x}_m^{(n-1)}(0,\omega,\alpha) + \ldots + c_1\bar{x}'_m(0,\omega,\alpha) + c_0\bar{x}_m(0,\omega,\alpha)\right\} + \mathcal{L}\{N(\bar{x}_m(t;\omega)) - \mathrm{g}(t;\omega)\}, \tag{15}$$

where $\underline{\Lambda}(s) = \mathcal{L}\{\underline{\lambda}(s,t)\}$ and $\bar{\Lambda}(s) = \mathcal{L}\{\bar{\lambda}(s,t)\}$. The iteration formula of Eqs.(14) and (15) are used to suggest the main scheme of the proposed approach involving the Lagrange multiplier.

Now, consider $\mathcal{L}\{N(\underline{x}_m(t;\omega)) - \mathrm{g}(t;\omega)\}$ and $\mathcal{L}\{N(\bar{x}_m(t;\omega)) - \mathrm{g}(t;\omega)\}$ in Eqs.(14) and (15) as a restricted variation, which makes Eqs.(12) and (13) stationary with respect to $\underline{x}_m$ and $\bar{x}_m$, respectively. Hence:

$$\mathcal{L}\{\delta\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\delta\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\{\underline{\lambda}(s,t)\}\{s^n\mathcal{L}\{\delta\underline{x}_m(t,\omega,\alpha)\}\}, \tag{16}$$

$$\mathcal{L}\{\delta\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\delta\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\{\bar{\lambda}(s,t)\}\{s^n\mathcal{L}\{\delta\bar{x}_m(t,\omega,\alpha)\}\}, \tag{17}$$

where δ is the classical first variation. The optimality condition for the extremum is $\mathcal{L}\{\delta\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\delta\bar{x}_{m+1}(t,\omega,\alpha)\} = 0$. Hence, Eqs.(16) and (17) lead to:

$\mathcal{L}\{\underline{\lambda}(s,t)\} = \mathcal{L}\{\bar{\lambda}(s,t)\} = \frac{-1}{s^n}, s > 0.$

The sequential approximations are obtained by taking the inverse Laplace transform of Eqs.(12) and (13) after substituting $\underline{\lambda}(s,t)$ and $\bar{\lambda}(s,t)$, this yields:

$$\underline{x}_{m+1}(t,\omega,\alpha) = \underline{x}_m(t,\omega,\alpha) - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\left[\underline{X}_m(t,\omega,\alpha) - s^{n-1}\underline{x}(0,\omega,\alpha) - \ldots - \underline{x}^{(n-1)}(0;\omega,\alpha) + \mathcal{L}\{N\underline{x}_m(t,\omega,\alpha) - \mathrm{g}(t;\omega)\}\right]\right\}, \tag{18}$$

$$\bar{x}_{m+1}(t,\omega,\alpha) = \bar{x}_m(t,\omega,\alpha) - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\left[\bar{X}_m(t,\omega,\alpha) - s^{n-1}\bar{x}(0,\omega,\alpha) - \ldots - \bar{x}^{(n-1)}(0;\omega,\alpha) + \mathcal{L}(N\bar{x}_m(t,\omega,\alpha) - \mathrm{g}(t;\omega))\right]\right\}, \tag{19}$$

where $\underline{X}_m(t,\omega,\alpha) = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\}$ and $\bar{X}_m(t,\omega,\alpha) = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\}$. Rearrange Eqs.(18) and (19), it implies to:

$$\underline{x}_{m+1}(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}(0,\omega,\alpha)}{s} + \ldots + \frac{\underline{x}^{(n-1)}(0,\omega,\alpha)}{s^n}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\left(\mathcal{L}(N\underline{x}_m(t,\omega,\alpha) - \mathrm{g}(t;\omega))\right)\right\}, \tag{20}$$

$$\bar{x}_{m+1}(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}(0,\omega,\alpha)}{s} + \ldots + \frac{\bar{x}^{(n-1)}(0,\omega,\alpha)}{s^n}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\left(\mathcal{L}(N\bar{x}_m(t,\omega,\alpha) - \mathrm{g}(t;\omega))\right)\right\}, \tag{21}$$

with initial approximation to Eqs.(20) and (21), the following equations can be considered:

$$\underline{x}_0(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}(0,\omega,\alpha)}{s} + \frac{\underline{x}'(0,\omega,\alpha)}{s^2} + \ldots + \frac{\underline{x}^{(n-1)}(0,\omega,\alpha)}{s^n}\right\}, \quad (22)$$

$$\bar{x}_0(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}(0,\omega,\alpha)}{s} + \frac{\bar{x}'(0,\omega,\alpha)}{s^2} + \ldots + \frac{\bar{x}^{(n-1)}(0,\omega,\alpha)}{s^n}\right\}. \quad (23)$$

After applying the inverse Laplace transform to Eqs.(22) and (23), one can get:

$$\underline{x}_0(t,\omega,\alpha) = \underline{x}(0,\omega,\alpha) + \underline{x}'(0,\omega,\alpha)t + \ldots + \frac{\underline{x}^{(n-1)}(0,\omega,\alpha)}{(n-1)!}t^{n-1}, \quad (24)$$

$$\bar{x}_0(t,\omega,\alpha) = \bar{x}(0,\omega,\alpha) + \bar{x}'(0,\omega,\alpha)t + \ldots + \frac{\bar{x}^{(n-1)}(0,\omega,\alpha)}{(n-1)!}t^{n-1}. \quad (25)$$

Therefore, the solution of Eq.(1) will be $\underline{x}(t,\omega,\alpha) = \lim_{m\to\infty}\underline{x}_m(t,\omega,\alpha)$ and $\bar{x}(t,\omega,\alpha) = \lim_{m\to\infty}\bar{x}_m(t,\omega,\alpha)$.

Case 2: If the coefficients $c_{n-1}, c_{n-2}, \ldots, c_{n-m}$ are positive and $c_{n-m-1}, c_{n-m-2}, \ldots, c_1, c_0$ are negative, then Laplace transformation of the lower and upper fuzzy solution $\tilde{x}$ takes the form:

$$\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}(\xi,t)\left[\frac{d^n}{d\xi^n}\underline{x}_m(\xi,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(\xi,\omega,\alpha) + \ldots + c_{n-m}\underline{x}_m^{(n-m)}(\xi,\omega,\alpha) + c_{n-m-1}\bar{x}_m^{(n-m-1)}(\xi,\omega,\alpha) + c_{n-m-2}\bar{x}_m^{(n-m-2)}(\xi,\omega,\alpha) + \ldots + c_1\bar{x}'_m(\xi,\omega,\alpha) + c_0\bar{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\right]d\xi\right\},$$

$$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \bar{\lambda}(\xi,t)\left[\frac{d^n}{d\xi^n}\bar{x}_m(\xi,\omega,\alpha) + c_{n-1}\bar{x}_m^{(n-1)}(\xi,\omega,\alpha) + \ldots + c_{n-m}\bar{x}_m^{(n-m)}(\xi,\omega,\alpha) + c_{n-m-1}\underline{x}_m^{(n-m-1)}(\xi,\omega,\alpha) + c_{n-m-2}\underline{x}_m^{(n-m-2)}(\xi,\omega,\alpha) + \ldots + c_1\underline{x}'_m(\xi,\omega,\alpha) + c_0\underline{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\right]d\xi\right\}.$$

Case 3: If all the coefficients $c_0, c_1, \ldots, c_{n-1}$ are negative, then:

$$\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}(\xi,t)\left[\frac{d^n}{d\xi^n}\underline{x}_m(\xi,\omega,\alpha) + \bar{x}_m^{(n-1)}(\xi,\omega,\alpha) + \cdots + c_1\bar{x}'_m(\xi,\omega,\alpha) + c_0\bar{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\right]\right\}d\xi,$$

$$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \bar{\lambda}(\xi,t)\left[\frac{d^n}{d\xi^n}\bar{x}_m(\xi,\omega,\alpha) + c_{n-1}\underline{x}_m^{(n-1)}(\xi,\omega,\alpha) + \ldots + c_1\underline{x}'_m(\xi,\omega,\alpha) + c_0\underline{x}_m(\xi,\omega,\alpha) - \mathrm{g}(\xi,\omega)\right]\right\}d\xi.$$

The same followed approach in the first case is used in cases 2 and 3 to get the final Lagrange multipliers:

$$\mathcal{L}\{\underline{\lambda}(s,t)\} = \mathcal{L}\{\bar{\lambda}(s,t)\} = \frac{-1}{s^n}.$$

Taking the inverse Laplace transform, it gives the optimal values of $\underline{\lambda}$ and $\bar{\lambda}$. Thus, the iteration formulations of the second and third cases take the form:

$$\underline{x}_{m+1}(t,\omega,\alpha) = \underline{x}_m(t,\omega,\alpha) - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\left[\underline{X}_m(t,\omega,\alpha) - s^{n-1}\bar{x}(0,\omega,\alpha) - \ldots - \bar{x}^{(n-1)}(0;\omega,\alpha) + \mathcal{L}(N\bar{x}_m(t,\omega,\alpha) - \mathrm{g}(t;\omega))\right]\right\},$$

$$\bar{x}_{m+1}(t,\omega,\alpha) = \bar{x}_m(t,\omega,\alpha) - \mathcal{L}^{-1}\left\{\frac{1}{s^n}\left[\bar{X}_m(t,\omega,\alpha) - s^{n-1}\underline{x}(0,\omega,\alpha) - \ldots - \underline{x}^{(n-1)}(0;\omega,\alpha) + \mathcal{L}(N\underline{x}_m(t,\omega,\alpha) - \mathrm{g}(t;\omega))\right]\right\}.$$

5. Illustrative Examples

Two illustrative examples will be considered in this section, which are carried out for two different generations of stochastic processes, namely the Wiener process or Brownian motion, which are 1000 and 10000, respectively. The same discretized signal processes of those two generations are used in the simulation of the considered examples. These generations are given in Figure 1.

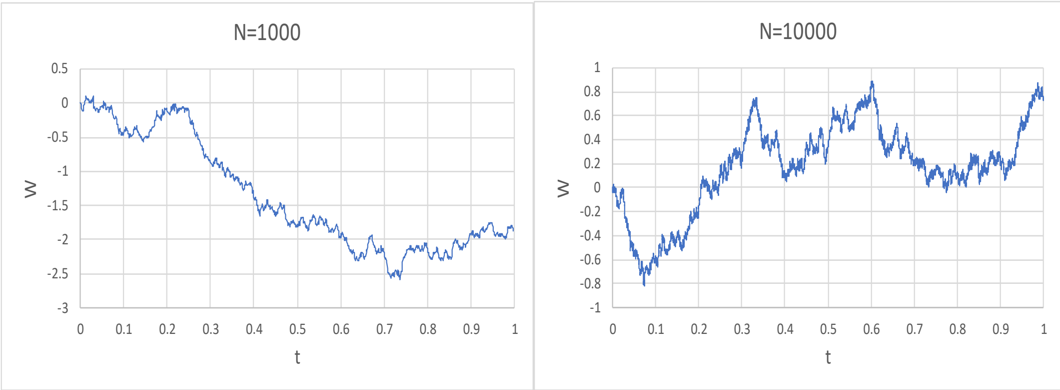

Figure 1: Discretized Brownian motion with $N = 1000$ and 10000 generations

Example 1: Consider the first order linear FRODE:

$\tilde{x}'(t,\omega,\alpha) + \tilde{x}(t,\omega,\alpha) = W_t(\omega)$, $t \in [0,1]$, W_t is the Wiener process (26)

with initial condition:

$\tilde{x}_0(0,\omega_0,\alpha) = (\alpha, 2-\alpha)$, $\alpha \in [0,1]$

Now, to solve Eq.(26) using the Laplace VIM, it is notable that this equation follows the first case because it contains only positive coefficients of the dependent variable. Therefore, using the Laplace VIM, we get:

$$\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}(\xi,t)\left[\underline{x}'_m(\xi,\omega,\alpha) + \underline{x}_m(\xi,\omega,\alpha) - W_\xi(\omega)\right]\right\} d\xi,$$
$$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \bar{\lambda}(\xi,t)\left[\bar{x}'_m(\xi,\omega,\alpha) + \bar{x}_m(\xi,\omega,\alpha) - W_\xi(\omega)\right]\right\} d\xi.$$

Which will give the general Lagrange multipliers $\mathcal{L}\{\underline{\lambda}(s,t)\} = \mathcal{L}\{\bar{\lambda}(s,t)\} = \frac{-1}{s}$.

Letting the fuzzy solution be given in terms of the lower and upper cases as in the interval $\tilde{x} = [\underline{x}, \bar{x}]$. Then, the lower-case solution $\underline{x}$ may be obtained iteratively as:

$\underline{x}_0(0,\omega_0,\alpha) = \alpha$

$\underline{x}_1(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}[\underline{x}_0(t,\omega,\alpha) - W_t(\omega)]\right\}$

$= \underline{x}_0(0,\omega_0,\alpha) + t\underline{x}_0(0,\omega_0,\alpha) - tW_t(\omega)$

$= \alpha + \alpha t - tW_t(\omega)$,

$\underline{x}_2(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}[\alpha + \alpha t - tW_t(\omega) - W_t(\omega)]\right\}$

$= \alpha + \alpha t - tW_t(\omega) + \frac{\alpha t^2}{2} - \frac{t^2 W_t(\omega)}{2}$,

$\underline{x}_3(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}\left[\alpha + \alpha t - tW_t(\omega) + \frac{\alpha t^2}{2} - \frac{t^2 W_t(\omega)}{2} - W_t(\omega)\right]\right\}$

$= \alpha + \alpha t - tW_t(\omega) + \frac{\alpha t^2}{2} - \frac{t^2 W_t(\omega)}{2} + \frac{\alpha t^3}{6} - \frac{t^3 W_t(\omega)}{6}$,

and so on by induction, one may get:

$\underline{x}_m(t,\omega,\alpha) = \alpha + \sum_{i=1}^{m} \frac{(t)^i}{i!}(\alpha - W_t(\omega))$, $m \in \mathbb{N}$.

Also, the upper-case solution $\bar{x}$ is obtained using the following iterative approximate solutions:

$\bar{x}_0(0,\omega_0,\alpha) = 2 - \alpha$,

$\bar{x}_1(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}[\bar{x}_0(t,\omega,\alpha) - W_t(\omega)]\right\}$

$= \bar{x}_0(0,\omega_0,\alpha) + t\bar{x}_0(0,\omega_0,\alpha) + tW_t(\omega)$

$= 2 - \alpha - \alpha t + 2t - tW_t(\omega)$,

$\bar{x}_2(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}[\bar{x}_1(t,\omega,\alpha) - W_t(\omega)]\right\}$

$= \mathcal{L}^{-1}\left\{\frac{\bar{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}[2 - \alpha - \alpha t + 2t - tW_t(\omega) - W_t(\omega)]\right\}$

$= 2 - \alpha - \alpha t + 2t - tW_t(\omega) + t^2 - \frac{\alpha t^2}{2} - \frac{t^2 W_t(\omega)}{2}$,

$$\bar{x}_3(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}_0(0,\omega_0,\alpha)}{s} - \frac{1}{s}\mathcal{L}[\bar{x}_2(t,\omega,\alpha) - W_t(\omega)]\right\}$$
$$= 2 - \alpha - \alpha t + 2t - tW_t(\omega) + t^2 - \frac{\alpha t^2}{2} - \frac{t^2 W_t(\omega)}{2} + \frac{t^3}{3} - \frac{\alpha t^3}{6} - \frac{t^3 W_t(\omega)}{6},$$
and so on by induction implies to:
$$\bar{x}_m(t,\omega,\alpha) = 2 - \alpha \; + \; \sum_{i=1}^{m} \frac{(t)^i}{i!}((2-\alpha) - W_t(\omega))$$

The signal simulation of the generated discretized Brownian motion (or Wiener process) with the total number of generations N = 1000 and 10000 (see Figure 1) over the unit time interval [0,1]. The corresponding eighth upper and lower approximate solutions of Eq.(26) for each generation are given in Figures 2 and 3, respectively with different α-levels which are 0, 0.2, 0.4, 0.6, 0.8 and 1.

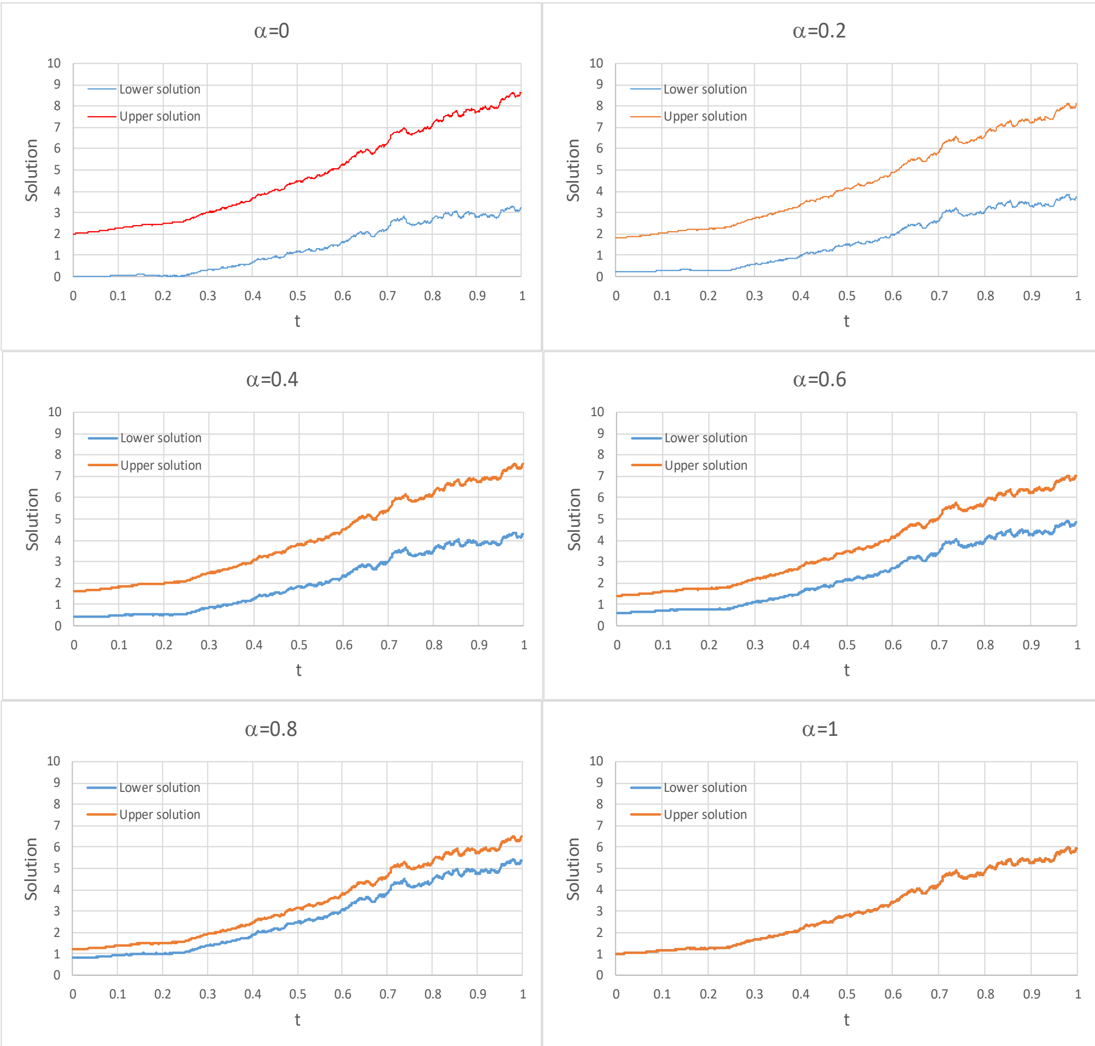

Figure 2: The eighth lower and upper iterative solutions of Example 1 using Laplace VIM for different values of α-levels and discretized Brownian motion with total signal processing number $N = 1000$.

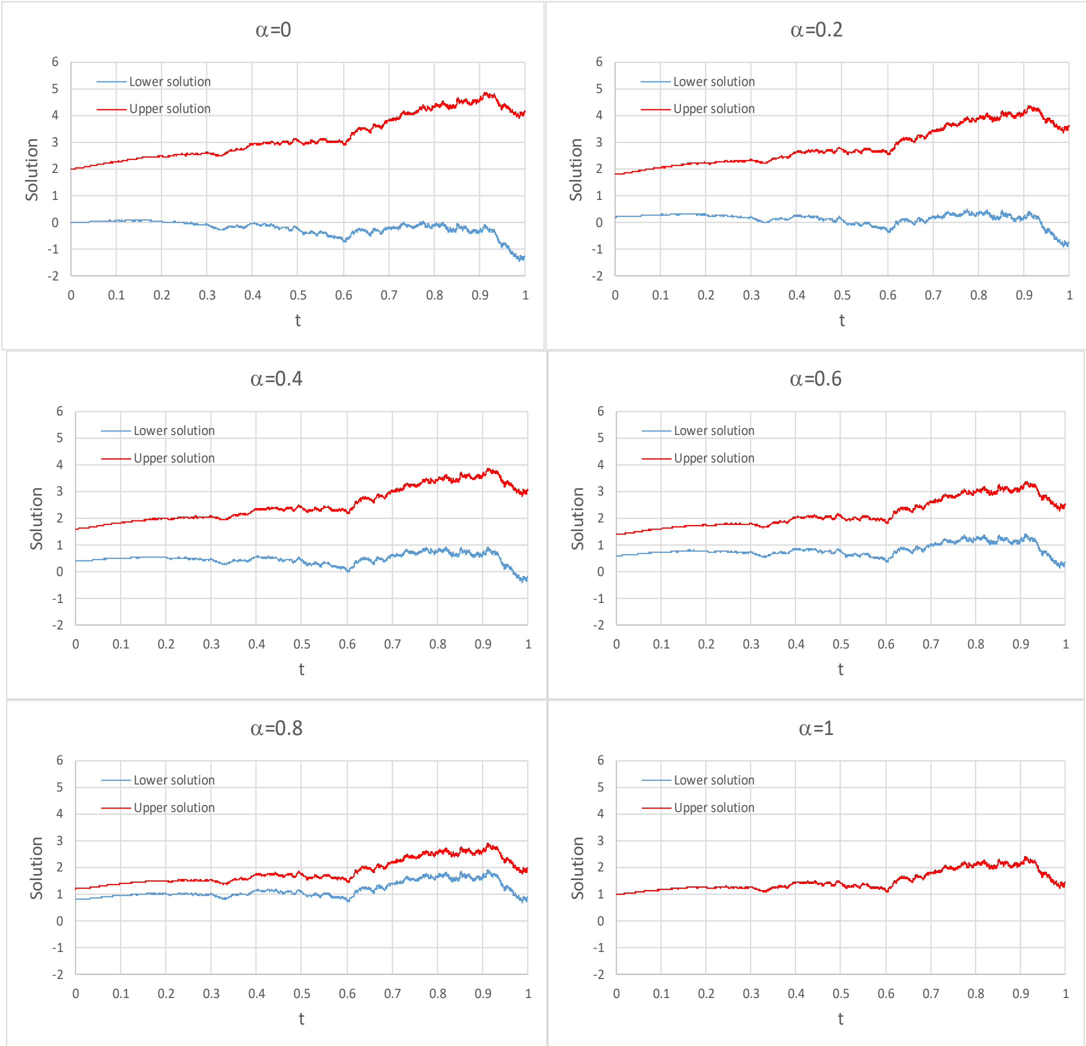

Figure 3: The eighth lower and upper iterative solutions of Example 1 using Laplace VIM for different values of α-levels and discretized Brownian motion with total signal processing number $N = 10000$.

Example 2: Consider the second order FRODE:

$$\tilde{x}''(t,\omega,\alpha) - \tilde{x}'(t,\omega,\alpha) + \tilde{x}(t,\omega,\alpha) = \cos(W_t(\omega)),\ t \in [0,1], \tag{27}$$

with initial conditions:

$\tilde{x}_0(0,\omega_0,\alpha) = \ (\alpha, 1.5 - 0.5\alpha),\ \tilde{x}'_0(0,\omega_0,\alpha) = \ (0.5\alpha, 1 - 0.5\alpha),\ \alpha \in [0,1],$

where W_t is the Brownian motion.

Now, to solve Eq.(27) using the Laplace VIM, it is notable that this equation is of the second case because it contains positive and negative coefficients simultaneously. Therefore:

$$\mathcal{L}\{\underline{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\underline{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \underline{\lambda}(\xi,t)[\underline{x}''_m(\xi,\omega,\alpha) + \bar{x}'_m(\xi,\omega,\alpha) + \underline{x}_m(\xi,\omega,\alpha) - \cos(W_t(\omega))]\right\} d\xi,$$

$$\mathcal{L}\{\bar{x}_{m+1}(t,\omega,\alpha)\} = \mathcal{L}\{\bar{x}_m(t,\omega,\alpha)\} + \mathcal{L}\left\{\int_0^t \bar{\lambda}(\xi,t)[\bar{x}''_m(\xi,\omega,\alpha) + \underline{x}'_m(\xi,\omega,\alpha) + \bar{x}_m(\xi,\omega,\alpha) - \cos(W_t(\omega))]\right\} d\xi.$$

Therefore, the general Lagrange multiplier $\mathcal{L}\{\underline{\lambda}(s,t)\} = \mathcal{L}\{\bar{\lambda}(s,t)\} = \frac{-1}{s^2}$. Hence, the lower and upper fuzzy solution $\tilde{x}$ are obtained using the successive approximate solutions as:

$\underline{x}_0(0,\omega_0,\alpha) = \alpha,\ \underline{x}'_0(0,\omega_0,\alpha) = 0.5\alpha,$

$\bar{x}_0(0,\omega_0,\alpha) = 1.5 - 0.5\alpha,\ \bar{x}'_0(0,\omega_0,\alpha) = 1 - 0.5\alpha,$

$$\underline{x}_1(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}_0(0,\omega_0,\alpha)}{s} + \frac{\underline{x}'_0(0,\omega_0,\alpha)}{s^2} - \frac{1}{s^2}\mathcal{L}[\bar{x}'_0(t,\omega,\alpha) + \underline{x}_0(t,\omega,\alpha) - \cos(W_t(\omega))]\right\}$$
$$= \alpha - 0.5\alpha t^2 + 0.5t^2\cos W_t(\omega) + 0.5\alpha t,$$
$$\bar{x}_1(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}_0(0,\omega_0,\alpha)}{s} + \frac{\bar{x}'_0(0,\omega_0,\alpha)}{s^2} - \frac{1}{s^2}\mathcal{L}[\underline{x}'_0(t,\omega,\alpha) + \bar{x}_0(t,\omega,\alpha) - \cos(W_t(\omega))]\right\}$$
$$= t - 0.5\alpha + 0.25\alpha t^2 + 0.5t^2 \, cos\, W_t(\omega) - 0.75t^2 - 0.5\alpha t + 1.5,$$
$$\underline{x}_2(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\underline{x}_0(0,\omega_0,\alpha)}{s} + \frac{\underline{x}'_0(0,\omega_0,\alpha)}{s^2} - \frac{1}{s^2}\mathcal{L}[\bar{x}'_1(t,\omega,\alpha) + \underline{x}_1(t,\omega,\alpha) - \cos(W_t(\omega))]\right\}$$
$$= \alpha - 0.25\alpha t^2 - 0.17\alpha t^3 + 0.042\alpha t^4 + 0.5t^2\cos W_t(\omega) - 0.17t^3\cos W_t(\omega) -$$
$$0.042t^4\cos W_t(\omega) - 0.5t^2 + 0.5\alpha t + 0.25t^3,$$
$$\bar{x}_2(t,\omega,\alpha) = \mathcal{L}^{-1}\left\{\frac{\bar{x}_0(0,\omega_0,\alpha)}{s} + \frac{\bar{x}'_0(0,\omega_0,\alpha)}{s^2} - \frac{1}{s^2}\mathcal{L}[\underline{x}'_1(t,\omega,\alpha) + \bar{x}_1(t,\omega,\alpha) - \cos(W_t(\omega))]\right\}$$
$$= t - 0.5\alpha + 0.25\alpha t^3 - 0.023\alpha t^4 + 0.5t^2\cos W_t(\omega) - 0.17t^3\cos W_t(\omega) -$$
$$0.042t^4\cos W_t(\omega) - 0.75t^2 - 0.17t^3 + 0.063t^4 - 0.5\alpha t + 1.5,$$

and so on, we can find the approximate lower and upper solutions up to the fifth iterations, which are simulated and presented in Figures 4 and 5 related to the above generations of Brownian motion given in Figure 1 with different α-levels.

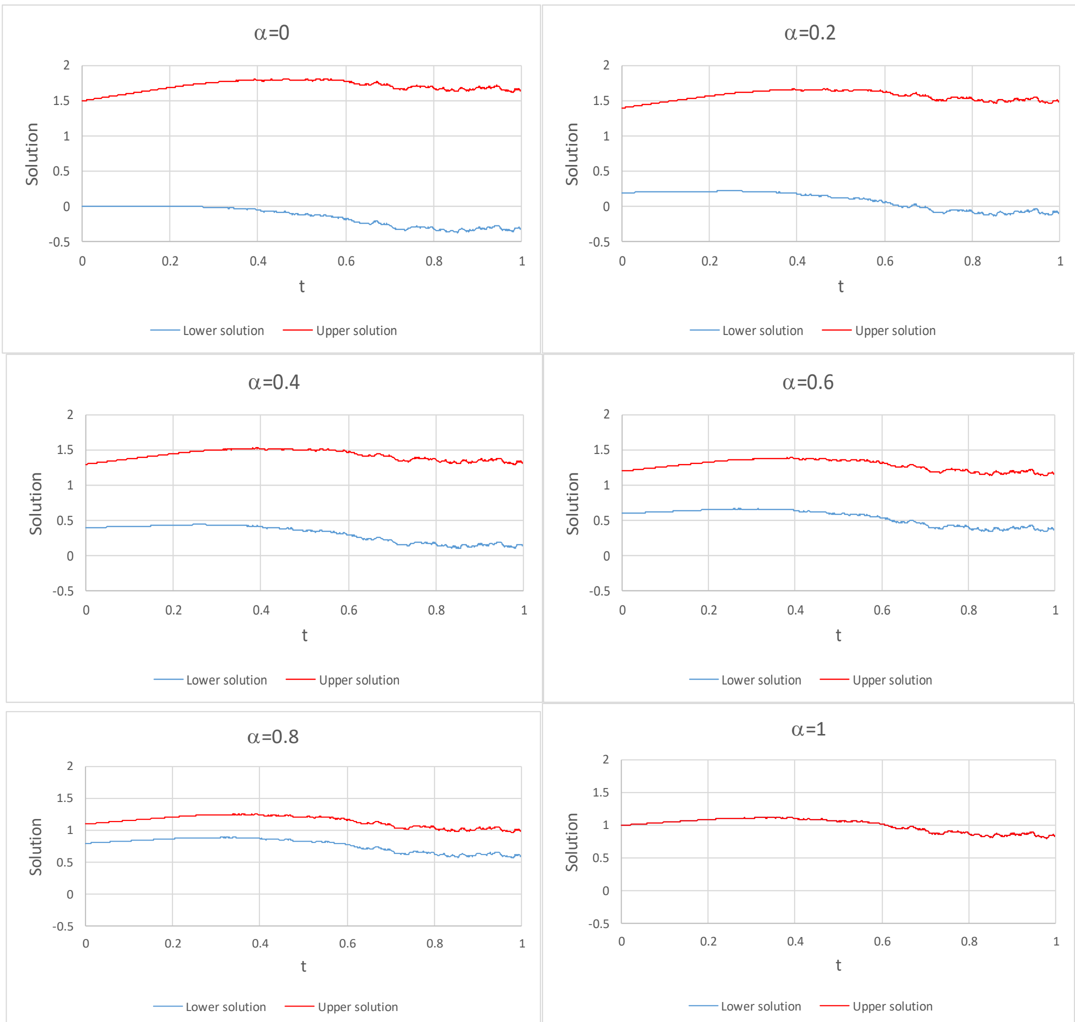

Figure 4: The fifth lower and upper iterative solutions of Example 2 using Laplace VIM for different values of α-levels and discretized Brownian motion with total signal processing number $N = 1000$.

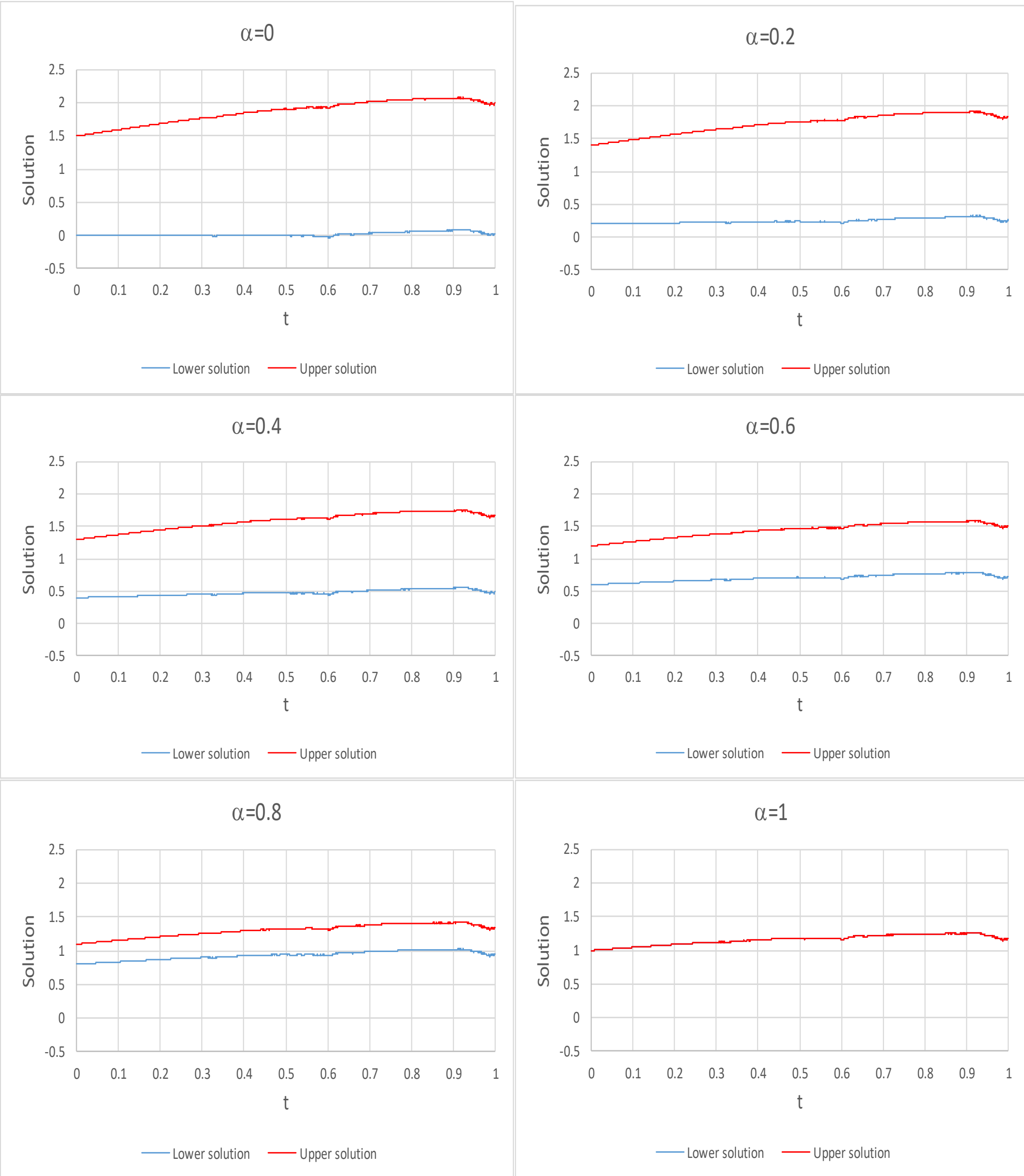

Figure 5: The fifth lower and upper iterative solutions of Example 2 using Laplace VIM for different values of α-levels and discretized Brownian motion with total signal processing number $N = 10000$.

6. Conclusions and Discussion of the Experimental Results

Laplace VIM is shown to be an effective and reliable method for solving linear FRODEs, where the analytical approximate solution may be obtained. Verification of the obtained results may be considered by comparing the approximate results for the lower and upper solutions for each α-level as well as the coincidence between them when $\alpha = 1$, which is the crisp solution.

The obtained results are presented in Figures 1-5 of the considered examples show that the stochastic process and fuzzy phenomena occurring in the differential equation will affect the behavior of the obtained solution, as well as the number generation of the Brownian motion. Also, the proposed approach that is used in this paper gives an analytical approximate solution, which may be helpful and needed later on in other real-life applications. In applications, such solutions, as an analytical function, are more reliable than numerical results that are obtained by other methods.

It is obvious to notice that the only limitation of the followed approach in this paper is its difficulty in solving nonlinear problems that can be treated or handled using the linearization approach or considered the Adomian decomposition method to treat nonlinear parts. This may be considered a recommendation for further work on this topic.

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