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Generalization of Gamma and Beta Functions with Certain Properties and Statistical Application

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Abstract

This work is devoted to define new generalized gamma and beta functions involving the recently suggested seven-parameter Mittag-Leffler function, followed by a review of all related special cases. In addition, necessary investigations are affirmed for the new generalized beta function including, Mellin transform, differential formulas, integral representations, and essential summation relations. Furthermore, crucial statistical application has been realized for the new generalized beta function.

Keywords: Gamma function, Beta function, Integral representation, Mellin transform, Beta distribution.
MSC 2010: 32A10, 33B15, 30G30.

تعميم لدوال گاما وبيتا مع خواص محددة وتطبيق احصائي

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الخلاصة

هذا البحث مخصص لتعريف تعميم جديد لدالتي گاما وبيتا باستخدام دالة ميتاج–لفلر ذات السبع معلمات المقترحة مؤخرًا، متبوعة باستعراض لجميع الحالات الخاصة المرتبطة. بالإضافة إلى ذلك، استقصاءات ضرورية تم اثباتها لدالة بيتا المعممة الجديدة، بما في ذلك صيغة جمعية، صيغ تفاضلية، تمثيلات تكاملية، تحويل ميلن، وبعض علاقات التكرار . علاوة على ذلك، تم استحصال تطبيق إحصائي مهم لوظيفة بيتا المعممة الجديدة.

1. Introduction

It is commonly agreed that the classical gamma and beta functions, which perform boundless functions, are the most significant members of the class of special functions. These well-known functions have attracted considerable attention to studying numerous properties owing to their miscellaneous applications in a range of scientific fields, specifically mathematical physics, statistics, and engineering, see [1-4].

The classical gamma and beta function usually known as the Euler's integral of first and second kind respectively [5],

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$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, dt, \tag{1.1}$$

where $Re(z) > 0$.

$$\beta(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \tag{1.2}$$

where $Re(z),\ Re(w) > 0$. In early of the 1990s, some academics and researchers began curious investigations to achieve extensions and generalizations of the classical gamma and beta functions, as well as, to certify a number of beneficial properties and applications.

In 1994, Chaudhry and Zubair [6], extended the gamma function for the first time by extended the regularizer e^{-t},

$$\Gamma_\rho(z) = \int_0^\infty t^{z-1} \exp\left(-t - \frac{\rho}{t}\right) dt, \tag{1.3}$$

where, $Re(z),\ Re(\rho) > 0$.

Analogously in 1997, Chaudry et al. [7], presented an extension for beta function

$$\beta_\rho(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} \exp\left(\frac{-\rho}{t(1-t)}\right) dt, \tag{1.4}$$

where, $Re(z),\ Re(w),\ Re(\rho) > 0$.

In 2011, Ozergin et al. [8], implied generalizations of gamma and beta functions by means of the confluent hypergeometric function

$$\Gamma_\rho^{(u,\tau)}(z) = \int_0^\infty t^{z-1}\, {}_1F_1\left(u;\ \tau;\ -t - \frac{\rho}{t}\right) dt, \tag{1.5}$$

where $Re(z),\ Re(u),\ Re(\tau),\ Re(\rho) > 0$.

$$\beta_\rho^{(u,\tau)}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, {}_1F_1\left(u;\ \tau;\ -\frac{\rho}{t(1-t)}\right) dt, \tag{1.6}$$

where $Re(z),\ Re(w),\ Re(u),\ Re(\tau),\ Re(\rho) > 0$.

Thereafter, shadab et al. [9], submitted another extension of beta function by virtue of standard Mittag-Leffler function

$$\beta_\sigma^\rho(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, E_\sigma\left(-\frac{\rho}{t(1-t)}\right) dt, \tag{1.7}$$

where $Re(z),\ Re(w),\ Re(\sigma),\ Re(\rho) > 0$.

Subsequently, Pucheta [10], involved the standard Mittag-Leffler function to introduce a modified extension of gamma and beta functions

$$\Gamma^\sigma(z) = \int_0^\infty t^{z-1}\, E_\sigma(-t) dt, \tag{1.8}$$

where $Re(z), Re(\sigma), Re(\rho) > 0$.

$$\beta_\rho^\sigma(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, E_\sigma\big(-\rho t(1-t)\big)\, dt, \tag{1.9}$$

where $Re(z),\ Re(w),\ Re(\sigma),\ Re(\rho) > 0$.

In 2018, Atash et al. [11] considered a new extension of gamma and beta functions by using Wiman's function

$$\Gamma_{\rho}^{(\sigma,\lambda)}(z)=\int_{0}^{\infty} t^{z-1}\,E_{\sigma,\lambda}\left(-t-\frac{\rho}{t}\right)\,dt, \tag{1.10}$$

where $Re(z),\ Re(\sigma), Re(\lambda),\ Re(\rho)>0$.

$$\beta_{\rho,q}^{(\sigma,\lambda)}(z,w)=\int_{0}^{1} t^{z-1}(1-t)^{w-1}E_{\sigma,\lambda}\left(-\frac{\rho}{t}\right)\,E_{\sigma,\lambda}\left(-\frac{q}{1-t}\right)dt, \tag{1.11}$$

where $Re(z),\ Re(w),\ Re(\sigma),\ Re(\lambda),\ \ Re(\rho),\ Re(q)>0$.

However, Al-Gonah and Mohammed [12], utilized three-parametric Mittag-Leffler function to introduce modern extensions of gamma and beta functions

$$\Gamma_{\rho}^{(\sigma,\lambda,u)}(z)=\int_{0}^{\infty} t^{z-1}\,E_{\sigma,\lambda}^{u}\left(-t-\frac{\rho}{t}\right)\,dt, \tag{1.12}$$

where $Re(z),\ Re(u),\ Re(\sigma),\ Re(\lambda),\ Re(\rho)>0$.

$$\beta_{\rho}^{(\sigma,\lambda,u)}(z,w)=\int_{0}^{1} t^{z-1}\,(1-t)^{w-1}E_{\sigma,\lambda}^{u}\left(-\frac{\rho}{t(1-t)}\right)\,dt, \tag{1.13}$$

where $Re(z), Re(w),\ Re(u),\ Re(\sigma),\ Re(\lambda),\ Re(\rho)>0$.

In 2020, Oraby and Rizq [13], gave additional generalized beta functions in terms of Wiman's function

$$\beta_{\rho,\lambda}^{\sigma,a}(z,w)=\int_{0}^{1} t^{z-1}(1-t)^{w-1}E_{\sigma,\lambda}\left(\frac{-\rho}{t^{a}(1-t)^{a}}\right)dt, \tag{1.14}$$

where $Re(z),\ Re(w),\ \ Re(\sigma),\ Re(\lambda),\ Re(a)>0,$ and $Re(\rho)\geq 0$.

Recently, Khan and Husain [14], supplied a new extension of beta function through Wiman's function

$$\beta_{\sigma,\lambda}^{\rho,a,b}(\mathrm{z},\mathrm{w})=\int_{0}^{1} t^{z-1}(-t)^{w-1}E_{\sigma,\lambda}\left(\frac{-\rho}{t^{a}(1-t)^{b}}\right)dt, \tag{1.15}$$

where $Re(z),\ Re(w)>0,\ Re(\rho)>0;\ \ \sigma,\lambda\in\mathbb{R}_{0}^{+}$ and $a,b\in\mathbb{R}^{+}$.

The major idea of this paper is to define new generalized gamma and beta functions by means of the seven-parameter Mittag-Leffler function proposed in [15], and review all related special cases. In addition, it develops certain vital properties and necessary functional formulas to the new generalized beta function. Further, attempt to obtain statistical application for the new generalized beta function that is beneficial in the area of special function's statistical applications.

2. Generalized Gamma and Beta Functions

The purpose of this segment is to introduce the new generalized gamma and beta functions in terms of the Mittag-Leffler function with seven complex parameters, as well as show all the special cases derived from them.

Consider the Mittag-Leffler function involving seven parameters proposed by Rasheed and Majeed in [15],

$$E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z)=\sum_{k=0}^{\infty}\frac{(u)_k(v)_k}{(\tau)_k\ \ k!}\frac{z^k}{\Gamma(\sigma_1 k+\lambda_1)\Gamma(\sigma_2 k+\lambda_2)}, \tag{2.1}$$

where $z\in\mathbb{C}, \min\{Re(u),Re(v),Re(\tau),Re(\sigma_1),Re(\sigma_2),Re(\lambda_1),Re(\lambda_2)\}>0$. Accordingly, the new generalized gamma and beta functions will be defined next.

Definition 2.1 Let min $\{Re(z),Re(u),Re(v),Re(\tau),Re(\sigma_1),Re(\sigma_2),Re(\lambda_1),Re(\lambda_2), Re(\rho)\}>0$,

$$\Gamma^{u,v,\tau,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z)=\int_0^{\infty}t^{z-1}\,E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(-t-\frac{\rho}{t}\right)dt. \tag{2.2}$$

Remark 2.2 The functions (1.12), (1.10), (1.8), (1.5), (1.3), and (1.1) are related special cases of the function (2.2), as reviewed respectively:

$$\Gamma^{u,v,v,\rho}_{\sigma_1,\lambda_{1,0,1}}(z)=\Gamma^{(\sigma_1,\lambda_1,u)}_{\rho}(z).$$
$$\Gamma^{1,v,v,\rho}_{\sigma_1,\lambda_{1,0,1}}(z)=\Gamma^{(\sigma_1,\lambda_1)}_{\rho}(z).$$
$$\Gamma^{1,v,v,0}_{\sigma_{1,1,0,1}}(z)=\Gamma^{\sigma_1}(z).$$
$$\Gamma^{u,1,\tau,\rho}_{1,1,0,1}(z)=\Gamma^{(u,\tau)}_{\rho}(z).$$
$$\Gamma^{1,v,v,\rho}_{1,1,0,1}(z)=\Gamma_{\rho}(z).$$
$$\Gamma^{1,v,v,0}_{1,1,0,1}(z)=\Gamma(z).$$

In addition to the associative special cases of previous functions.

Definition 2.3 Let min $\{Re(z),Re(w),Re(u),Re(v),Re(\tau),Re(\sigma_1),Re(\sigma_2),Re(\lambda_1), Re(\lambda_2),Re(z),Re(\rho)\}>0$, and $a,b\ \in\mathbb{R}^+$ then

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt. \tag{2.3}$$

Remark 2.4 The functions (1.15), (1.14), (1.13), (1.7), (1.6), (1.4), and (1.2) appear as special cases of the function (2.3) respectively:

$$\beta^{1,v,v,a,b,\rho}_{\sigma_1,\lambda_{1,0,1}}(z,w)=\beta^{a,b,\rho}_{\sigma_1,\lambda_1}(z,w).$$
$$\beta^{1,v,v,a,a,\rho}_{\sigma_1,\lambda_{1,0,1}}(z,w)=\beta^{\sigma_1,a}_{\rho,\lambda_1}(z,w).$$
$$\beta^{u,v,v,1,1,\rho}_{\sigma_1,\lambda_{1,0,1}}(z,w)=\beta^{(\sigma_1,\lambda_1,u)}_{\rho}(z,w).$$
$$\beta^{1,v,v,1,1,\rho}_{\sigma_{1,1,0,1}}(z,w)=\beta^{\rho}_{\sigma_1}(z,w).$$
$$\beta^{u,1,\tau,1,1,\rho}_{1,1,0,1}(z,w)=\beta^{(u,\tau)}_{\rho}(z,w).$$
$$\beta^{1,v,v,1,1,\rho}_{1,1,0,1}(z,w)=\beta_{\rho}(z,w).$$
$$\beta^{1,v,v,1,1,0}_{1,1,0,1}(z,w)=\beta(z,w).$$

Besides the implicit special cases of the preceding functions.

3. Certain Properties of $\boldsymbol{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}$ in the Complex Plane

This section will mainly focus on discussing a number of interesting properties and formulas of the functions (2.3), such as the Mellin transform, differentiation formulas, integral representations, and summation relations.

The notion that the function (2.3) is convergent and that it satisfies the symmetric relation with respect to the complex variables z and w is often immediately obvious. So then, we proceed to demonstrating that the function (2.3) has a significant relationship with the Euler beta function (1.2).

Theorem 3.1 The function (2.3) satisfy the following relation

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w) = E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(-\rho)\beta(z-ak,w-bk). \tag{3.1}$$

Proof.

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt$$

$$= \sum_{k=0}^{\infty} \frac{(u)_k(v)_k}{(\tau)_k\ k!}\frac{-\rho^k}{\Gamma(\sigma_1 k+\lambda_1)\Gamma(\sigma_2 k+\lambda_2)}\int_0^1 t^{z-ak-1}(1-t)^{w-bk-1}\ dt.$$

This reveals our gained result directly.

The next result dictate the Mellin transform for the function (2.3) as a notable relation that also provides a relationship between the function and the classical beta function (1.2).

Theorem 3.2 For $Re(s) > 0, Re(z+s) > 0$, and $Re(w+s) > 0$, the function (2.2) has the following Mellin Transform representation

$$M\left\{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)\right\}(s) = \beta(z+as-a-1,w+bs-b-1)\ \Gamma^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(s). \tag{3.2}$$

Proof.

$$M\left\{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)\right\}(s) = \int_0^\infty \rho^{s-1}\left(\int_0^1 t^{z-1}(1-t)^{w-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt\right)d\rho.$$

Because of the uniform convergence of the above integral, one can change the order of integration. Hence

$$M\left\{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)\right\}(s) = \int_0^1 t^{z-1}(1-t)^{w-1}\left[\int_0^\infty \rho^{s-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)d\rho\right]dt.$$

In order to simplify the above expression, set $\mu = \frac{-\rho}{t^a(1-t)^b}$, implies

$$M\left\{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)\right\}(s) = \int_0^1 t^{z+as-a}(1-t)^{w+bs-b}\left[\int_0^\infty \mu^{s-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(-\mu)d\mu\right]dt.$$

The proof is immediately evident by writing the preceding integral in terms of the classical beta function and the function (2.2). ■

Corollary 3.3 Using the inverse Mellin transform, give the following integral formula:

$$\beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \beta(z+as-a-1, w+bs-b-1)\, \Gamma_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}(s)\rho^{-s}\, ds. \quad (3.3)$$

The following theorem gives further integral representations of the function (2.2).

Theorem 3.4

$$\beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = 2\int_0^{\pi/2} cos^{2z-1}\theta\, sin^{2w-1}\theta\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}[-\rho\,(sec^2\theta)^a\,(csc^2\theta)^b]\, d\theta. \quad (3.4)$$

$$\beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = \int_0^{\infty} \frac{\gamma^{z-1}}{(1+\gamma)^{z+w}} E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho\,(1+\gamma)^{a+b}}{\gamma^a}\right) d\gamma. \quad (3.5)$$

$$\beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = 2^{1-z-w}\int_{-1}^{1} (1+\delta)^{z-1}(1-\delta)^{w-1}\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho\, 2^{a+b}}{(1+\delta)^a(1-\delta)^b}\right) d\delta \quad (3.6)$$

Proof. Formulas (3.4), (3.5), and (3.6) can immediately prove by setting $t = cos^2\theta$, $t = \gamma/1+\gamma$ and $t = \frac{1+\delta}{2}$ in (2.2) respectively. ■

A validation of the differentiation formulas for the function (2.3) is provided in the following theorem.

Theorem 3.5 For $m, n \in \mathbb{N}_0$, the following differentiation formulas are hold:

$$\frac{\partial^m}{\partial z^m}\, \beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = \int_0^1 t^{z-1}[\ln(t)]^m\,(1-t)^{w-1}\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho}{t^a(1-t)^b}\right) dt. \quad (3.7)$$

$$\frac{\partial^m}{\partial w^n}\, \beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = \int_0^1 t^{z-1}\,(1-t)^{w-1}\,[\ln(1-t)]^n\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho}{t^a(1-t)^b}\right) dt. \quad (3.8)$$

$$\frac{\partial^{m+n}}{\partial z^m \partial w^n}\, \beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) = \int_0^1 t^{z-1}\,[\ln(t)]^m\,(1-t)^{w-1}[\ln(1-t)]^n\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho}{t^a(1-t)^b}\right) dt. \quad (3.9)$$

Proof. Through using Leibniz rule, we authenticate each of the above formulas,

$$\begin{aligned}\frac{\partial^m}{\partial z^m}\, \beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w) &= \int_0^1 \left(\frac{\partial^m}{\partial z^m} t^z\right)\, t^{-1}\,(1-t)^{w-1}\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho}{t^a(1-t)^b}\right) dt \\ &= \int_0^1 t^{z-1}[\ln(t)]^m\,(1-t)^{w-1}\, E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau}\left(\frac{-\rho}{t^a(1-t)^b}\right) dt.\end{aligned}$$

In an analogous manner, we acquire the formulas (3.8) and (3.9). ■

What follows construct a functional and summation relations for the function (2.3), respectively.

Lemma 3.6 Let $Re(z+1) > 0$, and $Re(w+1) > 0$, then

$$\beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w+1) + \beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z+1,w) = \beta_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,\nu,\tau,a,b,\rho}(z,w). \quad (3.10)$$

Proof.

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w+1)+\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+1,w)$$
$$=\int_0^1 t^{z-1}(1-t)^{w}+t^{z}(1-t)^{w-1}\,E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt$$
$$=\int_0^1 t^{z-1}(1-t)^{w-1}\,E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt$$
$$=\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w).\ \blacksquare$$

Theorem 3.7 The following summation relation holds for the function (2.3)

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)=\sum_{\mathrm{n}=0}^{\mathrm{m}}\binom{m}{n}\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+n,w+m-n)\,,m\in\mathbb{N}_0. \tag{3.11}$$

Proof. From Lemma (3.6) we have

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)=\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+1,w)+\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w+1).$$

Employing the same logic to the two preceding terms, implies

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)$$
$$=\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+2,w)+\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w+2)+2\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+1,w+1).$$

Using mathematical induction to repeat the process, we set the desired result. ■

Theorem 3.8 For the function (2.3) the following summation relation holds

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,1-w)=\sum_{n=0}^{\infty}\frac{(w)_n}{n!}\,\beta^{a,b,c,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+n,1). \tag{3.12}$$

Proof. From the definition of the new generalized beta function we have,

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,1-w)=\int_0^1 t^{z-1}(1-t)^{-w}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt.$$

Recall the binomial series expansion

$$(1-t)^{-w}=\sum_{n=0}^{\infty}\frac{(w)_n\,t^n}{n!},|t|<1.$$

Hence,

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,1-w)=\sum_{n=0}^{\infty}\frac{(w)_n}{n!}\int_0^1 t^{z+n-1}\,E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt$$
$$=\sum_{n=0}^{\infty}\frac{(w)_n}{n!}\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+n,1).\ \blacksquare$$

Theorem 3.9 The function (2.3) satisfies the following summation relation

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)=\sum_{n=0}^{\infty}\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+n,w+1). \tag{3.13}$$

Proof. Apply the following expression in the function (2.3),

$$(1-t)^{w-1}=(1-t)^{w}\sum_{n=0}^{\infty}\mathrm{t}^n\,,|t|<1.$$

We get

$$\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)=\sum_{n=0}^{\infty}\int_0^1 t^{z+n-1}(1-t)^{w}\,E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt$$

which provide the result we desire. ■

4. Statistical Application for $\boldsymbol{\beta}^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(\boldsymbol{z},\boldsymbol{w})$

Parallel to the Euler's beta function, this section committee the statistical distribution for the new generalized beta function (2.3), as well as mean, variance, moment generating function, cumulative distribution, and reliability function.

The new generalized beta distribution is given as

$$f(t)=\begin{cases}\dfrac{1}{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}\ t^{z-1}(1-t)^{w-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\dfrac{-\rho}{t^a(1-t)^b}\right), & (0<t<1),\\ 0 & otherwise\end{cases} \quad (4.1)$$

$$(z,w,u,v,\tau,a,b,\rho,\sigma_1,\lambda_1,\sigma_2,\lambda_2,a,b\in\mathbb{R}^+).$$

Thus, the rth moment for the random variable Y given as

$$E(Y^r)=\frac{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+r,w)}{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}, \quad (4.2)$$

where the particular instances provide the mean and variance of the distribution respectively

$$\mu=E(Y)=\frac{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+1,w)}{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}, \quad (4.3)$$

$$\sigma^2=E(Y^2)-\{E(Y)\}^2=\frac{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)\ \beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+2,w)-\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+1,w)}{\left\{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)\right\}^2}. \quad (4.4)$$

The distribution's moment generating function can be expressed as

$$M(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}E(Y^n)=\frac{1}{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}\sum_{n=0}^{\infty}\frac{t^n}{n!}\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z+n,w). \quad (4.5)$$

The cumulative distribution of the function is defined as

$$F(y)=\frac{\beta^{u,v,\tau,a,b,\rho}_{y,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}, \quad (4.6)$$

where $\beta^{u,v,\tau,a,b,\rho}_{y,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)$ is the generalized incomplete beta function

$$\beta^{u,v,\tau,a,b,\rho}_{y,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)=\int_0^y t^{z-1}(1-t)^{w-1}E^{u,v,\tau}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}\left(\frac{-\rho}{t^a(1-t)^b}\right)dt. \quad (4.8)$$

$$(z,w,u,v,\tau,a,b,\rho,\sigma_1,\lambda_1,\sigma_2,\lambda_2,a,b\in\mathbb{R}^+).$$

We can directly obtain the reliability function.

$$R(t)=1-F(t)=\frac{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)-\ \beta^{u,v,\tau,a,b,\rho}_{y,\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}{\beta^{u,v,\tau,a,b,\rho}_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}(z,w)}. \quad (4.10)$$

5. Conclusion and Discussion

This paper defines a new generalization of gamma and beta functions using the seven-parameter Mittag-Leffler function that was recently presented and then reviews all the special functions deduced from these functions. Later, this paper was dedicated to discussing the most prominent properties related to the new generalized beta function, such as its relationship with the Euler's beta function, Mellin transform, some integral representations, differential formulas, and basic summation relations. Finally, we derive a statistical application of the new beta function by virtue of statistical facts.

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