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Inference for Generalized Inverted Exponential Distribution UnderProgressive Type-I Interval Censored Data

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Abstract

This article discusses the estimation methods for parameters of a generalized inverted exponential distribution with different estimation methods by using Progressive type-I interval censored data. In addition to conventional maximum likelihood estimation, the mid-point method, probability plot method and method of moments are suggested for parameter estimation. To get maximum likelihood estimates, we utilize the Newton-Raphson, expectation -maximization and stochastic expectation-maximization methods. Furthermore, the approximate confidence intervals for the parameters are obtained via the inverse of the observed information matrix. The Monte Carlo simulations are used to introduce numerical comparisons of the proposed estimators. In addition, we use the percentile bootstrapping technique that is used to calculate confidence intervals. The proposed methodology in a real-life using the survival times of guinea pigs inoculated with different doses of tubercle bacilli data are considered to offer the applicability of the suggested methods.

Keywords: Generalized Inverted Exponential Distribution (GIED), Progressive type-I interval censored, Probability plot, Stochastic expectation- maximization (SEM), Expectation-maximization(EM).

الاستدلال حول التوزيع الأسي المعكوس المعمم تحت النوع التدريجي الأول من بيانات الفاصل الزمني الخاضعة للرقابة

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الخلاصة

تناقش هذا البحث مسالة التقدير لمعلمات التوزيع الآسي المعكوس المعمم بطرق تقدير مختلفة باستخدام البيانات المتدرجة من النوع الأول خاضعة للرقابة. بالإضافة إلى تقدير الترجيح الأعظم التقليدية ، تم اقتراح طريقة النقطة الوسطى وطريقة مخطط الاحتمال وطريقة العزوم لتقدير المعلمات. للحصول على تقديرات الترجيح الأعظم، تم استخدام أسلوب نيوتن–رافسون وأسلوب تعظيم التوقعات وطرق تعظيم التوقع العشوائية. علاوة على ذلك ، تم الحصول على فترات الثقة التقريبية للمعلمات باستخدام معكوس مصفوفة المعلومات المشاهدة. تمُ استخدام محاكاة مونتي كارلو لعمل مقارنات عددية للمقدرات المقترحة بالإضافة إلى تقنية التمهيد المئوي ، والتي

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تُستخدم لحساب فترات الثقة. لتوضيح المنهجية المقترحة في الحياة الواقعية ، تم اعتبار بيانات أوقات بقاء خنازير غينيا الملقحة بجرعات مختلفة من عصيات السل بمثابة توضيح لإمكانية تطبيق الطرق المقترحة.

1 Introduction

The generalized inverted exponential distribution (GIED) is a generalization of the inverted exponential distribution (IED). IED is a continuous transformation of the reciprocal of the exponential distribution. Specifically, if X is a random variable that follows the exponential distribution, then $y = \frac{1}{X}$ follows an IED with c.d.f. and p.d.f. which are given by

$$\mathrm{F}(y) = e^{-\lambda/y}, y > 0, \lambda > 0$$

$$f(y) = \frac{\lambda}{y^2} e^{-\lambda/y}, y > 0, \lambda > 0$$

respectively. The IED was investigated by many authors, for example [1] , [2]. The IED can be generalized to include shape parameters and proposed a (GIED). A random variable X of the GIED with α shape parameter and λ scale parameter has the following expression of c.d.f. and p.d.f.

$$F(x) = 1 - (1 - e^{-\lambda/x})^{\alpha}, x > 0, \alpha > 0, \lambda > 0 \quad , \tag{1}$$

$$f(x) = \frac{\alpha\lambda}{x^2} e^{-\lambda/x} (1 - e^{-\lambda/x})^{\alpha-1}, x > 0, \alpha > 0, \lambda > 0 \tag{2}$$

respectively. It can be seen that the hazard function of the GIED

$$\frac{f(x)}{1 - F(x)} = \frac{\alpha\lambda}{x^2(e^{\lambda/x} - 1)}$$

can be increased or decreased based on the shape parameter. Also, it is clear that in many states, this distribution provides a better fit than the Weibull, Gamma, and GED distribution, for more details, see [3]. The GIED is used in such applications, for example; in sea currents, horse racing, and wind speeds. For more properties and applications of the GIED, one can refer to [4] [5] [6]- [7] , [8] .

In life and reliability testing studies, type-I and type-II censoring schemes are more common. However, it is important in some of these studies that a particular fraction of individuals may be removed from the experiment at every of several ordered failure times [9]. Clearly, type I and type II schemes do not have the ability to permit the removal of units at points other than the final point of the experiment. Aggarwala [10] proposed the progressive type I interval censored scheme which can be described as follows. Assume the units of n are put on a test at time $t_0 = 0$ and each unit is followed until it fails or is censored. Units can be observed at present time $t_1 < t_2 < \cdots < t_m$, where m is the pre-specified time to the end of the experiment which means the time axis is divided into intervals $I_j = [t_{j-1}, t_{j)}, j = 1, \ldots m$ with $t_m$is the time when the experimentation finishes. Let d_j be the units number which are failed in I_j and $r_j$be the units number which are removed from the experiment at time t_j particularly, if the units of n are put on a test at time t_0 and d_1 which are observed at time t_1, at this time $r_1$units that are not failed are removed from the experiment leaving $n_1 - d_1 - r_1$ items still there. At time t_2, when other d_2 items have failed, r_2 of items that are not failed are removed from the experiment with leaving $n - d_1 - r_1 - d_2 - r_2$ items still present and the same as for the rest. The experiment terminates after m number of repetitions.

Finally, at time t_m, the number of the removed items that are not failed is r_m . Note that $n = \sum_{i=1}^{m}(r_i + d_i)$. Figure 1 shows a progressive type I interval censored.

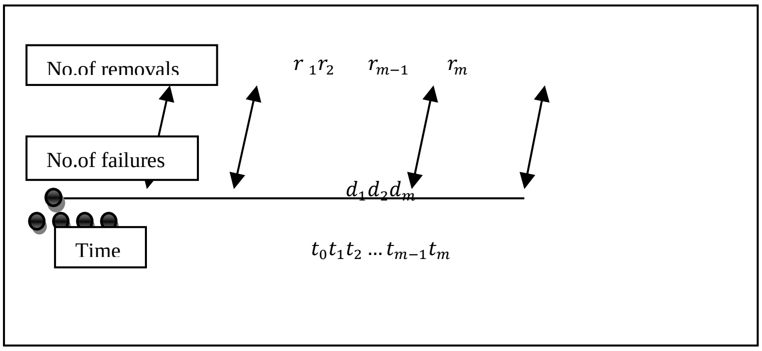

Figure 1: Progressive type I Interval Censored Scheme

Hence, our observations consist of $D = \{(t_i, d_i, r_i); i = 1, \dots, m\}$. The numbers of removal items $r_1, \dots, r_m$ are expressed as nonnegative integers. Alternatively, the removal numbers may be set by pre-specified percentages of the surviving units which are reminded as follows. Let $p = (p_1, p_2, \dots, p_m)$ be pre-specified percentages with $p_m = 1$ At time $t_i, \lceil p_i \times (number\ of\ surviving\ units\ at\ time\ t_i) \rceil$ from the remaining surviving units are removed from the experiment where $\lceil w \rceil$ denotes the largest integer, which is smaller than or equal to w.

In the paper, we utilize different estimation processes for estimating the parameters of the GIED based on progressive type I interval censored. The remainder of the paper is arranged as follows. In Section 2, we obtain the maximum likelihood function estimators (MLEs) of the unknown parameters α and λ.The standard errors for the MLEs and approximated 95% confidence intervals for the parameters are computed as well using the inverse of the observed information matrix. Further, the computing of the MLE using EM and stochastic EM algorithms is also investigated. The nonparametric bootstrap percentile technique is utilized to construct 95% confidence intervals of unknown parameters. The midpoint approximation method, probability plot and method of moments are studied in Sections 3, 4 and 5, respectively. A Monte Carlo simulation study is prepared in Section 6, which supplies a comparison of all the estimation procedures in terms of their biases, estimated standard errors, sampled standard error mean square errors, lengths of 95% confidence intervals and empirical 95% coverage probabilities. An analysis of a real data set is presented in Section 7. Finally, a conclusion is given in Section 8.

2 The Maximum likelihood estimation

Based on the observed progressive, type I interval censored sample $D = \{(t_i, d_i, r_i); i = 1, \dots, m\}$, the likelihood function of α and λ is written as

$$L(\alpha, \lambda | D) \propto \prod_{i=1}^{m} [F(t_i) - F(t_{i-1})]^{d_i} [1 - F(t_i)]^{r_i}$$

$$= \prod_{i=1}^{m} [(1 - e^{-\lambda/t_{i-1}})^{\alpha} - (1 - e^{-\lambda/t_i})^{\alpha}\ ]^{d_i}\ (1 - e^{-\lambda/t_i})^{\alpha r_i} \quad (3)$$

with corresponding log-likelihood function

$$l(\alpha, \lambda | D) \propto \sum_{i=1}^{m} d_i \log\ ((1 - e^{-\lambda/t_{i-1}})^{\alpha} - \left(1 - e^{-\lambda/t_i})^{\alpha}\right) + \alpha \sum_{i=1}^{m} r_i \log\left(1 - e^{-\lambda/t_i}\right) \quad (4)$$

Let, for $i = 1, \dots, m$,

$$A_i = (1 - e^{-\lambda/t_{i-1}})^\alpha - (1 - e^{-\lambda/t_i})^\alpha \quad (5)$$

$$B_i = 1 - e^{-\lambda/t_i} \quad (6)$$

Then log-likelihood (4) is expressed as

$$l(\alpha, \lambda|D) \propto \sum_{i=1}^{m} d_i \log(A_i) + \alpha \sum_{i=1}^{m} r_i \log(B_i) \quad (7)$$

The first-order partial derivatives of A_i and B_i with respect to α and λ are obtained by

$$A_{i,\alpha} := \frac{\partial A_i}{\partial \alpha} = (1 - e^{-\lambda/t_{i-1}})^\alpha \log(1 - e^{-\lambda/t_{i-1}}) - (1 - e^{-\lambda/t_i})^\alpha \log(1 - e^{-\lambda/t_i}) \quad (8)$$

$$A_{i,\lambda} := \frac{\partial A_i}{\partial \lambda} = \frac{\alpha}{t_{i-1}} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}})^{\alpha-1} - \frac{\alpha}{t_i} e^{-\lambda/t_i} (1 - e^{-\lambda/t_i})^{\alpha-1} \quad (9)$$

$$B_{i,\lambda} := \frac{\partial B_i}{\partial \lambda} = \frac{1}{t_i} e^{-\lambda/t_i} \quad (10)$$

and the second-order partial derivatives are given by

$$A_{i,\alpha\alpha} := \frac{\partial^2 A_i}{\partial \alpha^2} = (\log(1 - e^{-\lambda/t_{i-1}}))^2 (1 - e^{-\lambda/t_{i-1}})^\alpha - (\log(1 - e^{-\lambda/t_i}))^2 (1 - e^{-\lambda/t_i} \quad (11)$$

$$A_{i,\alpha\lambda} := \frac{\partial^2 A_i}{\partial \alpha \partial \lambda} = \frac{1}{t_{i-1}} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}})^{\alpha-1} [1 + \alpha \log(1 - e^{-\lambda/t_{i-1}})] \quad (12)$$

$$A_{i,\lambda\lambda} := \frac{\partial^2 A_i}{\partial \lambda^2} = \frac{\alpha}{t_{i-1}} \left(\frac{\alpha-1}{t_{i-1}} (e^{-\lambda/t_{i-1}})^2 (1 - e^{-\lambda/t_{i-1}})^{\alpha-2} - \frac{1}{t_{i-1}} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}})^{\alpha-1}\right)$$

$$- \frac{\alpha}{t_i} \left(\frac{\alpha-1}{t_i} (e^{-\lambda/t_i})^2 (1 - e^{-\lambda/t_i})^{\alpha-2} - \frac{1}{t_i} e^{-\lambda/t_i} (1 - e^{-\lambda/t_i})^{\alpha-1}\right) \quad (13)$$

$$B_{i,\lambda\lambda} := \frac{\partial^2 B_i}{\partial \lambda^2} = \frac{1}{t_i^{\ 2}} e^{-\lambda/t_i} \quad (14)$$

Hence, the first and the second order partial derivatives of the log-likelihoodEq. (7) with respect to α and λ can be computed by

$$l_\alpha := \frac{\partial l(\alpha,\lambda \backslash D)}{\partial \alpha} = \sum_{i=1}^{m} d_i \frac{A_{i,\alpha}}{A_i} + \sum_{i=1}^{m} r_i \log(B_i) \quad (15)$$

$$l_\lambda := \frac{\partial l(\alpha,\lambda \backslash D)}{\partial \lambda} = \sum_{i=1}^{m} d_i \frac{A_{i,\lambda}}{A_i} + \alpha \sum_{i=1}^{m} r_i \frac{B_{i,\lambda}}{B_i} \quad (16)$$

$$l_{\alpha\alpha} := \frac{\partial^2 l(\alpha,\lambda \backslash D)}{\partial \alpha^2} = \sum_{i=1}^{m} d_i \frac{A_i A_{i,\alpha\alpha} - A_{i,\alpha}^{\ 2}}{A_i^{\ 2}} \quad (17)$$

$$l_{\alpha\lambda} := \frac{\partial^2 l(\alpha,\lambda \backslash D)}{\partial \alpha \partial \lambda} = \sum_{i=1}^{m} d_i \frac{A_i A_{i,\alpha\lambda} - A_{i,\alpha} A_{i,\lambda}}{A_i^{\ 2}} + \sum_{i=1}^{m} r_i \frac{B_{i,\lambda}}{B_i} \quad (18)$$

$$l_{\lambda\lambda} := \frac{\partial^2 l(\alpha,\lambda \backslash D)}{\partial \lambda^2} = \sum_{i=1}^{m} d_i \frac{A_i A_{i,\lambda\lambda} - A_{i,\lambda}^{\ 2}}{A_i^{\ 2}} + \alpha \sum_{i=1}^{m} r_i \frac{B_i B_{i,\lambda\lambda} - B_{i,\lambda}^{\ 2}}{B_i^{\ 2}} \quad (19)$$

.

To calculate the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, for the unknown parameters α and λ, we need to solve the equations $l_\alpha = 0$ and $l_\lambda = 0$,where l_α and l_λ are given in Eq.(15) and Eq.(16). It can be seen that there is no closed form of the MLEs. Hence, to obtain the MLEs of

α and λ, we use a simple numerical procedure, namely the Newton-Raphson method. The iterative equation is given by

$$\begin{pmatrix}\alpha^{(k+1)}\\ \lambda^{(k+1)}\end{pmatrix}=\begin{pmatrix}\alpha^{(k)}\\ \lambda^{(k)}\end{pmatrix}-\begin{pmatrix}l_{\alpha\alpha} & l_{\alpha\lambda}\\ l_{\alpha\lambda} & l_{\lambda\lambda}\end{pmatrix}^{-1}\begin{pmatrix}l_{\alpha}\\ l_{\lambda}\end{pmatrix}\Bigg|_{\alpha=\alpha^{(k)},\lambda=\lambda^{(k)}}$$

or equivalently

$$\alpha^{(k+1)}=\alpha^{(k)}-\frac{l_{\alpha}\,l_{\lambda\lambda}-l_{\lambda}l_{\alpha\lambda}}{l_{\alpha\alpha}l_{\lambda\lambda}-l^{2}{}_{\alpha\lambda}}\Big|_{\alpha=\alpha^{(k)},\lambda=\lambda^{(k)}}\ , \tag{20}$$

$$\lambda^{(k+1)}=\lambda^{(k)}-\frac{l_{\lambda}\,l_{\alpha\alpha}-l_{\alpha}l_{\alpha\lambda}}{l_{\alpha\alpha}l_{\lambda\lambda}-l^{2}{}_{\alpha\lambda}}\Big|_{\alpha=\alpha^{(k)},\lambda=\lambda^{(k)}}\ , \tag{21}$$

where $\alpha^{(k)}$ and $\lambda^{(k)}$ are the amounts of α and λ at k-th iteration and $l_{\alpha}, l_{\lambda}, l_{\alpha\alpha}, l_{\alpha\lambda}$ and $l_{\lambda\lambda}$ are given in Eq.(15),Eq.(16),Eq.(17),Eq.(18) and Eq.(19), respectively. The iteration procedure continues until convergence that means $|\alpha^{(k+1)}-\alpha^{(k)}|+|\lambda^{(k+1)}-\lambda^{(k)}|<\varepsilon$ for some pre-specified $\varepsilon>0$.

The standard error of the MLEs can be computed by the inverse of the observed information matrix. Hence, the estimated standard error of α and λ can be calculated by the square root of the diagonal elements of the inverting of the observed information matrix assessed at $(\hat{\alpha},\hat{\lambda})$ as follows $se(\hat{\alpha})=\sqrt{-\frac{\hat{l}_{\lambda\lambda}}{\hat{l}_{\alpha\alpha}\,\hat{l}_{\lambda\lambda}-\hat{l}^{2}{}_{\alpha\lambda}}}$ and $se(\hat{\lambda})=\sqrt{-\frac{\hat{l}_{\alpha\alpha}}{\hat{l}_{\alpha\alpha}\,\hat{l}_{\lambda\lambda}-\hat{l}^{2}{}_{\alpha\lambda}}}$,

Where $\hat{l}_{\alpha\alpha}$, $\hat{l}_{\alpha\lambda}$ and $\hat{l}_{\lambda\lambda}$ are given in Eq.(17),Eq.(18) and Eq.(19), respectively, with α and λ are replaced by $\hat{\alpha}$ and $\hat{\lambda}$, respectively. The asymptotic normality of the MLE is used to calculate the approximate confidence intervals for parameters α and λ. Subsequently, $100(1-\gamma)\%$ Wald confidence intervals for α and λ are computed by

$(\hat{\alpha}-z_{\gamma/2}se(\hat{\alpha}),\hat{\alpha}+z_{\gamma/2}se(\hat{\alpha})$ and $(\hat{\lambda}-z_{\gamma/2}se(\hat{\lambda}),\hat{\lambda}+z_{\gamma/2}se(\hat{\lambda}))$,

respectively, where z_{γ} is the upper $\gamma-th$ percentile of the standard normal distribution.

Next, we calculate the 95% confidence interval for α and λ using the nonparametric percentile bootstrap (Boot-p) method. Bootstrap methods are extremely used to get confidence intervals for the parameters. In [11], the authors suggested the Boot-p method which is used to construct confidence intervals for the parameters in addition to hazard functions and reliability.To construct the Boot-p confidence interval, one has to follow the next steps.

Step(1): Compute the MLEs$\hat{\alpha}$ and $\hat{\lambda}$ under the original progressively type I interval censored sample $D=\{(t_i,d_i,r_i); i=1,\dots,m\}$

Step(2) : Based on the computed MLEs in Step(1), $\hat{\alpha}$ and $\hat{\lambda}$ generate a bootstrap sample D^* of size m contains of $D=\{(t_i,d_i,r_i); i=1,\dots,m\}$ using $\hat{\alpha}$ and $\hat{\lambda}$.

Step(3): Calculate the MLEs, $\hat{\alpha}^*$ and $\hat{\lambda}^*$ under the generated bootstrap sample in Step(2).

Step(4): Reiterate Step(2) and Step(3), for B times, where B is a pre-specified quantity.

Define $\hat{\alpha}_B(x)=G_{\alpha}^{*-1}(x)$, where $G_{\alpha}^{*}(x)$ is the empirical cumulative distribution of $\hat{\alpha}^*$.In a similar way, we define $\hat{\lambda}_B(x)=G_{\lambda}^{*-1}(x)$, where $G_{\alpha}^{*}(x)$ is the empirical cumulative distribution of $\hat{\lambda}^*$ Now, calculate the approximate $100(1-\gamma)\%$ bootstrap-p confidence interval of α

and λ as following
$(\hat{\alpha}_B(\gamma/2), -\hat{\alpha}_B(1-\gamma/2))$ and $(\hat{\lambda}_B\,\gamma/2), -\hat{\lambda}_B(1-\gamma/2))$. respectively.

2.1 The EM Algorithm

It can be seen that utilizing the Newton-Raphson method to compute the MLEs requires the computation of the second derivatives of the associated log-likelihood. In this subsection, we propose the EM algorithm to avoid such computations for obtaining the MLEs of α and λ. The EM algorithm suggested by [12] is a very powerful technique used in parameter estimation under incomplete or missing information data. The EM algorithm contains of two main steps; the expectation step (E-step) and Maximization step (M-step). In the E-step, we calculate the conditional expectation of the complete log-likelihood function condition on the observed values and in the M-step, we maximize the resulting function with respect to the unknown parameters. Now, we define $Z_{ij}, j = 1, \ldots, d_i$ to represent the complete survival times by subintervals $I_i = [t_{i-1}, t_i)$ and we also define $W_{ik}, k = 1, \ldots, r_i$ to represent the complete survival times of those withdrawn items at t_i where $i = 1, \ldots, m$ Using $Z = (Z_{11}, \ldots, Z_{md_m})$ and $W = (W_{11}, \ldots, W_{mr_m})$ the complete log-likelihood function can be expressed by

$$l^c(\alpha,\lambda\backslash Z,W) \propto \sum_{i=1}^{m}\left(\sum_{j=1}^{d_i}\log\left(f(Z_{ij})\right) + \sum_{k=1}^{r_i}\log(f(W_{ik}))\right)$$

$$= n\log(\alpha) + n\log(\lambda) - 2\sum_{i=1}^{m}\sum_{j=1}^{d_i}\log(Z_{ij}) - 2\sum_{i=1}^{m}\sum_{K=1}^{R_i}\log(W_{iK})$$
$$-\lambda\sum_{i=1}^{m}\sum_{j=1}^{d_i}\left(1/Z_{ij}\right) - \lambda\sum_{i=1}^{m}\sum_{k=1}^{r}\left(1/W_{ik}\right) + (\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{d_i}\log\left(1-e^{-\lambda/Z_{ij}}\right)$$
$$+(\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{r_i}\log\left(1-e^{-\lambda/W_{ik}}\right) \qquad (22)$$

Now, for $i = 1, 2, \cdots, m,$ the following conditional expectations , define

$$E_{11i}(\alpha,\lambda) = E(\log(X)\backslash t_{i-1} < X \le t_i) = \frac{\alpha\lambda\int_{t_{i-1}}^{t_i}\log(x)x^{-2}e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}dx}{(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}} \qquad (23)$$

$$E_{21i}(\alpha,\lambda) = E(\log(X)\backslash t_i < X) = \frac{\alpha\lambda\int_{t_i}^{\infty}\log(x)x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_i})^{\alpha}} \qquad (24)$$

$$E_{12i}(\alpha,\lambda) = E(\mathrm{X}^{-1}\backslash t_{i-1} < X \le t_i) = \frac{\alpha\lambda\int_{t_{i-1}}^{t_i}x^{-3}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}} \qquad (25)$$

$$E_{22i}(\alpha,\lambda) = E(\mathrm{X}^{-1}\backslash t_i < X) = \frac{\alpha\lambda\int_{t_i}^{\infty}x^{-3}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_i})^{\alpha}} \qquad (26)$$

$$E_{13i}(\alpha,\lambda) = E\left(\log(1-e^{-\lambda/x})\backslash t_{i-1} < X \le t_i\right) = \frac{\alpha\lambda\int_{t_{i-1}}^{t_i}\log(1-e^{-\lambda/x})x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_{i-1}})^{\alpha} - (1-e^{-\lambda/t_i})^{\alpha}} \qquad (27)$$

$$E_{23i}(\alpha,\lambda) = E\left(\log(1-e^{-\lambda/x})\backslash t_i < X\right) = \frac{\alpha\lambda\int_{t_i}^{\infty}\log(1-e^{-\lambda/x})x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_i})^{\alpha}}$$

(28) Then the conditional expectation of the complete log-likelihood function l^c, which is given the observed values, $D,$ is written as

$$E(l^c(\alpha,\lambda\backslash Z,W)\backslash D) = n\log(\alpha) + n\log(\lambda) - 2\sum_{i=1}^{m} d_i\ E_{11i}(\alpha,\lambda) - 2\sum_{i=1}^{m} r_i\ E_{21i}(\alpha,\lambda) -\lambda\sum_{i=1}^{m} d_i E_{12i}(\alpha,\lambda) - \lambda\sum_{i=1}^{m} r_i\ E_{22i}(\alpha,\lambda) + (\alpha-1)\sum_{i=1}^{m} d_i\ E_{13i}(\alpha,\lambda) +(\alpha-1)\sum_{i=1}^{m} r_i\ E_{23i}(\alpha,\lambda). \quad (29)$$

By computing the first partial derivatives of the log-likelihood function with respect to unknown parameters α and λ and by equating the resulted equations with zero, we obtain

$$\alpha = -\frac{n}{\sum_{i=1}^{m} d_i E_{13i}(\alpha,\lambda) + \sum_{i=1}^{m} r_i\ E_{23i}(\alpha,\lambda)} \quad (30)$$

$$\lambda = -\frac{n}{\sum_{i=1}^{m} d_i E_{12i}(\alpha,\lambda) + \sum_{i=1}^{m} r_i\ E_{22i}(\alpha,\lambda)} \quad (31)$$

Therefore, the EM algorithm works as follows. Set as $\alpha^{(0)}$ and $\lambda^{(0)}$.
Be the initial values of α and λ ,respectively.

Step(i) At k-th iteration, let $(\alpha^{(k)},\lambda^{(k)})$ be an estimate of (α,λ).
Step(ii) Using the expressions (25)-(28), compute $E_{12}(\alpha^{(k)},\lambda^{(k)})$, $E_{22}(\alpha^{(k)},\lambda^{(k)})$, $E_{13}(\alpha^{(k)},\lambda^{(k)})$ and $E_{23}(\alpha^{(k)},\lambda^{(k)})$ where α and λ are replaced by $\alpha^{(k)}$ and $\lambda^{(k)}$, respectively.
Step(iii) Using Eq.(30) and Eq.(31) to compute $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$
Step(iv) If $|\alpha^{(k+1)} - \alpha^{(k)}| + |\lambda^{(k+1)} - \lambda^{(k)}| < \varepsilon$, for some pre-specified quantity ε, then set $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$ as the MLEs of α and λ. Otherwise, put $k = k + 1$ and go to **Step(ii)**.

2.2 The Stochastic EM Algorithm

The Stochastic EM algorithm (SEM) is an alternative method of the EM algorithm where the expectation in the E-step is calculated using Monte Carlo simulations. It is useful for cases when the E-step is hard to calculate exactly. The approximating of the E-step in the EM algorithm by the Monte-Carlo technique was first proposed by [13]. As mentioned by [14], the approximations have more time-consuming. Later, in [15] the authors modified their idea by replacing the E-step with a stochastic step through the simulation technique. For more details about SEM, see for example, [16], [17] , [18].

The main idea of the SEM method can be described as follows. Observe that the conditional survival functions of X $a < X \le b$ can be written as

$$S(t\backslash a < t \le b) = P(X > t\backslash a < X \le b) = \frac{S(t) - S(b)}{S(a) - S(b)} \quad (32)$$

Now, we state the procedure for simulating random variate from the GIED in the interval $[a,b]$. Let $u{:}\,U(0,1)$. Observe that, by solving the expression

$$\frac{\left(1 - e^{-\lambda/t}\right)^{\alpha} - \left(1 - e^{-\lambda/b}\right)^{\alpha}}{\left(1 - e^{-\lambda/a}\right)^{\alpha} - \left(1 - e^{-\lambda/b}\right)^{\alpha}} = u$$

with respect to t , we obtain

$$t = \frac{-\lambda}{\log\left[1 - \left[u\left(\left(1 - e^{-\lambda/a}\right)^{\alpha} - \left(1 - e^{-\lambda/b}\right)^{\alpha}\right) + \left(1 - e^{-\lambda/b}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right]} \quad (33)$$

Note that, when b approaches to ∞ , the above expression reduces to $t = \frac{-\lambda}{\log\left(1-\left[u\left(\left(1-e^{-\lambda/a}\right)^{\alpha}\right)\right]^{\frac{1}{\alpha}}\right)}$. (34)

Now, we first generate independent d_i number of samples $Z_{ij}, i = 1,2,\dots,m; j = 1,\dots,d_i$ from the conditional survival function given in Eq.(32) with a and b are substituted by t_{i-1} and t_i, respectively. Next, we generate r_i number of samples of $W_{ij}, i = 1,2,\dots,m; j = 1,\dots,r_i$ from the conditional survival function given in Eq.(32) with a is replaced by t_i. Using these simulated samples, Eq.(30) and Eq.(31) reduce to

$$\alpha = \frac{n}{\sum_{i=1}^{m}\sum_{j=1}^{d_i}\log\left(1-e^{-\lambda/Z_{ij}}\right)+\sum_{i=1}^{m}\sum_{j=1}^{r_i}\log\left(1-e^{-\lambda/W_{ij}}\right)}. \quad (35)$$

$$\lambda = \frac{n}{\sum_{i=1}^{m}\sum_{j=1}^{d_i}\log\left(1/Z_{ij}\right)+\sum_{i=1}^{m}\sum_{j=1}^{r_i}\log\left(1/W_{ij}\right)}$$

(36)

Therefore, the SEM algorithm works as follows. Set $\alpha^{(0)}$ and $\lambda^{(0)}$ be the initial values of α and λ, respectively.

Step(i) At k-th iteration, let $(\alpha^{(k)},\lambda^{(k)})$ be the estimate of (α,λ).
Step(ii) Using the expression (33), simulate $Z_{ij} \equiv Z_{ij}\left(\alpha^{(k)},\lambda^{(k)}\right), i = 1,\dots,m; j = 1,\dots,d_i$ and using the expression (34), simulate $W_{ij} \equiv W_{ij}\left(\alpha^{(k)},\lambda^{(k)}\right), i = 1,\dots,m; j = 1,\dots,r_i$ where α and λ are replaced by $\alpha^{(k)}$ and $\lambda^{(k)}$, respectively.
Step(iii) Using Eq.(35) and Eq.(36) to compute $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$
Step(iv) If $\left|\alpha^{(k+1)}-\alpha^{(k)}\right|+\left|\lambda^{(k+1)}-\lambda^{(k)}\right| < \varepsilon$, for some pre-specified quantity ε, then put $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$ as the MLEs of α and λ . Otherwise, put $k = k+1$ and go to **Step(ii).**

2.3 The Midpoint Approximation Method

In this subsection, we estimate the unknown parameters of the GIED using the midpoint approximation method. The main thought of this method is to approximate the data of the progressive type I interval censored by type I censored. We assume that d_i number of failures is noticed at the center $a_i = (t_{i-1},t_i)/2$ of i-th interval $(t_{i-1},t_i]$ and too r_i number of units are censored at the inspection time $t_i, i = 1,2,\dots,m$. The log-likelihood function of α and λ based on this type of observations is written as

$$l^m(\alpha,\lambda|data) = \sum_{i=1}^{m}[\,d_i\log[f(a_i)+r_i\log[1-F(t_i)]]$$
$$= \log(\alpha)\sum_{i=1}^{m}d_i + \log(\lambda)\sum_{i=1}^{m}d_i - 2\sum_{i=1}^{m}d_i\log(\alpha_i) - \lambda\sum_{i=1}^{m}d_i/a_i$$
$$+(\alpha-1)\sum_{i=1}^{m}d_i\log\left(1-e^{-\lambda/a_i}\right)+\alpha\sum_{i=1}^{m}r_i\log\left(1-e^{-\lambda/t_i}\right).$$

(37)

After that, we need to resolve the following system of equations to get the midpoint estimates of unknown parameters

$$\sum_{i=1}^{m}\frac{d_i}{\alpha}+\sum_{i=1}^{m}d_i\log\left(1-e^{-\lambda/a_i}\right)+\sum_{i=1}^{m}r_i\log\left(1-e^{-\lambda/t_i}\right) = 0 \quad , \quad (38)$$

and

$$\sum_{i=1}^{m}\frac{d_i}{\lambda}-\sum_{i=1}^{m}\frac{d_i}{a_i}+(\alpha-1)\sum_{i=1}^{m}\frac{d_i\,e^{-\lambda/a_i}/a_i}{1-e^{-\lambda/a_i}}+\alpha\sum_{i=1}^{m}\frac{r_i\,e^{-\lambda/t_i}/t_i}{1-e^{-\lambda/t_i}} = 0. \quad (39)$$

The likelihood Eq.(38) and Eq.(39) cannot be solved analytically because of their nonlinear nature. Therefore, we may adopt a numerical method ,Newton-Raphson method ,to get the estimates of α and λ.

3 The estimation using Probability Plot

Let $(r_i, d_i, t_i), i = 1, \dots, m,$ with $n = \sum_{i=1}^{m}(d_i + r_i)$ be a progressive type I interval censored sample from a GIED. The cumulative distribution function at t_i is estimated under this sample as follows:

$$\widehat{F}(t_i) = 1 - \prod_{j=1}^{i}(1 - \hat{p}_j), \qquad (40)$$

where

$$\hat{p}_j = \frac{d_j}{n - \sum_{k=1}^{j-1}(d_k + r_k)}; j = 1, \dots, m.$$

Estimating the parameters using the probability plot method can be performed by finding the amounts of α and λ that minimize the function

$$S = \sum_{i=1}^{m}(F(t_i) - \widehat{F}(t_i))^2$$
$$= \sum_{i=1}^{m}(1 - (1 - e^{-\lambda/t_i})^{\alpha} - \widehat{F}(t_i))^2$$

So, we want to find the solution the following system of equations $\frac{\partial S}{\partial \alpha} = 0$ and $\frac{\partial S}{\partial \lambda} = 0$ where

$$\frac{\partial S}{\partial \alpha} = -2\sum_{i=1}^{m}\left(1 - \left(1 - e^{-\lambda/t_i}\right)^{\alpha} - \widehat{F}(t_i)\right)(1 - e^{-\lambda/t_i})^{\alpha}\log(1 - e^{-\lambda/t_i})$$
$$\frac{\partial S}{\partial \lambda} = -2\alpha\sum_{i=1}^{m}\left(1 - \left(1 - e^{-\lambda/t_i}\right)^{\alpha} - \widehat{F}(t_i)\right)(1 - e^{-\lambda/t_i})^{\alpha-1}\frac{1}{t_i}e^{-\lambda/t_i}.$$

These estimates are computed numerically by some nonlinear optimization technique.

4 Method of moments estimation

The k th population moment of a GIED with pdf that is given in Eq.(2) has not an explicit form and can be computed by

$$E_{\alpha,\lambda}(X^k) = \alpha\lambda\int_0^{\infty} x^{k-2}e^{-\lambda/x}(1 - e^{-\lambda/x})^{\alpha-1}dx$$

$$= k\int_0^{\infty} x^{k-1}(1 - e^{-\lambda/x})^{\alpha}dx, k \in I^+,$$

where I^+ is the set of positive integers. Substituting $w = e^{-\lambda/x}$ in the above integral gives us

$$E_{\alpha,\lambda}(X^k) = \alpha\lambda^k(-1)^k\int_0^1 \frac{(1-w)^{\alpha-1}}{(\log w)^k}dw.$$

Clearly, the above integral converges if $\alpha > k$. Therefore, we consider the moments with negative integer powers. Let $Y = 1/X$. Then Y follows general exponential distribution and consequently

$$E_{\alpha,\lambda}(X^{-1}) = E_{\alpha,\lambda}(Y) = (\psi(\alpha + 1) - \psi(1))/\lambda$$

$$E_{\alpha,\lambda}(X^{-2}) = E_{\alpha,\lambda}(Y^2) = (\psi^{'}(1) - \psi^{'}(\alpha+1) - \frac{\psi(\alpha+1)-\psi(1))^2}{\lambda^2},$$

where ψ is the digamma function and $\psi^{'}$ is its derivative, see [19] . Now, the k_th negative population moment of a doubly truncated GIED distribution in the interval $[a,b), 0 < a < b$ can be given by

$$E_{\alpha,\lambda}(X^{-k}|X \in [a,b]) = \frac{\int_a^b x^{-k}}{F(b;\alpha,\lambda) - F(a;\alpha,\lambda)}$$

$$= \frac{\alpha\lambda \int_a^b x^{-k-2}\, e^{-\lambda/x}\,(1-e^{-\lambda/x})^{\alpha-1} dx}{(1-e^{-\lambda/a})^{\alpha} - (1-e^{-\lambda/b})^{\alpha}}. \quad (41)$$

By equating the first negative sample moments and the second with the moments of the corresponding population, we obtain the following two equations

$$\frac{(\psi(\alpha+1)-\psi(1))}{\lambda} = \frac{1}{n}\left[\sum_{i=1}^{m} d_i E_{\alpha,\lambda}[X^{-1}|X \in [t_{i-1}, t_i]] + \sum_{i=1}^{m} r_i E_{\alpha,\lambda}[X^{-1}|X \in [t_i, \infty)]\right]$$
(42)

and

$$\frac{\psi^{'}(1) - \psi^{'}(\alpha+1) - (\psi(\alpha+1)-\psi(1))^2}{\lambda^2} = \frac{1}{n}\left[\sum_{i=1}^{m} d_i E_{\alpha,\lambda}[X^{-2}|X \in [t_{i-1}, t_i]] + \sum_{i=1}^{m} r_i E_{\alpha,\lambda}[X^{-2}|X \in [t_i, \infty)]]\right] \quad (43)$$

Since we can not obtain the closed form of the solution to Eq.(42) and Eq.(43) , we can employ the iterative procedure as follows. Set $\alpha^{(0)}$ and $\lambda^{(0)}$ as initial values of α and λ.

Step(i) At k -th iteration, let $(\alpha^{(k)},)$ be an estimate of (α,λ).

Step(ii) Compute $\alpha^{(k+1)}$ by solving the following equation for α

$$\frac{n(\lambda^{(k)}\psi(\alpha+1)-\psi(1))^2}{\psi^{'}(1) - \psi^{'}(\alpha+1) - (\psi(\alpha+1)-\psi(1))^2}$$

$$= \frac{(\sum_{i=1}^{m} d_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-1}|X \in [t_{i-1}, t_i]] + \sum_{i=1}^{m} r_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-1}|X \in [t_i, \infty)])^2}{\sum_{i=1}^{m} d_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-2}|X \in [t_{i-1}, t_i]] + \sum_{i=1}^{m} r_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-2}|X \in [t_i, \infty)]}$$

Step(iii) Compute $\lambda^{(k)}$, using

$$\lambda^{(k+1)} = \frac{n(\psi(\alpha^{(k+1)}+1) - \psi(1))}{\sum_{i=1}^{m} d_i E_{\alpha^{(k+1)},\lambda^{(k)}}[X^{-1}|X \in [t_{i-1}, t_i]] + \sum_{i=1}^{m} r_i E_{\alpha^{(k+1)},\lambda^{(k)}}[X^{-1}|X \in [t_i, \infty)]}$$

Step(iv) If $|\alpha^{(k)} - \alpha^{(k+1)}| + |\lambda^{(k)} - \lambda^{(k+1)}| < \varepsilon$, for ε pre-specified quantity, set $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$ as the method of moments estimators of α and λ . Otherwise, put $k = k+1$ and go to **Steps**(ii).

5 Simulation

In this section, a simulation study behaves in order to scout the performance of the proposed methods to estimate the GIED parameters based on progressive type I interval censored data. the parameter values and sample sizes are considered as $(\alpha,\lambda) = (0.5,0.5), (1.5,1)$, respectively, for $n = 25,50,100$ and we consider $m = 5$ for all the cases. Four different progressives type I interval censored schemes are adopted here, namely
$p_1 = (0.25,0.25,0.5,0.5,1)$

$p_2 = (0.5,0.5,0.25,0.25,1)$
$p_3 = (0,0,0,0,1)$
$p_4 = (0.25,0,0,0\ ,1)$.

The above schemes are picked to specify the surviving units percentage to be withdrawn at the censoring and monitoring points. Observe that, in Scheme 1, in the first two intervals the removal is lighter as compared to the last two intervals and in Scheme 2 is the reverse scenario of Scheme 1. Moreover, in Scheme 3, there is no removal done prior to termination which is a case similar to conventional type I interval censored. In Scheme 4, we conduct the removal at the left-most and right-most ends.

Data is simulated by employing an algorithm proposed by [20] to generate a number of failures $d_1, d_2, \dots, d_m$ in every interval $(t_{i-1}, t_i]$, for $i = 1, \dots, m$ from the sample of size n. The data generation algorithm is described as follows. Given n, m and $p = (p_1, \dots, p_m)$ where $0 \leq p_i \leq 1$ and $p_m = 1$.

Step (i) Generate $t^*{}_1, \dots, t^*{}_m$ from GIED (α, λ) using $t^*{}_i = -\frac{\lambda}{\log\left(1-U_i{}^{1/\alpha}\right)}$, where U_i: $U(0,1)$.
Step(ii) Arrange $t^*{}_1, \dots, t^*{}_m$ as $t_1 < t_2 < \cdots < t_{1m}$.
Step(iii)Compute $F_i = F(t_i), i = 1, \dots, m$ using (1).
Step(iv) Set $d_0 = r_0 = F_0 = 0$ and $i = 1$.
Step(v) Generate $d_i | (d_0, \dots, d_{i-1}, r_0, \dots, r_{i-1})$:binomial $\left(n - \sum_{j=0}^{i-1}(d_j + r_j), q_i\right)$, Where $q_i = \frac{F_i - F_{i-1}}{1 - F_{i-1}}$.
Step(vi) Compute $r_i = \left\lceil p_i\left(n - \sum_{j=0}^{i} d_j - \sum_{j=0}^{i-1} r_j\right)\right\rceil$, where $\lceil x \rceil$ indicates the largest integer not greater than x.
Step(vii) If $i < m$, replace i by $i + 1$ and go to **Step(v)**. Otherwise, stop.

For the bootstrap confidence intervals, the size of the bootstrap samples is taken to be 5000.
At each iteration, we estimate the parameters using the MLE via Newton-Raphson, EM and SEM, probability plot (PP), mid-point (MP) and method of moments (MM) methods. For each of these methods, we have computed the absolute average bias (Bias), the root mean square error (RMSE), the sample standard deviation (SSE), the estimated standard deviation (ESE) via the observed information matrix. Moreover, we have evaluated the widths (Len) of 95% Wald's confidence intervals by using the observed information matrix (CI) and 95% Boot-p (BT) confidence intervals with their empirical coverage probabilities (CP). The process for the estimation is repeated 1000 times and the results of the estimation are reported in Tables 1-7.
From Tables 1-6, it is observed that the Bias For every estimators, in general, it is rationally small which references that the estimated values are close to the true parameter values. However, the MP method, as expected, presents more bias estimates than the other methods. In addition, the SEM algorithm performs worse than NR and EM based on this aspect. Clearly, the RMSE of MP is higher than that of the other methods. Moreover, the values of SSE and ESE of NR and EM methods are close, especially for large n. This indicates that ESE based on the inverse of the observed information matrix is considered as a reasonable estimate of the SSE. As expected, the Bias, RMSE, SSE an ESE of all estimators are decreasing when are increasing sample sizes for every case. With respect to the 95% confidence interval, from Table 7, the length of the confidence intervals is decreasing when is increasing the value of the sample size. Moreover, the empirical coverage probabilities of 95% confidence intervals (CP) are very close to the nominal level for every case. Subsequently, the performances of all

proposed methods except for the MP method are satisfying in terms of the biases and standard errors of the estimates.

Table 1: Simulation results of the proposed methods of estimation for $n = 25$

		$\alpha = 1.5$				$\lambda = 1$			
Scheme	**Method**	**Bias**	**RMSE**	**ESE**	**SSE**	**Bias**	**RMSE**	**ESE**	**SSE**
p_1	*NR*	0.459	1.717	1.026	1.228	0.162	0.277	0.501	0.501
	EM	0.460	1.713	1.027	1.226	0.163	0.276	0.502	0.499
	SEM	0.685	1.734	1.201	1.125	0.317	0.255	0.571	0.394
	PP	0.406	1.661	-	-	0.142	0.272	-	-
	MM	0.369	1.495	-	-	0.117	0.264	-	-
	MP	2.287	10.578	-	-	0.195	0.078	-	-
p_2	*NR*	0.782	3.768	1.622	1.778	0.222	0.417	0.619	0.607
	EM	0.784	3.745	1.623	1.770	0.225	0.411	0.622	0.601
	SEM	1.067	3.795	1.623	1.631	0.400	0.391	0.601	0.480
	PP	0.726	4.450	-	-	0.189	0.448	-	-
	MM	0.639	3.100	-	-	0.162	0.388	-	-
	MP	1.842	7.026	-	-	0.209	0.071	-	-
p_3	*NR*	0.387	1.269	0.817	1.059	0.132	0.244	0.439	0.476
	EM	0.387	1.267	0.817	1.058	0.132	0.244	0.439	0.476
	SEM	0.592	1.361	0.926	1.006	0.275	0.219	0.487	0.379
	PP	0.384	1.507	-	-	0.119	0.266	-	-
	MM	0.349	1.268	-	-	0.104	0.259	-	-
	MP	2.969	15.731	-	-	0.214	0.089	-	-
p_4	*NR*	0.453	1.777	0.970	1.254	0.160	0.268	0.477	0.493
	EM	0.453	1.774	0.971	1.253	0.160	0.267	0.478	0.492
	SEM	0.675	1.878	1.113	1.193	0.313	0.254	0.532	0.396
	PP	0.386	1.824	-	-	0.127	0.273	-	-
	MM	0.339	1.343	-	-	0.106	0.256	-	-
	MP	2.642	12.356	-	-	0.247	0.103	-	-

Table 2: Simulation results of the proposed methods of estimation for $n = 50$

		$\alpha = 1.5$				$\lambda = 1$			
Scheme	**Method**	**Bias**	**RMSE**	**ESE**	**SSE**	**Bias**	**RMSE**	**ESE**	**SSE**
p_1	*NR*	0.228	0.543	0.620	0.701	0.088	0.139	0.349	0.363
	EM	0.229	0.542	0.621	0.700	0.088	0.138	0.349	0.361
	SEM	0.427	0.532	0.621	0.592	0.219	0.118	0.349	0.265
	PP	0.199	0.565	-	-	0.073	0.141	-	-
	MM	0.204	0.556	-	-	0.071	0.149	-	-
	MP	1.673	4.714	-	-	0.143	0.040	-	-
	NR	0.311	1.042	0.860	0.973	0.093	0.188	0.420	0.424
	EM	0.312	1.031	0.860	0.967	0.094	0.186	0.421	0.421

p_2	*SEM*	0.555	1.010	0.860	0.838	0.246	0.157	0.421	0.311
	PP	0.253	1.001	-	-	-	-	0.069	0.186
	MM	0.259	1.020	-	-	0.063	0.193	-	-
	MP	1.340	2.827	-	-	0.167	0.041	-	-
p_3	*NR*	0.156	0.305	0.468	0.530	0.069	0.100	0.299	0.309
	EM	0.156	0.304	0.468	0.530	0.069	0.100	0.299	0.309
	SEM	0.311	0.288	0.522	0.438	0.183	0.086	0.324	0.230
	PP	0.139	0.355	-	-	0.057	0.106	-	-
	MM	0.133	0.339	-	-	0.051	0.112	-	-
	MP	2.255	7.432	-	-	0.168	0.050	-	-
p_4	*NR*	0.188	0.504	0.553	0.685	0.061	0.117	0.322	0.336
	EM	0.188	0.503	0.553	0.684	0.062	0.116	0.322	0.336
	SEM	0.385	0.505	0.635	0.598	0.195	0.097	0.354	0.243
	PP	0.181	0.638	-	-	0.052	0.128	-	-
	MM	0.169	0.546	-	-	0.045	0.129	-	-
	MP	2.118	6.760	-	-	-	-	0.203	0.065

Table 3: Simulation results of the proposed methods of estimation for $n = 100$

		$\alpha = 1.5$				$\lambda = 1$			
Scheme	**Method**	**Bias**	**RMSE**	**ESE**	**SSE**	**Bias**	**RMSE**	**ESE**	**SSE**
p_1	*NR*	0.087	0.171	0.397	0.404	0.033	0.064	0.244	0.251
	EM	0.087	0.170	0.397	0.403	0.034	0.064	0.244	0.250
	SEM	0.268	0.168	0.461	0.311	0.153	0.053	0.267	0.172
	PP	0.076	0.190	-	-	0.027	0.065	-	-
	MM	0.088	0.208	-	-	0.029	0.073	-	-
	MP	1.326	2.250	-	-	0.110	0.020	-	-
p_2	*NR*	0.145	0.393	0.540	0.610	0.044	0.088	0.294	0.294
	EM	0.146	0.388	0.540	0.606	0.045	0.087	0.295	0.291
	SEM	0.354	0.378	0.661	0.503	0.177	0.075	0.338	0.210
	PP	0.111	0.390	-	-	0.028	0.087	-	-
	MM	0.121	0.417	-	-	0.028	0.097	-	-
	MP	1.162	1.824	-	-	0.148	0.029	-	-
p_3	*NR*	0.065	0.108	0.304	0.322	0.034	0.045	0.208	0.210
	EM	0.065	0.107	0.303	0.321	0.034	0.045	0.208	0.210
	SEM	0.196	0.104	0.336	0.257	0.127	0.040	0.222	0.156
	PP	0.060	0.128	-	-	0.030	0.049	-	-
	MM	0.056	0.134	-	-	0.025	0.053	-	-
	MP	1.929	4.510	-	-	0.142	0.030	-	-
p_4	*NR*	0.093	0.156	0.357	0.384	0.036	0.057	0.225	0.236
	EM	0.093	0.156	0.357	0.384	0.036	0.057	0.225	0.236
	SEM	0.241	0.150	0.399	0.303	0.138	0.050	0.241	0.178
	PP	0.077	0.171	-	-	0.027	0.058	-	-
	MM	0.073	0.174	-	-	0.021	0.062	-	-
	MP	1.848	4.199	-	-	0.187	0.047	-	-

Table 4: Simulation results of the proposed methods of estimation for $n = 25$

		$\alpha = 0.5$				$\lambda = 0.5$			
Scheme	**Method**	**Bias**	**RMSE**	**ESE**	**SSE**	**Bias**	**RMSE**	**ESE**	**SSE**
p_1	*NR*	0.086	0.089	0.269	0.286	0.104	0.152	0.374	0.376
	EM	0.090	0.089	0.271	0.284	0.108	0.151	0.377	0.373
	SEM	0.178	0.095	0.269	0.252	0.229	0.151	0.374	0.315
	PP	0.103	0.110	-	-	0.136	0.207	-	-
	MM	0.087	0.111	-	-	0.095	0.158	-	-
	MP	0.416	0.361	-	-	0.285	0.100	-	-
p_2	*NR*	0.197	0.352	0.411	0.560	0.190	0.271	0.449	0.485
	EM	0.207	0.351	0.417	0.555	0.202	0.269	0.457	0.478
	SEM	0.292	0.372	0.417	0.536	0.312	0.275	0.457	0.422
	PP	0.216	0.807	-	-	0.240	1.016	-	-
	MM	0.180	0.321	-	-	0.171	0.264	-	-
	MP	0.542	0.613	-	-	0.382	0.174	-	-
p_3	*NR*	0.049	0.038	0.183	0.189	0.078	0.109	0.312	0.321
	EM	0.049	0.038	0.183	0.189	0.078	0.109	0.312	0.320
	SEM	0.114	0.040	0.212	0.165	0.178	0.097	0.368	0.255
	PP	0.051	0.043	-	-	0.085	0.123	-	-
	MM	0.051	0.053	-	-	0.072	0.122	-	-
	MP	0.367	0.272	-	-	0.227	0.068	-	-
p_4	*NR*	0.064	0.058	0.213	0.233	0.102	0.127	0.336	0.341
	EM	0.066	0.058	0.214	0.232	0.104	0.126	0.337	0.340
	SEM	0.140	0.061	0.253	0.203	0.210	0.124	0.397	0.283
	PP	0.069	0.063	-	-	0.115	0.145	-	-
	MM	0.075	0.082	-	-	0.104	0.146	-	-
	MP	0.398	0.359	-	-	0.274	0.097	-	-

Table 5: Simulation results of the proposed methods of estimation for $n = 50$

		$\alpha = 0.5$				$\lambda = 0.5$			
Scheme	Method	Bias	**RMSE**	**ESE**	**SSE**	**Bias**	**RMSE**	**ESE**	**SSE**
p_1	*NR*	0.032	0.034	0.177	0.181	0.050	0.075	0.262	0.269
	EM	0.034	0.033	0.178	0.179	0.052	0.074	0.263	0.267
	SEM	0.105	0.031	0.214	0.143	0.153	0.065	0.313	0.203
	PP	0.035	0.037	-	-	0.058	0.094	-	-
	MM	0.038	0.042	-	-	0.052	0.083	-	-
	MP	0.318	0.162	-	-	0.258	0.074	-	-
p_2	*NR*	0.071	0.071	0.239	0.258	0.079	0.105	0.307	0.315
	EM	0.076	0.069	0.242	0.252	0.085	0.102	0.312	0.307
	SEM	0.158	0.073	0.239	0.220	0.195	0.101	0.307	0.250
	PP	0.083	0.082	-	-	0.104	0.134	-	-
	MM	0.071	0.081	-	-	0.075	0.117	-	-

	MP	0.427	0.297	-	-	0.348	0.133	-	-
p_3	*NR*	0.019	0.016	0.121	0.127	0.022	0.043	0.211	0.207
	EM	0.019	0.016	0.122	0.126	0.022	0.043	0.212	0.207
	SEM	0.071	0.015	0.138	0.101	0.113	0.034	0.246	0.147
	PP	0.019	0.017	-	-	0.025	0.047	-	-
	MM	0.020	0.023	-	-	0.020	0.053	-	-
	MP	0.299	0.136	-	-	0.201	0.048	-	-
p_4	*NR*	0.035	0.025	0.143	0.155	0.049	0.057	0.230	0.234
	EM	0.036	0.025	0.143	0.155	0.049	0.057	0.230	0.233
	SEM	0.091	0.025	0.162	0.130	0.131	0.047	0.262	0.174
	PP	0.036	0.027	-	-	0.053	0.061	-	-
	MM	0.041	0.038	-	-	0.049	0.072	-	-
	MP	0.342	0.182	-	-	0.256	0.075	-	-

Table 6: Simulation results of the proposed methods of estimation for $n = 100$

		$\alpha = 0.5$				$\lambda = 0.5$			
Scheme	**Method**	**Bias**	**RMSE**	**ESE**	**SSE**	**Bias**	**RMSE**	**ESE**	**SSE**
p_1	*NR*	0.022	0.017	0.124	0.129	0.029	0.036	0.186	0.187
	EM	0.022	0.017	0.124	0.128	0.030	0.035	0.186	0.185
	SEM	0.080	0.017	0.144	0.104	0.113	0.032	0.212	0.140
	PP	0.028	0.020	-	-	0.041	0.046	-	-
	MM	0.020	0.020	-	-	0.026	0.040	-	-
	MP	0.305	0.123	-	-	0.256	0.070	-	-
p_2	*NR*	0.018	0.023	0.158	0.152	0.043	0.216	0.206	0.206
	EM	0.019	0.022	0.159	0.149	0.020	0.041	0.217	0.201
	SEM	0.094	0.025	0.193	0.126	0.119	0.038	0.254	0.155
	PP	0.021	0.029	-	-	0.024	0.058	-	-
	MM	0.026	0.031	-	-	0.024	0.051	-	-
	MP	0.364	0.169	-	-	0.327	0.111	-	-
p_3	*NR*	0.009	0.007	0.084	0.086	0.018	0.023	0.150	0.151
	EM	0.009	0.007	0.084	0.086	0.018	0.023	0.150	0.150
	SEM	0.052	0.007	0.093	0.066	0.092	0.022	0.168	0.115
	PP	0.009	0.008	-	-	0.019	0.024	-	-
	MM	0.012	0.011	-	-	0.020	0.029	-	-
	MP	0.269	0.093	-	-	0.192	0.040	-	-
p_4	*NR*	0.016	0.011	0.098	0.104	0.025	0.030	0.161	0.172
	EM	0.016	0.011	0.098	0.104	0.025	0.030	0.161	0.172
	SEM	0.062	0.010	0.109	0.079	0.094	0.024	0.179	0.122
	PP	0.017	0.012	-	-	0.027	0.032	-	-
	MM	0.023	0.016	-	-	0.031	0.038	-	-
	MP	0.303	0.118	-	-	0.243	0.064	-	-

Table 7: Widths of 95% confidence interval of α and λ and their coverage probabilities.

			$\alpha = 1.5$		$\lambda = 1$		$\alpha = 0.5$		$\lambda = 0.5$	
n	**Scheme**		**Len**	**CP**	**Len**	**CP**	**Len**	**CP**	**Len**	**CP**
25	p_1	*CI*	4.830	97.0	2.265	93.0	1.204	96.0	2.080	94.0
		BT	5.973	92.8	2.081	94.3	1.440	95.5	1.605	95.7
	p_2	*CI*	8.774	96.0	3.010	92.0	2.008	95.0	2.768	93.0
		BT	7.248	93.3	2.317	93.5	2.911	92.5	2.035	92.1
	p_3	*CI*	3.649	94.0	1.925	92.0	0.770	96.0	1.529	94.0
		BT	4.908	92.5	1.863	92.9	0.892	95.5	1.335	95.0
	p_4	*CI*	4.517	96.0	2.122	93.0	0.914	95.0	1.692	93.0
		BT	5.756	92.0	2.013	93.6	1.139	94.0	1.458	95.0
50	p_1	*CI*	2.644	95.0	1.473	93.0	0.744	96.0	1.226	94.0
		BT	3.294	93.0	1.438	94.0	0.778	96.1	1.050	95.6
	p_2	*CI*	3.897	96.0	1.837	92.0	1.049	95.0	1.546	92.0
		BT	4.900	93.0	1.716	94.0	1.230	94.2	1.270	94.7
	p_3	*CI*	1.930	95.0	1.239	93.0	0.493	96.0	0.931	96.0
		BT	2.373	93.0	1.237	93.0	0.517	94.9	0.853	95.9
	p_4	*CI*	2.327	95.0	1.346	94.0	0.586	95.0	1.025	94.0
		BT	3.031	93.0	1.354	95.0	0.630	94.6	0.930	95.5
100	p_1	*CI*	1.618	95.0	0.992	95.0	0.504	96.0	0.797	95.0
		BT	1.780	94.0	0.979	94.6	0.512	95.4	0.732	96.0
	p_2	*CI*	2.270	95.0	1.219	94.0	0.656	96.0	0.961	95.0
		BT	2.632	93.6	1.192	95.0	0.678	96.4	0.847	96.3
	p_3	*CI*	1.219	95.0	0.837	95.0	0.334	94.0	0.624	96.0
		BT	1.331	94.1	0.835	94.2	0.342	94.0	0.596	95.8
	p_4	*CI*	1.445	95.0	0.912	94.0	0.392	94.0	0.673	94.0
		BT	1.627	94.0	0.912	94.3	0.402	94.0	0.639	94.4

6 Application

In that section, we can analyze a data set as a real life application of the GIED under progressive type I interval censored observations. The data set can be provided by [21], and it represents the survival times (in days) of guinea pigs inoculated with different doses of tubercle bacilli. It can be known that guinea pigs have a high predisposition to human tuberculosis and for this reason, they are used in this specific study. The regimen number is the common logarithm of the number of bacillary units in ml. of challenge solution; i.e., regimen corresponds to bacillary units per ml. [22]. This data are used to fit the inverse Weibull distribution. In agreement to regimen 6.6, there are 72 observations listed below:

12,15,22,24,24,32,32,33,34,38,38,43,44,48,52,53,54,54,55,56,57,58,58,59,60,60,

60,60,61,62,63,65,65,67,68,70,70,72,73,75,76,76,81,83,84,85,87,91,95,96,98,99,

109,110,121,127,129,131,143,146,146,175,175,211,233,258,258,263,297,341,341,376.

First, we check whether the GIED is suitable for the data based on the complete data set. We propose three measures for fitting the data set with GIED and these measures are the Akaikes information criterion (AIC), the Bayesian information criterion (BIC) and the minimum distance of Kolmogorov-Simrnov (KS). These measures are defined by

$$\mathrm{AIC} = -2l(\hat{\alpha}, \hat{\lambda}|D) + 4.$$

$$\mathrm{BIC} = -2l(\hat{\alpha}, \hat{\lambda}|D) + 2\log(n).$$

and

$$\mathrm{KS} = \sup_{0 \le t < \infty} |\hat{F}(t) - F(t; \hat{\alpha}, \hat{\lambda})|,$$

where $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of α and λ, l is the log-likelihood function that can get it from Eq.(4), $\hat{F}$ is the empirical c.d.f. and F is the population c.d.f. given in (1). The values AIC, BIC and KS of some two-parameter lifetimes distributions, namely; the GIED, BurrXII, generalized exponential (GExp), Weibull and inverse Weibull (Iweibull) are reported in Table 9. In addition, the curves of the population c.d.f. of GIED, $F(t; \hat{\alpha}, \hat{\lambda})$, and the empirical c.d.f. data set, $\hat{F}$ is depicted in Figure 2. Clearly, from Table 9 and Figure 2, it is shown that the GIED is the best fitted distribution of the data compared with BurrXII, GExp, Weibull and Iweibull distributions.

Next, we estimate α and λ, of GIED based on the real data set using the proposed methodology. For analyzing the above data set,we take $m = 5$ and inspection times $t = (40,90,150,190,220)$. In addition, we consider the same censoring schemes presented in the simulation section, namely p_1, p_2, p_3 and p_4. According to the censoring schemes, the values of (d_i, r_i) within the intervals $I_0 = (0, t_1]$ and $I_i = (t_{i-1}, t_i], i = 1,2, \dots, m$ can be reported inTable 8.

Table 8: Values of (r_i, d_i) within each interval $I_i, i = 1,2, \dots, m$ for the data set

	p_1		p_2		p_3		p_4	
I	d	r	d	r	d	r	d	r
(0,40]	11	16	11	31	11	0	11	16
(40,90]	20	7	5	13	36	0	20	0
(90,150]	7	6	1	3	14	0	14	0
(150,190]	0	3	0	2	2	0	2	0
(190,220]	0	2	0	6	1	8	1	8

To propose initial values of the parameters, the Cantor plot of the log-likelihood function under the real data set is plotted and is presented in Figure 3. Table 10 presents the estimates and standard errors while Table 11 presents the confidence intervals of the parameters, α and λ. for real data sets. From the obtained results, one can see that the values of the MLEs computed using NR and EM methods are very close except for the censoring scheme p_2 Similar conclusion can be observed for the ESE values. With respect to the length of the confidence intervals, both methods; CI and BT have introduced almost the same lengths except for the scheme p_2.

Table 9: The values of MLEs, AIC, BIC and KS of real data set

Distribution	MLEs (α, λ)	AIC	BIC	KS
GIED	(1.435207,86.308831)	155.063097	59.616430	0.088796
BurrXII	(1.37295,0.1)	221.50750	226.06083	0.24498
GExp	(152.39614,0.1)	454.18751	458.74084	0.45205
Weibull	(0.197891,28.571845)	163.320348	167.873680	0.089201
IWeibull	(1.244539,182.158051)	154.737928	159.291260	0.089019

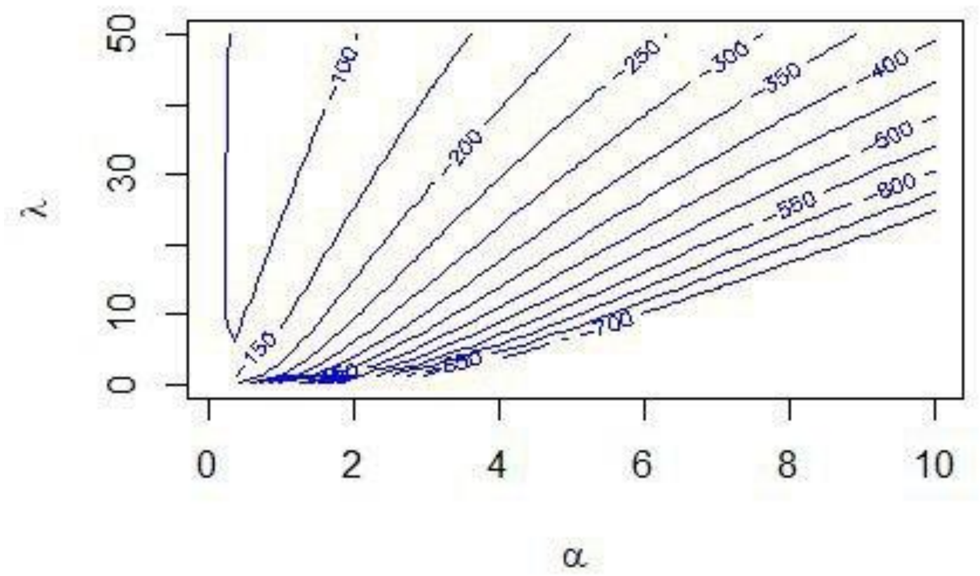

Figure 2: represents the population CDF and Empirical c.d.f. of GIED. Solid line: population c.d.f and dashed lines: empirical c.d.f

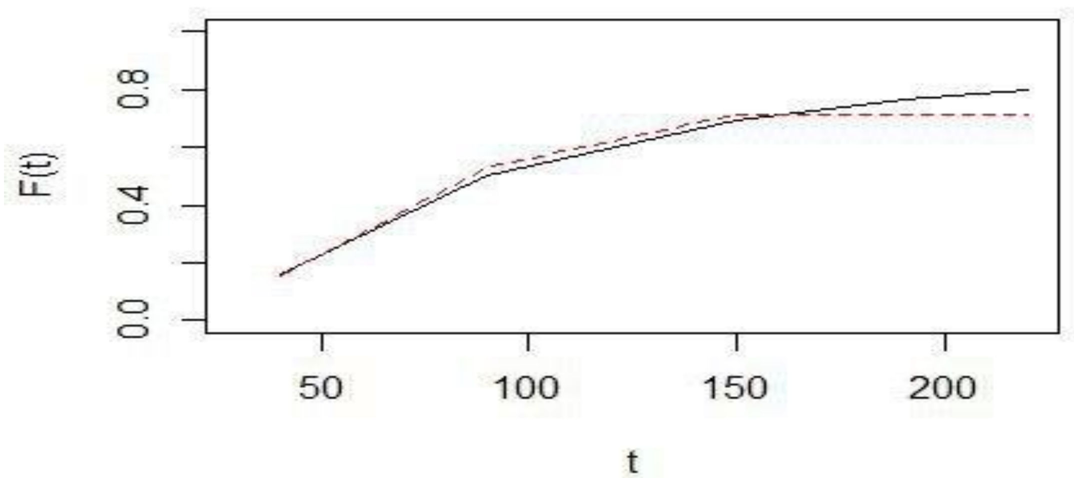

Figure 3: Log-likelihood contour plot of the GIED

Table 10: Estimates of α and λ of the real data set.

		α		λ	
Scheme	**Method**	**Estim**	**ESE**	**Estim**	**ESE**
p_1	*NR*	1.435	0.438	86.309	18.439
	EM	1.432	0.437	86.162	18.419
	SEM	1.273	0.372	80.390	17.299
	PP	1.126	-	71.319	-
	MM	1.629	-	92.913	-
	MP	0.869	-	36.566	-
p_2	*NR*	0.229	0.119	26.146	17.041
	EM	0.298	0.169	34.587	20.960
	SEM	0.277	0.149	32.967	19.815
	PP	0.186	-	18.252	-
	MM	0.266	-	30.645	-
	MP	0.254	-	27.562	-
p_3	*NR*	2.560	0.582	105.410	16.231
	EM	2.557	0.581	105.330	16.218
	SEM	2.329	0.515	99.172	15.406
	PP	2.647	-	106.016	-
	MM	3.085	-	116.689	-
	MP	2.528	-	44.583	-
P_4	*NR*	1.969	0.507	100.692	17.731
	EM	1.969	0.507	100.659	17.726
	SEM	1.972	0.614	89.278	18.562
	PP	2.070	-	103.790	-
	MM	1.996	-	101.449	-
	MP	1.574	-	42.313	-

Table 11: 95% Wald's confidence intervals and 95%Boot-p confidence intervals α and λ of the real data set.

Scheme	**Method**	α	λ
P_1	*CI*	(0.789,2.610)	(56.781,131.192)
	BT	(0.775,2.588)	(53.878,130.019)
P_2	*CI*	(0.083,0.632)	(7.288,93.800)
	BT	(0.100,0.569)	(4.087,65.316)
P_3	*CI*	(1.639,3.998)	(77.949,142.546)
	BT	(1.644,4.201)	(75.553,142.914)
P_4	*CI*	(1.189,3.263)	(71.302,142.195)
	BT	(1.213,3.306)	(68.215,140.461)

7 Concluding remarks

In this article, statistical inference of the unknown parameters of GIED under progressive type I interval censored data is considered. The MLEs, probability plot, mid-point and method of moments as well as associated standard error, root mean square error and confidence intervals are obtained. MLEs are obtained by using the Netwon-Raphson method, expectation minimization (EM) algorithm and stochastic expectation minimization (SEM) algorithm. The Simulation results showed that all the estimators, except MP method, present reasonably small amounts of biases and RMSEs. Moreover, the ESE based on the inverse of the observed information matrix is considered as a reasonable estimate of the SSE for NR and EM methods, especially for large n. with respect to 95% confidence interval, the length of the confidence intervals is decreasing when increasing the value of sample size and the estimated CP of 95% confidence intervals are very close to the nominal level for every case.

In real data analysis, we analyze the survival times of guinea pigs inoculated with different doses of tubercle bacilli based on the proposed methodology. Fitting the data set with the GIED is first implemented and then the GIED parameters are estimated based on the proposed methods.

We hope that the methodologies proposed in this work will be useful to applied statisticians. It will be entertaining to study the methods of estimation based on hybrid censored data.The work is in advancement and it will be announced later.

References

[1] G.Praksh, "Intverted exponential distribution under a bayesian viewpoint," *Journal of Modern Applied Statistical Methods,* vol. 11, no. 1, p. 16, 2012.

[2] S. K. Singh, U. Singh, and D. Kumar, " Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors," *Journal of Statistical computation and simulation,* vol. 83, no. 12, pp. 2258-2269, 2013.

[3] A. Abouammoh and A. M. Alshingiti, "Reliability estimation of generalized inverted exponential distribution," *Journal of Statistical Computation and Simulation* , vol. 79, no. 11, p. 1301–1315, 2009.

[4] H. Krishna and K. Kumar, "Reliability estimation in generalized inverted exponential distribution with progressively type ii censored sample," *Journal of StatisticalComputation and Simulation,* vol. 83, no. 6, p. 1007–1019, 2013.

[5] S. Dey and T. Dey, "Generalized inverted exponential distribution: Different methods of estimation," *American Journal of Mathematical and Management Sciences* , vol. 33, no. 3, pp. 194–215,, 2014.

[6] S. Dey and T. Dey, "On progressively censored generalized inverted exponential distribution," *Journal of Applied Statistics* , vol. 41, no. 12, p. 2557–2576, 2014.

[7] S. Singh, Y. M. Tripathi, and C.-H. Jun, " Sampling plans based on truncated life test for a generalized inverted exponential distribution," *Industrial Engineering and Management Systems,* vol. 14, no. 2, p. 183–195, 2015.

[8] M. Dube, H. Krishna, and R. Garg, "Generalized inverted exponential distribution under progressive first-failure censoring," *Journal of Statistical Computation and Simulation* , vol. 86, no. 6, p. 1095–1114, 2016.

[9] C. Cheng, J. Chen, and Z. Li, "A new algorithm for maximum likelihood estimation with progressive type-I interval censored data," *Communications in StatisticsU° Simulation and Computation R* , vol. 39, no. 4, p. 750–766, 2010.

[10] R. Aggarwala, "Progressive interval censoring: some mathematical results with applications to inference," *Communications in Statistics-Theory and Methods* , vol. 30, no. 8-9, p. 1921–1935, 2001.

[11] B. Efron ,R. Tibshirani, "Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy," *Statistical science,* p. 54–75, 1986.

[12] A. P. Dempster, N. M. Laird, D. B. Rubin, "Maximum likelihood from incomplete data via the em algorithm," *Journal of the Royal Statistical Society: Series B (Methodological),* vol. 39, no. 1, p. 1–22, 1977.

[13] G. C. Wei , M. A. Tanner, "A monte carlo implementation of the em algorithm and the poor man's data augmentation algorithms," *Journal of the American statistical Association ,* vol. 85, no. 411, p. 699–704, 1990.

[14] F.-K. Wang ,Y. Cheng, "Em algorithm for estimating the burr xii parameters with multiple censored data," *Quality and Reliability Engineering International ,* vol. 26, no. 6, p. 615–630, 2010.

[15] J. Diebolt , G. Celeux, "Asymptotic properties of a stochastic em algorithm for estimating mixing proportions," *Stochastic Models,* vol. 9, no. 4, p. 599–613, 1993.

[16] . Tregouet, S. Escolano, L. Tiret, A. Mallet, J. Golmard, "A new algorithm for haplotype-based association analysis: the stochastic-em algorithm," *Annals of human genetics,* vol. 68, no. 2, p. 165–177, 2004.

[17] X. Zhang , M. Haenggi, "A stochastic geometry analysis of inter-cell interference coordination and intra-cell diversity," *IEEE Transactions on Wireless Communications,* vol. 13, no. 12, p. 6655–6669, 2014.

[18] R. Arabi Belaghi, M. Noori Asl, S. Singh, "On estimating the parameters of the burr xii model under progressive type-i interval censoring," *Journal of Statistical Computation and Simulation, ,* vol. 87, no. 16, p. 3132–3151, 2017.

[19] R. D. Gupta ,D. Kundu, "Theory & methods: Generalized exponential distributions,," *Australian & New Zealand Journal of Statistics ,* vol. 41, no. 2, p. 173–188, 1999.

[20]] R. Aggarwal ,K. T. Jacques, "The impact of fdicia and prompt corrective action on bank capital and risk: Estimates using a simultaneous equations model," *Journal of Banking & Finance, ,* vol. 25, no. 6, p. 1139–1160, 2001.

[21] T. Bjerkedal et al., "Acquisition of resistance in guinea pies infected with different doses of virulent tubercle bacilli.," *American Journal of Hygiene,* vol. 130–48, no. 1, p. 72, 1960..

[22] D. Kundu,H. Howlader, “"Bayesian inference and prediction of the inverse weibull distribution for type-ii censored data," *Computational Statistics & Data Analysis, ,* vol. 54, no. 6, p. 1547–1558, 2010.