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On Subspace Codisk-Cyclicity

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Abstract.
Let $\mathcal{N}$ be a subspace of an infinite dimensional complex separable on a Hilbert space $\mathcal{H}$. The operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $\mathcal{N}$-codisk-cyclic, if there is a nonzero vector y in $\mathcal{H}$, then $\mathcal{N} \cap \{\beta T^n y: \beta \in \mathbb{C}, |\beta| \geq 1, n \in \mathbb{N}\}$ is dense in $\mathcal{N}$. This paper, introduces the properties of the concepts $\mathcal{N}$-codisk-cyclic and $\mathcal{N}$-codisk-cyclic transitive. The existence of a subspace codisk-cyclic operator on n -dimensional complex Hilbert space is illustrated and a criterion of $\mathcal{N}$-codisk-cyclic operator in infinite dimensional is obtained.

Keywords: Codisk-cyclic operators, Hilbert spaces, Codisk -cyclic transitive, dense sets.

حول المؤثرات الفضاء الجزئي القرصي المشترك

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الخلاصه
ليكن $\mathcal{N}$ فضاء جزئي من فضاء غير منته البعد قابل للفصل عقدي هلبرت $\mathcal{H}$. يقال للمؤثر $T \in \mathcal{B}(\mathcal{H})$ انه مؤثر قرصي مشترك من النمط $\mathcal{N}$ إذا وجد متجه غير صفري y في $\mathcal{H}$ بحيث انه المجموعة
$\mathcal{N} \cap \{\beta T^n y: \beta \in \mathbb{C}, |\beta| \geq 1, n \in \mathbb{N}\}$
تكون كثيفة في $\mathcal{N}$.
هذا البحث يتناول خواص المؤثرات القرصية المشتركة وخواص المؤثرات القرصية المشتركة المتعدية. يتم توضيح وجود مؤثر قرصي مشترك فضاء جزئي في فضاء هيلبرت المركب ذي الأبعاد n ويتم الحصول على معيار مؤثر قرصي مشترك من النمط $\mathcal{N}$ في الأبعاد اللانهائية.

1. INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable infinite dimensional Hilbert space $\mathcal{H}$, $T \in \mathcal{B}(\mathcal{H})$ is said to be hypercyclic operator, if the orbit of T with a nonzero vector y in $\mathcal{H}$, then $\text{orbt}(T, y) := \{T^n y: n \geq 0\}$ is dense in $\mathcal{H}$. Thus y is said to be a hypercyclic vector for T [1]

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The motivation of the studying of the scalar multiples of an orbit is due to the example of Rolewicz [2]. The operator T is called a supercyclic operator, if there exists a nonzero vector y where the cone generated by $orbt(T, y)$ is dense in $\mathcal{H}$ its definition is created by Hilden and Wallen in 1974 [3] Hypercyclicity are extensively studied by many researchers, for more detail see [1] [4].

Because the operator $\lambda B;\ |\lambda| \leq 1$ is not hypercyclic, where B is the backward shift operator on $\ell^p(\mathbb{N})$, one may wonder if there is an operator T such that its disk or co-disk orbital is dense in $\mathcal{H}$. The codisk-cyclicity concepts was presented by Jamil, 2002 [5]. The operator $T \in B(\mathcal{H})$ is a codisk-cyclic operator if there is a nonzero vector $y \in \mathcal{H}$, such that the codisk orbit, $C\mathbb{D}orbt(T, y) \coloneqq \{\beta T^n y: \beta \in \mathbb{C}, |\beta| \geq 1, n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, that vector y is said to be codisk-cyclic for T [5] Recently, several authors [6] [7] have studied a codisk-cyclic operators.

In 2010, Jamil in [8] shown that, if the $C\mathbb{D}orbt(T, y)$ is somewhere dense, then it is everywhere dense, that is closure of $int\big(C\mathbb{D}orbt(T, y)\big) \neq \emptyset$, then T must be codisk-cyclic. Hence, to discuss codisk-cyclicity for closed sets M, it must have empty interior, e.g., M is a nontrivial subspace.

The concepts of subspaces codisk-cyclicity and codisk – transitivity are presented in this paper. We give an example to ensure that not every subspace codisk-cyclic operator is codisk-cyclic. Some necessary and sufficient conditions of subspace codisk-transitive operators are investigated. Moreover, a subspace codisk-cyclic criterion is established, and discussed when these two concepts (subspaces codisk-cyclicity and codisk – transitivity) are equivalence.
We will abbreviate the set $\{z \in \mathbb{C}: |z| \geq 1\}$ by $\mathbb{B}^c$, $\{z \in \mathbb{C}: |z| \leq 1\}$ by $\mathbb{D}$ and $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

2. Subspace Codisk-Cyclic

In this section, we introduce a subspace codisk-cyclic operator and study some of its properties.

Definition 2.1. Let $\mathcal{N}$ be a nontrivial subspace of $\mathcal{H}$. A subspace codisk-cyclic operator T for $\mathcal{N}(\mathcal{N} -$ codisk-cyclic) that means, there is a non-zero $y \in H$, such that $\mathcal{N} \cap C\mathbb{D}orb(T, y)$ is dense in $\mathcal{N}$. This vector y is said to be a subspace codisk-cyclic vector for T.
Let us denote the set of all $\mathcal{N}$-codisk-cyclic vectors for T by
$C\mathbb{D}(T, \mathcal{N}) \coloneqq \{ y \in H : \mathcal{N} \cap \beta orb(T, y)$ is dense in $\mathcal{N}$; $\beta \in \mathbb{B}^c\}$.
and the set of all $\mathcal{N}$-codisk-cyclic operators by
$C\mathbb{D}(\mathcal{H}, \mathcal{N}) \coloneqq \{T \in B(\mathcal{H}): \mathcal{N} \cap \beta orb(T, y)$ is dense in $\mathcal{N}; y \in \mathcal{H}$ and $\beta \in \mathbb{B}^c\}$.
In general, the subspace codisk-cyclicity does not imply codisk-cyclicity as shown in the next example.

Remark 2.2. Let $T \in B(\mathcal{H})$ be a codisk-cyclic operator and y be a codisk-cyclic vector. Let $I \in B(\mathcal{H})$ be the identity operator. Hence, $I \oplus T \in B(\mathcal{H} \oplus \mathcal{H})$ is $\{0\} \oplus \mathcal{H}$ – codisk-cyclic operator for the subspace codisk-cyclic vector $0 \oplus y$. This is due to, the fact that $I \oplus T|_{\{0\}\oplus\mathcal{H}}$ is codisk-cyclic operator, where $\{0\} \oplus \mathcal{H}$ is $I \oplus T$ - invariant subspace. Note that, $I \oplus T$ is not codisk-cyclic.

Remark 2.3. Let y be a codisk-cyclic vector for $T \in B(\mathcal{H})$. Assume that $F \in B(\mathcal{H})$ is nonzero with closed range of $\mathcal{N}$. If $S \in B(\mathcal{H})$ satisfies $SF = FT$, then one can easily prove that S is $\mathcal{N}$-codisk-cyclic with $\mathcal{N}$-codisk-cyclic vector Fy.
Now, we prove that every n- dimensional Hilbert space contains $\mathcal{N}$- codisk-cyclicity.

Proposition 2.4. Every complex finite dimensional Hilbert space has a subspace codisk-cyclic operator.

Proof. Since any two n-dimensional complex Hilbert spaces are isomorphic, then it is enough to show the existence of any complex n-dimensional operator.
Let $\mathcal{N} = \{\hat{y}: \hat{y} = (a, 0, \dots ,0); \hat{y} \in \mathbb{C}_n\}$ be a subspace of $\mathbb{C}_n$; $n \in \mathbb{N}$ and $Tx = kx$; $x \in \mathbb{C}_n$, $k \in \mathbb{C}$, such that $|k| < 1$. Then let $x = (1,0, \dots ,0)$. Thus,
$C\mathbb{D}(T, x) \cap \mathcal{N} = \{(\beta k^n, 0, \dots ,0): |\beta| \geq 1, n \geq 0\}$.
Let $\hat{z} = (b, 0, \dots ,0) \in \mathcal{N}$ and let us choose an $n \in \mathbb{N}$, such that $|k^n| \leq |b|$, then $\hat{z} = \left(\left(\frac{b}{k^n}\right) k^n, 0, \dots ,0\right) \in C\mathbb{D}(T, x) \cap \mathcal{N}$. Hence T is $\mathcal{N}$-codisk-cyclic operator.

Proposition 2.5. If $T \in B(H)$, $\mathcal{N}$ is a non-zero subspace of H, then $C\mathbb{D}(T, \mathcal{N}) = \cap_k \left(\cup_{\beta \in \mathbb{D}} \cup_n \ T^{-n}(\beta V_k)\right)$.

Proof: Let $\{V_k\}_{k=1}^{\infty}$ be a countable basis for the relative topology on $\mathcal{N}$. $x \in C\mathbb{D}(T, \mathcal{N})$ if and only if, for all k larger than zero, there exist $n \in \mathbb{N}$, and $\alpha \in \mathbb{B}^c$ such that $\alpha T^n x \in V_k$, if and only if $x \in \cap_k \left(\cup_{\beta \in \mathbb{D}} \cup_n \ T^{-n}(\beta V_k)\right)$.

3. Subspace Codisk-Cyclic transitive

In this section, we aim to introduce the subspace Codisk – cyclic transitive and study some of its properties.

Definition 3.1. Let $\mathcal{N}$ be a nonzero subspace of $\mathcal{H}$. The operator $T \in B(\mathcal{H})$ is $\mathcal{N} -$ codisk transitive, if for each nonempty relatively open set V, U in $\mathcal{N}$, there is n $\in \mathbb{N}_0$ and $\alpha \in \mathbb{B}^c$, such that $U \cap T^n(\alpha V)$ has a nonempty relatively open set of $\mathcal{N}$.

Proposition 3.2: If an operator T is $\mathcal{N}$ $-$codisk transitive, then for any nonempty relatively open sets $V, U \subseteq \mathcal{N}$, there are $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{B}^c$, such that $U \cap T^n(\alpha V)$ is non-empty and $\mathcal{N}$ is invariant under T^n.

Proof: Since T be $\mathcal{N}$ - codisk-cyclic transitive and U and V be non-empty relatively open sets of $\mathcal{N}$, then Definition 3.1 implies that, there exists $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{B}^{\,c}$, such that $U \cap T^n\,(\alpha V)$ has a nonempty relatively open set of $\mathcal{N}$. Then $U \cap T^n\,(\alpha V) \neq \emptyset$. Thus $V \cap T^{-n}\left(\frac{1}{\alpha} U\right) \neq \emptyset$, say that Y.
Now, let x in $\mathcal{N}$. Since $Y \subseteq T^{-n}\left(\frac{1}{\alpha} U\right)$, hence $T^n(\alpha Y) \subseteq U \subseteq \mathcal{N}$. Take x_0 in Y, then $T^n(\alpha x_0) \in \mathcal{N}$. Since Y is nonempty relatively open set in $\mathcal{N}$ and $x \in \mathcal{N}$, thus for small enough $r > 0$, we get, $x_0 + rx \in Y$, thus $T^n(\alpha x_0) + r\alpha T^n(x) \in T^n(\alpha Y) \subseteq \mathcal{N}$. This leads to $T^n(x) \in \mathcal{N}$. Therefore, $T^n(\mathcal{N}) \subseteq \mathcal{N}$.
The converse of the Proposition 3.2 is not true unless T is open mapping or has inverse on $\mathcal{N}$. In fact, since $T^n|\mathcal{N}: \mathcal{N} \to \mathcal{N}$ is bounded and U is relatively open in $\mathcal{N}$, hence $T^{-n}\left(\frac{1}{\alpha} U\right)$ is relatively open set in $\mathcal{N}$, thus $V \cap T^{-n}\left(\frac{1}{\alpha} U\right)$ is relatively open set in $\mathcal{N}$. The following result is done

Proposition 3.3. The following statements are equivalent, where $T \in B(\mathcal{H})$ is open mapping or bijective on a nonzero subspace $\mathcal{N}$ of $\mathcal{H}$:
1) The operator T is $\mathcal{N}$ $-$codisk transitive.

2) For any nonempty relatively open sets $V, U \subseteq \mathcal{N}$, there are $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{B}^c$, such that $U \cap T^n(\alpha V)$ is a nonempty and $T^n(\mathcal{N}) \subseteq \mathcal{N}$.
3) For all non-empty relatively open sets $V, U \subseteq \mathcal{N}$, there exist $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{B}^c$, such that $U \cap T^n(\alpha V)$ is a non-empty relatively open set of $\mathcal{N}$.
Now we turn our attention to discuss the necessary condition for an operator to be $-$codisk transitive .

Lemma 3.4. Let $\mathcal{N}$ be a nonzero subspace of $\mathcal{H}$, and $\{V_j\}$ be a countable open basis for the relative topology of $\mathcal{N}$. If $T \in B(H)$ is a $\mathcal{N} -$ codisk transitive, then

$$\bigcap_{j=1}^{\infty} \bigcup_{\alpha \in \mathbb{B}^c} \bigcup_{n=0}^{\infty} T^n(\alpha V_j) \bigcap \mathcal{N}$$

is a dense subset of $\mathcal{N}$.
Proof: By Definition 3.1, for each j and, there exist $n_{j,k} \in \mathbb{N}_0$ and $\alpha \in \mathbb{B}^c$, such that the set $T^{n_{j,k}}(\alpha V_j) \cap V_k$ has a nonempty relatively open set. Hence the set

$$A_j \coloneqq \bigcup_{\alpha \in \mathbb{B}^c} \bigcup_{k=1}^{\infty} \left(T^{n_{j,k}}(\alpha V_j) \bigcap V_k\right)$$

has a nonempty relatively open set in $\mathcal{N}$, say $\widehat{A_j}$.
Moreover, each $\widehat{A_j}$ is dense of $\mathcal{N}$, since it intersects each B_k. Thus, by using Baire Category theorem, we get,

$$\bigcap_{j=1}^{\infty} \bigcup_{\alpha \in \mathbb{B}^c} \bigcup_{n=0}^{\infty} T^n(\alpha V_j) \bigcap \mathcal{N} \supseteq \bigcap_{j=1}^{\infty} \left(\bigcup_{\alpha \in \mathbb{B}^c} \bigcup_{k=1}^{\infty} T^{n_{j,k}}(\alpha V_j) \bigcap V_k \right)$$
$$= \bigcap_{j=1}^{\infty} A_j \supseteq \bigcap_{j=1}^{\infty} \widehat{A_j},$$

is a dense subset of $\mathcal{N}$.
Clearly, the following Proposition is implied from combining Lemma 3.4 and Proposition 2.5.

Proposition 3.5: If $T \in B(\mathcal{H})$ is an open mapping (or bijective) $\mathcal{N} -$ codisk transitive, then T is $\mathcal{N}$ $-$codisk-cyclic operator.

Proposition 3.6: Let $T \in B(\mathcal{H})$ and $\mathcal{N}$ be a nonempty subspace of H such that:
1) T is an open mapping or bijective on $\mathcal{N}$.
2) There are dense sets Y, X in $\mathcal{N}$ and S is an operator on $\mathcal{N}$(not need to be bounded), such that $S(Y) \subset Y$ and $TS = I_Y$.
3) There is a sequence $\{n_k\}$ in $\mathbb{N}$, such that,
a) $\lim_{n\to\infty} \inf \|T^{n_k}x\| = 0$ for all $x \in X$.
b) $\lim_{n\to\infty} \inf \|T^{n_k}x\| \|S^{n_k}y\| = 0$ for all $x \in X, y \in Y$.
Then T is $\mathcal{N}$- codisk-cyclic transitive, hence T is a $\mathcal{N}$ $-$codisk-cyclic operator.

Proof: Let V and U be two relatively open sets in $\mathcal{N}$. Since Y and X are dense sets in $\mathcal{N}$, then from the condition (2), there are $x \in X \cap V$ and $y \in Y \cap U$, such that for some sequence $\{n_k\}$ in $\mathbb{N}$ and $0 < \mathcal{E} < 1$,

$$\|T^{n_k}x\| < \frac{\mathcal{E}}{2} \dots (I)$$
$$\|T^{n_k}x\| \|S^{n_k}y\| < \frac{\mathcal{E}}{4} \dots (II)$$

If $\|T^{n_k}x\| \neq 0$, put $c = 2\,\|T^{n_k}x\|$. Thus, $0 < c < 1$. Take $u = x + cS^{n_k}\,y$, and $\alpha = \frac{1}{c}$. Hence by the equation (II), $\|u - x\| = \|cS^{n_k}\,y\| < \frac{\varepsilon}{2}$. Therefore, $u \in V$.
Now, since $T^{n_k}u = T^{n_k}x + cy$, thus, $\|\alpha T^{n_k}u - y\| = \frac{1}{c}\|T^{n_k}x\| < 1$.
Then, $\alpha T^{n_k}u \in U$. Therefore, $U \cap T^{n_k}(\alpha V)$ is a nonempty relatively open set of $\mathcal{N}$.
Now, if $\|T^{n_k}x\| = 0$, then choose $0 < c < 1$ small enough, such that $c\|S^{n_k}\,y\| < \frac{1}{2}$, and let $\alpha = \frac{1}{c}$. Thus, $\|u - x\| < \frac{1}{2}$ and $\|\alpha T^{n_k}u - y\| < 1$. Hence, $U \cap T^{n_k}(\alpha V)$ is a nonempty relatively open set of $\mathcal{N}$.

4-Conclusions:

A bounded linear operator T on subspace, say $\mathcal{N}$, of an infinite dimensional complex separable Hilbert space $\mathcal{H}$ is said to be $\mathcal{N}$-codisk-cyclic if satisfy:

$$\mathcal{N} \cap \{\beta T^n y: \beta \in \mathbb{C}, |\beta| \geq 1, n \in \mathbb{N}\},$$

is dense in $\mathcal{N}$, for some nonzero vector y in $\mathcal{H}$. This paper, presented two new concepts, $\mathcal{N}$-codisk-cyclic and $\mathcal{N}$-codisk-cyclic transitive. We prove that their existence of a $\mathcal{N}$ - codisk-cyclic operator on n -dimensional complex Hilbert space, also prove a criterion of $\mathcal{N}$-codisk-cyclic operator in infinite dimensional. Finally, we discussed the relation between these two concepts.

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