DOI: 10.24996/ijs.2024.65.3.22

ISSN: 0067-2904

On Certain Subclass of Meromorphic Multivalent Functions Associated with Fractional Calculus Operator

Sattar K. Hussein[1] *, Kassim A. Jassim[2]

[1]*Department of Heet Education, General Directorate of Education in Anbar, Ministry of Education, Hit, Anbar,31007 Iraq*

[2]*Department of Mathematics , College of Science , University of Baghdad,Iraq*

Received: 29/9/2022 Accepted: 29/3/2023 Published: 30/3/2024

Abstract:

In this paper, the class of meromorphic multivalent functions of the form by using fractional differ-integral operators is introduced. We get Coefficients estimates, radii of convexity and star likeness. Also closure theorems and distortion theorem for the class $\sum_K^+(\lambda,\mu,\nu,\eta,\gamma,\alpha,\beta)$ is calculaed.

Keywords and phrases: Meromorphic multivalent functions, Fractional calculus, Radius of Starlikeness and Convexity.

حول فئة جزئية محددة من الدوال الميرومورفية متعددة التكافؤ المرتبطة بمؤثر التفاضل الكسري

ستار كامل حسين[1] * , قاسم عبد الحميد جاسم[2]

[1]دائرة تربية هيت,المديرية العامة للتربية في الانبار,وزارة التربية ,هيت,الانبار, 31007 العراق

[2]قسم الرياضيات, كليةالعلوم, جامعة بغدا د,العراق

الخلاصة:

في هذا البحث , قدم صنف من الدوال الميرومورفية متعددة التكافؤ المعرفة بواسطة حساب التفاضل والتكامل الكسري, تم الحصول على تقديرات المعاملات,انصاف اقطارالتحدبية و النجمية .ايضا نظريات الانغلاق ونظرية التشوه تم حسابها للصنف

$\sum_K^+(\lambda,\mu,\nu,\eta,\gamma,\alpha,\beta)$

1. Introduction:

The $\sum_k$ symbol the class of meromophic functions is defined by:

$$f(w)=w^{-k}+\sum_{i=K}^{\infty}a_iw^i, k \in \mathbb{N}, \quad (1)$$

which is analytic and k-valent in the puncture unit disk $U^*=\{w \in \mathbb{C} : 0<|w|<1\}$.

The function f is said to be class $\sum_k^*(\alpha)$ of meromorphic k-valenty star like function of range α [1] if :

$$\text{Re}\left\{\frac{w\,f'(w)}{f(w)}\right\} > \alpha, (w \in U^*, 0 \le \alpha < k, k \in \mathbb{N}). \quad (2)$$

A function f can say in the class $\sum_k^m(\alpha)$ of meromorphic

*Email: sattar1979heet@gmail.com

k-valently convex function of rangeα [2] if :

$$-Re\left\{1+\frac{w\,f''(w)}{f'(w)}\right\}>\alpha,(w\in\ U^*,0\leq\alpha<k,k\in\mathbb{N}). \quad (3)$$

By making use of the fractional differ-integral operator contained in :

Definition 1.1:

$$M_{0,w}^{\lambda,\mu,v,\eta}f(w)$$

$$=\begin{cases}\frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)}w^{-k+\eta+1}J_{0,w}^{\lambda,\mu,v,\eta}[w^{\mu+p}f(w)](0\leq\lambda<1),\\ \frac{\Gamma(\mu+v+\eta-\lambda)\Gamma(\eta)}{\Gamma(\mu+\eta)\Gamma(v+\eta)}w^{-k-\eta+1}I_{0,w}^{-\lambda,\mu,v,\eta}[w^{\mu+p}f(w)](-\infty\leq\lambda<0).\end{cases} \quad (4)$$

Where $J_{0,w}^{\lambda,\mu,v,\eta}$ is the generalized fractional derivative operator of order α given by

$$J_{0,w}^{\lambda,\mu,v,\eta}f(w)=\frac{1}{\Gamma(1-\lambda)}\frac{d}{dw}\left\{w^{\lambda-\mu}\int_0^z t^{\eta-1}(w-t)^{-\lambda}{}_2F_1\left(\mu-\lambda,1-v;1-\lambda;1-\frac{t}{2}\right)f(t)dt\right\}, \quad (5)$$

where ($0\leq\lambda<1$).

A function f is analytic in a simply connected region of the w-plane in the multiplicity of $(w-t)^{-\lambda}$is removed by requiring log (w−t) to be real when w−t > 0, that

$$f(w)=0(|w|^r) \quad (w\to0), \quad (6)$$

and $I_{0,w}^{-\lambda,\mu,v,\eta}$ is the generalized fractional integral operator of order $-\lambda$ $(-\infty<-\lambda<0)$ of the form

$$I_{0,w}^{\lambda,\mu,v,\eta}f(w)=\frac{w^{-(\lambda+\mu)}}{\Gamma(\lambda)}\int_0^w t^{\eta-1}(w-t)^{\lambda-1}{}_2F_1\left(\lambda+\mu,-v;\lambda;1-\frac{t}{2}\right)f(t)dt \quad (7)$$

($\lambda>0$, $\mu,\eta\in R$, $r\in R^+$ and $r>(\max\{0,\mu\}-\eta)$),

by (5) and (7) we get:

$$J_{0,w}^{\lambda,\mu,v,1}f(w)=J_{0,w}^{\lambda,\mu,v}f(w), \quad (8)$$

and

$$I_{0,w}^{\lambda,\mu,v,1}f(w)=I_{0,w}^{\lambda,\mu,v}f(w). \quad (9)$$

Clearly, $J_{0,w}^{\lambda,\mu,v}$ and $I_{0,w}^{\lambda,\mu,v}$ are well-known (see, [3-5]).

Now,

$$J_{0,w}^{\lambda,v,1}f(w)=D_w^{\lambda}f(w),\qquad(0\leq\lambda<1) \quad (10)$$

and

$$I_{0,w}^{-\lambda,v,1}f(w)=D_w^{-\lambda}f(w),\qquad(\lambda>0). \quad (11)$$

Where D_w^{λ} and $D_w^{-\lambda}$ are the famous Owa- Saigo-Srivastana and this can be shown in [6] and [7].

By using the Gamma function, we have

$$J_{0,w}^{\lambda,\mu,v,\eta}w^k=\frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta-\lambda+v)}w^{k+\eta-\mu-1},(0\leq\lambda<1) \quad (12)$$

and

$$I_{0,w}^{\lambda,\mu,v,\eta}w^k=\frac{\Gamma(k+\eta)\Gamma(k+\eta-\mu+v)}{\Gamma(k+\eta-\mu)\Gamma(k+\eta+\lambda+v)}w^{k+\eta-\mu-1}\ (\lambda>0). \quad (13)$$

Where $\mu,\eta\in R,v\in R^+$ and $k>(\max\{0,\mu\}-\eta)$.

By using (1), (12) and (13) in (4), we find

$$M_{o,w}^{\lambda,\mu,v,\eta}f(w)=w^{-k}+\sum_{i=k}^{\infty}\Gamma_i^{\lambda,\mu,v,\eta}a_iw^i, \quad (14)$$

$$(-\infty<\lambda<1),\mu+v+\eta>\lambda,\mu>-\eta,\eta>0,m\in\mathbb{N},f\in\textstyle\sum_m$$

and

$$\Gamma_i^{\lambda,\mu,v,\eta} = \frac{(\mu+\eta)_{i+k}(v+\eta)_{i+k}}{(\mu+v+\eta-\lambda)_{i+k}(\eta)_{i+k}}. \quad (15)$$

The operator $M_{0,w}^{\lambda,\mu,v,\eta}f(w)$ reduces to the famous Ruscheweyh derivative $D^{\lambda}f(w)$ for meromorphich univalent functions [8].

several authors studied a subclass of meromorphic multivalent functions for other conditions, like [9-11].

Definition 1.2: A function $f \in \sum_k$ of the class $\sum_k (\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ it is satisfied the condition:

$$\left| \frac{\frac{w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))''}{(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'}+(1+\gamma)}{\frac{w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))''}{(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'}+(1+2\alpha-\gamma)} \right| < \beta,$$

(16)

for some α $(o\leq\alpha < 1), \beta(0 \leq \beta < 1), \gamma(0 \leq \gamma \leq 1), k \in \mathbb{N}, -\infty < \lambda < 1, \mu + v + \eta > \lambda, \mu > -\eta, v > -\eta$ and $\eta > 0$.

Definition 1.3: Let $\sum_k^+$ denote the subclass of $\sum_k$defined by

$$f(w) = w^{-k} + \sum_{i=k}^{\infty} a_i w^i, \quad (a_i\geq 0 ,k\in\mathbb{N}). \quad (17)$$

Then we define a new subclass $\sum_k^+ (\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ by

$$\sum_k^+ (\lambda,\mu,v,\eta,\gamma,\alpha,\beta) = \sum_k^+ \cap \sum_k (\lambda,\mu,v,\eta,\gamma,\alpha,\beta). \quad (18)$$

2.Coefficient Estimates:

Theorem 2.1: Suppose that $f \in \sum_k$ and

$\sum_{i=k}^{\infty} i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}|a_i| \leq k(k-\gamma)+\beta[k(k+\gamma-2\alpha)]$.(18)

where $\Gamma_i^{\lambda,\mu,v,\eta}$ is defined by (15) and the conditions with (16) holds.

Then $f \in \sum_k(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Proof: Suppose that the inequality (18) is true.

Assume that

$$\Omega\ (f) = \left|w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'' + (1+\gamma)(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'\right| - \beta\left|w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'' + (1+2\alpha-\gamma)(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'\right|.$$

By using (14), we have

$$M_{o,w}^{\lambda,\mu,v,\eta}f(w) = w^{-k} + \sum_{i=k}^{\infty} \Gamma_i^{\lambda,\mu,v,\eta} a_i w^i.$$

Then we find that

$$\Omega\ (f) = \left|k(k+1)w^{-k-1} + \sum_{i=k}^{\infty} i(i-1)\,\Gamma_i^{\lambda,\mu,v,\eta}a_i w^{i-1} + (1+\gamma)(-kw^{-k-1} + \sum_{i=k}^{\infty} i\,\Gamma_i^{\lambda,\mu,v,\eta}a_i w^{i-1})\right|$$
$$-\beta\left|k(k+1)w^{-k-1} + \sum_{i=k}^{\infty} i(i-1)\,\Gamma_i^{\lambda,\mu,v,\eta}a_i w^{i-1} + (1+2\alpha-\gamma)(-kw^{-k-1} + \sum_{i=k}^{\infty} i\,\Gamma_i^{\lambda,\mu,v,\eta}a_i w^{i-1})\right|$$

$$\leq \left| k(k+1)w^{-k-1} + \sum_{i=k}^{\infty} i(i-1)\,\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i-1} - k(1+\gamma)w^{-k-1} + \sum_{i=k}^{\infty} i(1+\gamma)\ \Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i-1} \right|$$

$-\beta \left| k(k+1)w^{-k-1} + \sum_{i=k}^{\infty} i(i-1)\,\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i-1} - k(1+2\alpha-\gamma)w^{-k-1} + \sum_{i=k}^{\infty} i\,(1+2\alpha-\gamma)\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i-1} \right| \leq 0$

$\leq \left| k\,(k-\gamma)\,w^{-k-1} + \sum_{n=k}^{\infty} i(i+\gamma)\,\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i-1} \right| - \beta \left| k(k+\gamma-2\alpha)w^{-k-1} + \sum_{i=k}^{\infty} i(i+2\alpha-\gamma)\,\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{n-1} \right| \leq 0$

$\leq \quad |k(\,k-\gamma)w^{-k}| + \left|\sum_{i=k}^{\infty} i(i+\gamma)\,\Gamma_i^{\lambda,\mu,v,\eta} a_i w^i\right| - \beta(|k(k+\gamma-2\alpha)w^{-k}| - \left|\sum_{i=k}^{\infty} i(i+2\alpha-\gamma)\ \Gamma_i^{\lambda,\mu,v,\eta} a_i w^i\right|) \leq 0$

$\leq k(\,k-\gamma)r^{-k} + \sum_{i=k}^{\infty} i(i+\gamma)\,\Gamma_i^{\lambda,\mu,v,\eta} |a_i| r^i - \beta(k(k+\gamma-2\alpha))r^{-k}$

$$+\sum_{i=k}^{\infty} \beta i\,(i+2\alpha-\gamma)\Gamma_i^{\lambda,\mu,v,\eta}|a_i|r^i \leq 0.$$

$\sum_{i=k}^{\infty} i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}|a_i| \leq k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]$. (19)

Since the above inequality holds for all r, $0 < r < 1$. Letting $r \to 1^-$ in (19) we get that $\Omega(f) \leq 0$, hence $f \in \sum_k(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

Theorem 2.2 : Let $f \in \sum_k^+$ .Then $f \in \sum_k^+\ (\,\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ if and only if

$\sum_{i=k}^{\infty} i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\ \Gamma_i^{\lambda,\mu,v,\eta}|a_i| \leq k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]$. (20)

Where $\Gamma_i^{\lambda,\mu,v,\eta}$ is defined by (15) and all the parameters are constrained in Theorem (2.1).

Proof: In the of Theorem 2.1, it is sufficient to prove the only if part.
If assume that $f \in \sum_k^+(\,\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$,then

$$\left| \frac{\dfrac{w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))''}{(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'} + (1+\gamma)}{\dfrac{w(M_{0,w}^{\lambda,\mu,v,\eta}f(w))''}{(M_{0,w}^{\lambda,\mu,v,\eta}f(w))'} + (\,1+2\alpha-\gamma)} \right| < \beta,$$

$$= \left| \frac{k(k-\gamma)+\sum_{i=k}^{\infty} i(i+\gamma)\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i+p}}{\beta[k(k+\gamma-2\alpha)]+\sum_{i=k}^{\infty} \beta i(i+2\alpha-\gamma)\Gamma_i^{\lambda,\mu,v,\eta} i w^{i+k}} \right| < \beta.$$

Since $\mathrm{Re}(w) \leq |w|$ for all w, it follows that

$$\mathrm{Re}\left\{ \frac{k(k-\gamma)+\sum_{i=k}^{\infty} i(i+\gamma)\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i+k}}{\beta[k(k+\gamma-2\alpha)]-\sum_{i=k}^{\infty} \beta\ i(i+2\alpha-\gamma)\Gamma_i^{\lambda,\mu,v,\eta} a_i w^{i+k}} \right\} < \beta.$$

Now letting $r \to 1^-$, through real values, we obtain the result (20).

3. Distortion Theorems:

Theorem 3.1: Let $f \in \sum_k^+(\,\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$. Then

$$\frac{1}{|w|^k} - \frac{k\,(k-\gamma)+\beta[k\,(k\ +\ \gamma-2\alpha)]}{k[k(1+\beta)+\gamma(1-\beta)+2\,\beta\alpha]}|w|^k \leq \left|M_{0,w}^{\lambda,\mu,v,\eta}f(w)\right| \leq \frac{1}{|w|^k} + \frac{k\,(k-\gamma\,)+\beta[k\,(k\ +\ \gamma-2\alpha)]}{k[k(1+\beta)+\gamma(1-\beta)+2\,\beta\alpha]}|w|^k,$$

for all the parameters are constrained in Theorem 2.1.

Proof: Since $f \in \sum_k^+(\,\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$.

In the of Theorem 2.2, we have

$$\sum_{i=k}^{\infty} a_i\, \Gamma_i^{\lambda,\mu,\nu,\eta} \le \frac{k\,(k-\gamma\,) + \beta[k\,(k\,+\,\gamma - 2\alpha)]}{k[k(1+\beta) + \gamma(1-\beta) + 2\,\beta\alpha]}. \tag{21}$$

Now,

$\left|M_{0,w}^{\lambda,\mu,\nu,\eta} f(w)\right| \le \frac{1}{|w|^k} + \sum_{i=k}^{\infty} a_i\, \Gamma_i^{\lambda,\mu,\nu,\eta} |w|^i \le \frac{1}{|w|^k} + |w|^k \sum_{i=k}^{\infty} a_i\, \Gamma_i^{\lambda,\mu,\nu,\eta}$.

By using (21), we get

$$\left|M_{0,w}^{\lambda,\mu,\nu,\eta} f(w)\right| \le \frac{1}{|w|^k} + \frac{k\,(k-\gamma\,) + \beta[k\,(k\,+\,\gamma - 2\alpha)]}{k[k(1+\beta) + \gamma(1-\beta) + 2\,\beta\alpha]} |w|^k.$$

And

$\left|M_{0,w}^{\lambda,\mu,\nu,\eta} f(w)\right| \ge \frac{1}{|w|^k} - \sum_{i=k}^{\infty} a_i\, \Gamma_i^{\lambda,\mu,\nu,\eta} |w|^i \ge \frac{1}{|w|^k} - |w|^k \sum_{i=k}^{\infty} a_i\, \Gamma_i^{\lambda,\mu,\nu,\eta}$.

Also use of (21), we get

$$\left|M_{0,w}^{\lambda,\mu,\nu,\eta} f(w)\right| \ge \frac{1}{|w|^k} - \frac{k\,(k-\gamma\,) + \beta[k\,(k\,+\,\gamma - 2\alpha)]}{k[k(1+\beta) + \gamma(1-\beta) + 2\,\beta\alpha]} |w|^k.$$

4. Radii of Starlikeness and Convexity:

Theorem 4.1: Let$f \in \sum_k^{+}(\lambda,\mu,\nu,\eta,\gamma,\alpha,\beta)$. Then f is meromorphically k-valentstarlike of order $\psi(0 \le \psi < k)$ in |w|< R_1, where

$$R_1 = \inf_i \left\{\frac{(k-\psi)\, i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,\nu,\eta}}{(i+2k-\psi)k\,(k-\gamma\,)+\beta[k\,(k\,+\gamma-2\alpha)]}\right\}^{\frac{1}{i+k}}, \tag{22}$$

for every parameter are constrained as in Theorem 2.1.

proof: It is sufficient to show that for $(0 \le \psi < k)$,

$$\left|\frac{w\ f'(w)}{f(w)} + k\right| < k - \psi. \tag{23}$$

That is

$$\left|\frac{-kw^{-k} + \sum_{i=k}^{\infty} i\, a_i w^i + kw^{-k} + \sum_{i=k}^{\infty} k a_i w^{\,i}}{w^{\,-k} + \sum_{i=k}^{\infty} a_i w^i}\right| = \left|\frac{\sum_{i=k}^{\infty}(i+k)\, a_i w^{i+k}}{1 + \sum_{i=p}^{\infty} a_i w^{\,i+k}}\right|$$

$$\le \frac{\sum_{i=k}^{\infty}(i+k)\, a_i |w|^{\,i+k}}{1 - \sum_{i=k}^{\infty} a_i |w|^{i+k}} < k - \psi,$$

or equivalently

$$\sum_{i=k}^{\infty} \left(\frac{i+2k-\psi}{k-\psi}\right) a_i |w|^{i+k} \le 1.$$

It is enough letting

$$|w|^{i+k} \le \left\{\frac{(k-\psi)i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,\nu,\eta}}{(i+2k-\psi)k\,(k-\gamma\,)+\beta[k\,(k\,+\,\gamma-2\alpha)]}\right\}.$$

Therefore,

$$|w| \le \left\{\frac{(k-\psi)\, i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,\nu,\eta}}{(i+2k-\psi)k\,(k-\gamma)+\beta[k(k\,+\gamma-2\alpha)]}\right\}^{\frac{1}{i+k}}. \tag{24}$$

Setting |w|=$R_1(\lambda,\mu,\nu,\eta,\gamma,\alpha,\beta,\psi)$in (24), getting the radius of starlikeness.
Noting the fact that f is convex if and only if w $f'$is starlike [2], we have

Theorem 4.2: Let $f \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$. Then f is meromorphically k-valently convex of order $\psi(0\leq\psi<k)$, in $|w|<R_2$ where

$$R_2=\inf_i\left\{\frac{k(k-\psi)i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}}{i(i+2k-\psi)k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}\right\}^{\frac{1}{i+k}}. \qquad (25)$$

proof: Let $f \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$. Then by Theorem 2.2

$$\sum_{i=k}^{\infty}\frac{i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}a_i}{k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}\leq 1.$$

For $(0\leq\psi<k)$, we show that

$$\left|\frac{wf''(w)}{f'(w)}+(1+k)\right|\leq k-\psi.$$

That is

$$\left|\frac{k(k+1)w^{-(k+1)}+\sum_{i=k}^{\infty}i\,(i-1)a_iw^{i-1}-k(k+1)w^{-(k+1)}+\sum_{i=k}^{\infty}i(k+1)a_iw^{i-1}}{-kw^{-(k+1)}+\sum_{i=k}^{\infty}i\,a_iw^{i-1}}\right|$$

$$=\left|\frac{\sum_{i=k}^{\infty}i(i+k)a_i\,w^{\,i-1}}{-kw^{-(k+1)}+\sum_{i=k}^{\infty}ia_iw^{\,i-1}}\right|\leq\frac{\sum_{i=k}^{\infty}i(i+k)a_i|w|^{i+k}}{k-\sum_{i=k}^{\infty}ia_i|w|^{i+k}}<k-\psi,$$

or, equivalently

$$\sum_{i=k}^{\infty}\frac{i(i+2k-\psi)}{k(k-\psi)}a_i|w|^{i+k}\leq 1.$$

It is enough to consider

$$|w|^{i+k}\leq\left\{\frac{k(k-\psi)\,i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}}{i(i+2k-\psi)k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}\right\}.$$

Therefore,

$$|w|\leq\left\{\frac{k(k-\psi)\,i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}}{i(i+2k-\psi)k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}\right\}^{\frac{1}{i+k}}. \qquad (26)$$

Setting $|w|=R_2(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ in (26), we get the radius of convexity

5. Closure Theorems:

Theorem 5.1: Let the functions $f_k(w), (k = 1,2, \dots, s)$, of the from:

$$f_m(w)=w^{-p}+\sum_{i=k}^{\infty}a_{i,m}w^n, \left(w\in U^*, a_{i,m}\geq 0\right). \qquad (27)$$

be in the class$\sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Then the function $F \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$,where

$$F(w)=\sum_{m=1}^{s}b_mf_m(w);\ (b_m\geq 0 \text{ and} \sum_{m=1}^{s}b_m=1). \qquad (28)$$

Proof: By using (28), we can write

$$F(w)=w^{-k}+\sum_{i=k}^{\infty}\left(\sum_{m=1}^{s}b_ma_{i,m}\right)w^i. \qquad (29)$$

Since $f_m \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)(m = 1,2, \dots, s)$, therefore,

$$\sum_{i=k}^{\infty}i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}\left(\sum_{m=1}^{s}b_ma_{i,m}\right)w^i$$

$$=\sum_{m=1}^{s}b_m\left(\sum_{i=k}^{\infty}i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\Gamma_i^{\lambda,\mu,v,\eta}a_{i,m}\right)$$

$$\leq\sum_{m=1}^{s}b_m\left(k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]\right)=k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)].$$

By Theorem 2.2, we have F∊ $\sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.
Theorem 5.2: The class $\sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$ is closed under convex linear combination.
Proof: If the function $f_m(m=1,2)$ given by (28) be in the class $\sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$,then it is enough to show that the function

$$g(w)=\sigma f_1(w)+(1-\sigma)f_2(w), (0 \leq \sigma \leq 1), \quad (30)$$

is also in the class $\sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.
Since, for $(0 \leq \sigma \leq 1)$, g (w)=$w^{-k}+\sum_{i=k}^{\infty}[\sigma a_{i,1}+(1-\sigma)a_{i,2}]w^i$,
we observe that
$\sum_{i=k}^{\infty} i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}\{\sigma a_{i,1}+(1-\sigma)a_{i,2}\}$
$=\sigma\sum_{i=k}^{\infty} i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}a_{i,1}+(1-\sigma)\sum_{i=k}^{\infty} i\,[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}a_{i,2}$
$\leq k(k-\gamma)+\beta[k(k+\gamma-2\alpha)]$,
by Theorem 2.2, we have g∊$\sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.

Theorem 5.3: Let $f_{k-1}(w)=w^{-k}$,

$$f_k(w)=w^{-k}+\frac{k(k-\gamma)+\beta[k(k+\gamma-2\alpha)]}{i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}}w^i. \quad (31)$$

For every parameter in Theorem 2.1,
then
$f \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$if and only if f can be expressed in the form

$$f(w)=\sigma_{k-1}f_{k-1}(w)+\sum_{k}^{\infty}\sigma_i f_i(w), \quad (32)$$

Where $\sigma_{k-1}\geq 0, \sigma_i \geq 0$ and $\sigma_{k-1}+\sum_{i=k}^{\infty}\sigma_i=1$.
Proof: Let

$$f(w)=\sigma_{k-1}f_{k-1}(w)+\sum_{i=k}^{\infty}\sigma_i f_i(w)$$

$=w^{-k}+\sum_{i=k}^{\infty}\frac{k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}{i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}}\sigma_i w^i$.
Then

$$\sum_{i=k}^{\infty}\frac{k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}}{i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}\sigma_i$$

$=\sum_{i=k}^{\infty}\sigma_i=1-\sigma_{k-1}\leq 1$.
By Theorem 2.1, we have $f \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.
Conversely, let $f \in \sum_k^+(\lambda, \mu, v, \eta, \gamma, \alpha, \beta)$.
Since

$$a_i \leq \frac{k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}{i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}}, \text{for } i \geq k.$$

We may take

$$\sigma_i=\frac{i[i(1+\beta)+\gamma(1-\beta)+2\beta\alpha]\,\Gamma_i^{\lambda,\mu,v,\eta}}{k\,(k-\gamma)+\beta[k\,(k+\gamma-2\alpha)]}i, \text{ for } i \geq kp$$

and $\sigma_{k-1}=1-\sum_{i=k}^{\infty}\sigma_i$. Then

$$f(w)=\sigma_{k-1}\,f_{k-1}(w)+\sum_{i=k}^{\infty}\sigma_i\,f_i(w).$$

6. Conclusions: We obtain the class $\sum_{K}^{+}(\lambda,\mu,v,\eta,\gamma,\alpha,\beta)$ of multivalent functions by using fractional differ-integral operators . And some geometric properties are calculated.

Refrences

[1] M .Daras, "Meromorphic functions with positive coefficients" , *Int. J. Math.&Math.Sci*.,vol.6, pp.319-324, 2004.

[2] P.L. Duren , Univalent Functions , *Grundlehren der Mathematischen Wissenschaften* ,vol. 259, Springer-Verlag, New York,1983.

[3] S. Owa, M. Saigo and H.M. Srivastava , "some characterization theorems for starlike and convex functions involving a certain fractional integral operator", *J . Math .Anal. APPL*., vol. 140, pp.419-426, 1989.

[4] H.M. srivastava , M. Saigo and S. Owa , "A class of distortion theorems involving certain operators of fractional calculus", *J.Math .Anal. Appl*., vol. 31, pp. 412-420, 1988.

[5] H.M. Srvastava and S. Owa, "Current Topics in Analytic Function Theory", *World Scientific publishing company , Singapore , New Jersey , London and Hong kong* , 1992.

[6] S. Owa , "On the distortion theorems" , *I. Kyung Pook Math .J*., vol. 18, pp.53-59, 1978.

[7] H. M. srivastava and S. Owa, Univalent Functions , Fractional calculus and their Applications , *Halsted press (Ellis Horwood Limited , chichester), John wiley and Sons , New York , Chichester, Brisbane and Toronto*,1989.

[8] S. Ruscheweyh , "New criteria for univalent functions", *Proc. Amer. Math. Soc*., vol. 49, pp.109-115, 1975.

[9] S.K. Hussein and K.A.Jassim, "On A Class of Meromorphic Multivalent Functions Convoluted with Higher Derivatives of Fractional Calculus Operator", *Iraqi Journal of Science*,vol.60,no.10, pp.79-94, 2019.

[10] A. A. Hussein and K.A. Jassim, "On A Certain Class of Meromorphic Multivalent Functions Defined by Fractional Calculus Operator", *Iraqi Journal of Science*, vol. 60, no.12, pp.2685-2696, 2019.

[11] W. G. Atshan, L. A. Alzopee and M. M. Alcheikh, "On Fractional Calculas Operators of a class of Meromorphic Multivalent Functions", *Gen. Math. Notes*, vol. 18, no. 2, pp. 92-103, 2013.