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2-Quasi-Prime Modules

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Abstract

We introduce in this paper, the notion of a 2-quasì-prime module as a generalization of quasi-prime module, we know that a module E over a ring R is called quasi-prime module, if (0) is quasi-prime submodule. Now, we say that a module E over ring R is a 2-quasi-prime module if (0) is 2-quasi-prime submodule, a proper submodule K of E is 2-quasi-prime submodule if whenever a, b ∈ R, $x \in W$ and abx ∈ K, then either $a^2x \in K$ or $b^2x \in K$.
Many results about these kinds of modules are obtained and proved, also, we will give a characterization of these kinds of modules.

Keywords: quasi-prime module, 2-quasi-prime module, 2-quasi-prime submodule, 2-prime module.

المقاسات الاولية الظاهرية من النمط−2

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الخلاصة

في هذا البحث نقدم تعريف مفهوم المقاسات الاولية الظاهرية من النمط −2− كتعميم لمفهوم للمقاسات الاولية الظاهرية, نحن نعلم بأن المقاس E على الحلقة R يدعى مقاس اولي ظاهري اذا كان (0) هو مقاس جزئي اولي ظاهري . الان نقول عن المقاس المعرف على الحلقة R بأنه مقاس اولي ظاهري من النمط −2− اذا كان (0) هو مقاس جزئي اولي ظاهري من النمط−2− , المقاس الجزئي الفعلي K من المقاس E يعرف بأنه مقاس جزئي اولي ظاهري من النمط−2− اذا كان $a,b \in R$, $x \in W$ و $abx \in K$ فأن $a^2x \in K$ او $b^2x \in K$. في هذا البحث العديد من النتائج اثبتت حول هذه الانواع من المقاسات و ايضاً اعطينا تعميم لهذه الانواع من المقاسات.

1. Introduction.

Let E a module over a ring R with identity. Deasle defined in [2] prime module, if (0) is prime submodule, a proper submodule H of E called prime submodule. If rx∈H, where r∈R, x∈E, then either x∈H or r ∈[H: E].

Fatima and Alaa introduced in [3] a 2-prime module as a generalization of a prime module as the following we say E is 2-prime R-module if (0) is 2-prime submodule of E, where every

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prime module is 2-prime module. A proper submodule H of a module E is 2-prime submodule if whenever r∈R, x∈E, rx ∈E, implies x∈H or r^2 ∈[H: E]. [4].

Montana Abdul-Razak introduced in [5] Quasi-prime module, which is a generalization of prime modules. We say E is Quasi-prime R-module if (0) is quasi-prime submodule of E. where a proper submodule H of E is called quasi-prime submodule if whenever $a, b \in R$, x∈E, abx ∈E, implies either $ax \in H$ or $bx \in H$. As a generalization for this notion, we will introduce 2-quasi-prìme R-module, if (0) is 2-quasi-prime submodule, where H is called 2-quasi-prime submodule if whenever $a, b \in R$, x∈E, abx ∈E, implies either $a^2x \in H$ or $b^2x \in H$,[6]. In this paper, many properties for this new kind of modules are proved such that every 2-prime module is 2-quasi-prime but the converse is not true.

2. Basic Properties of 2-quasi-prime module

We introduce in this section the concept of a 2-quasi-prime R-modules and we give a characterization of these kinds of modules.

Definition (2.1).

Let E be a module over a ring R, E is called a 2-quasi-prime R-module if (0) is 2-quasi-prime submodule of E.

In the upcoming proposition, we give a characterization for 2-quasi-prime module.

Proposition (2.2). Let E be an R-module. Then E is a 2-quasi-prime R-module if and only if $ann(x)$ is 2-prime ideal for every non-zero $x \in E$.

Proof:(⇒) suppose that E is 2-quasi-prime module. $ab \in ann(x)$ where $abx = 0$, so $abx \in (0)$ for every $x \in E$. But (0) is 2-quasi-prime submodule, then either $a^2x \in (0)$ or $b^2x \in (0)$. Thus either $a^2 \in ann(x)$ or $b^2 \in ann(x)$, hence $ann(x)$ is 2-prime ideal.

(⇐)Suppose $abw \in (0)$ where $w \in E$ Thus $abw = 0$, then $ab \in ann$(w). But $ann(w)$ is 2-prime ideal then either $a^2 \in ann(w)$ or $b^2 \in ann(w)$, so $a^2w = 0$ or $b^2w = 0$ then either $a^2w \in (0)$ or$b^2w \in (0)$ therefore (0) is 2-quasi-prime submodule of E and hence E is 2-quasi-prime R-module.

Corollary (2.3). Let E be module over ring R. Then E is a 2-quasi-prime R-module if and only if $ann(K)$ is 2-prime ideal for every non-zero submodule K of E.

Remarks and Examples 2.4.

1. The Z-module Z_n is a quasi-prime module if and only if n is prime number by [5] that is not true in 2-quasi-prime R-module, the converse is not true Z_8 is 2-quasi-prime R-module but n is not prime
2. Z_6 as Z-module which is not 2-quasi-prime R-module. Because (0) is not 2-quasi-prime submodule since $3 \cdot 2 \cdot \bar{1} = 0 \in (0)$ and $3^2 \cdot 1 = 9 \notin (0)$ also $2^2 . 1 = 4 \notin (0)$
3. Every quasi-prime module if 2-quasi-prime module.

Proof: Let $abx \in (0)$ where $a, b \in R, x \in E$. Now, since E is quasi-prime module. Thus, (0) is quasi-prime submodule so, either $ax \in (0)$ or $bx \in (0)$and hence either $a^2x \in (0)$ or $b^2x \in (0)$, hence (0) is 2-quasi-prime submodule then E is 2-quasi-prime R-module.

4. The converse of (3) is not true as the following example show Z_8 as Z-module is not quasi-prime R-module by [5] since $2 \cdot 1 \cdot \bar{4} \in (0)$ but $2 \cdot 1 \notin (0)$, but it is 2-quasi-prime R-module
5. Each prime R-module is 2-quasi-prime R-module.
6. The converse of (5) is not true for example: Z_4 as Z-module is not prime since $2 \cdot \bar{2} = 4 \in (\bar{0}), \bar{4} \notin (\bar{0})$ and $2 \notin [(0) :_Z Z_4] = 4Z$
7. Each 2-prime R-module is 2-quasi-prime R-module.

Proof: Let E be 2-prime R-module (0) is 2-prime submodule and by [6] every 2-prime submodule is 2-quasi-prime submodule. Thus (0) is 2-quasi-prime submodule, and hence E is 2-quasi-prime R-module.

8. The converse (7) is not true as the following example shows: Z_8 as Z-module is not 2-prime module but it is 2-quasi-prime submodule.

9. Each non-zero submodule of 2-quasi-prime R-module is 2-quasi-prime R-module.

Proof: Let K be non-zero submodule of 2-quasi-prime R-module E. Now, let $abx \in (0)$, where $a, b \in R$ and $x \in K$ thus either $a^2x \in (0)$ or $b^2x \in (0)$ since E is 2-quasi-prime R-module therefore K is 2-quasi-prime R-module.

10. Each module over a field is 2-quasi-prime R-module.

Proof: Clear by [5]

11. The homomorphic image of 2-quasi-prime module is not necessary to be a 2-quasì-prìme module, for example: clear that Z-module Z is a 2-quasì-prime [5]. But $\frac{Z}{6Z} \cong Z_6$ is not 2-quasi-prime Z-module since $2 \cdot 3 \cdot \bar{1} = \bar{0} \in (0)$ but $2^2 \cdot 1 = 4 \in (0)$ and $3^2 . 1 = 9 \notin (0)$.

12. The Z-module $E = Z \oplus Z_n$ is 2-quasi-prime module

Proof: Since E is prime module by [5] and by (5) E is 2-quasi-prime R-module.

We need the following proposition to prove proposition (2.6).

Proposition (2.5). Let $f: E_1 \longrightarrow E_2$ be an epimorphsim, where E_1 and $E_2$are two R-modules. If K is a 2-quasi-prime submodule of E_2, then $f^{-1}(K)$ is a 2-quasi-prime submodule of E_1.

Proof: To prove $f^{-1}(K)$ is 2-quasi-prime submodule of E_1. We want to prove $[f^{-1}(K): L]$ is a 2-prime ideal. $\forall K \leq E_2$ such that $f^{-1}(K) \subsetneq L$ let $ab \in [f^{-1}(K) \dot{_R} L]$ and so $abL \subseteq f^{-1}(K)$. Thus $f(abL) \subseteq f(f^{-1}(K))$, so $ab(f(L) \subseteq K$, therefore $ab \in [K: f(L)]$. But K is 2-quasi-prime submodule of E_2. Then either $a^2 \in [K: f(L)]$ or $b^2 \in [K: f(L)]$. Thus either $a^2 f(L) \subseteq K$ or $b^2 f(L) \subseteq K$, i.e. either $a^2 L \subseteq f^{-1}(K)$ or $b^2 L \subseteq f^{-1}(K)$ therefore either $a^2 \in [f^{-1}(K): L]$ or $b^2 \in [f^{-1}(K): L]$ i.e. $[f^{-1}(K): L]$ is 2-prime ideal. So $f^{-1}(K)$ is 2-quasi-prime submodule of E_1

Proposition (2.6). Let H is a proper submodule of an R-module E, then H is 2-quasi-prime submodule of E if and only if $\frac{E}{H}$ is a 2-quasi-prime R-module.

Proof: Since $\frac{E}{H}$ is 2-quasi-prime R-module, so by definition (2.1) $0_{\frac{E}{H}}$ is 2-quasi-prime submodule of $\frac{E}{H}$.

Now, let $\pi: E \longrightarrow \frac{E}{H}$ be natural projection. Now, by using Proposition (2.5), and by our assumption we have $\pi^{-1}(0_{\frac{E}{H}}) = H$ is 2-quasi-prime submodule of E.

Conversely, suppose H is a 2-quasi-prime submodule of E. to show that $\frac{E}{H}$ is a 2-quasi-prime submodule: let $r_1 r_2 (x + H) \in 0_{\frac{E}{H}}$, where $r_1, r_2 \in R, (x + H) \in \frac{E}{H}$. So $(r_1 r_2 x + H)$ this implies that $r_1 r_2 x \in H$. But H is a 2-quasi-prime submodule of E, thus either ${r_1}^2 x \in H$ or ${r_2}^2 x \in H$. This means either ${r_1}^2 x + H = H$ or ${r_2}^2 x + H = H$ i.e., ${r_1}^2 (x + H) = H$ or ${r_2}^2 (x + H) = H$ therefore, either ${r_1}^2 (x + H) \in 0_{\frac{E}{H}}$ ${r_2}^2 (x + H) \in 0_{\frac{E}{H}}$.This means that $0_{\frac{E}{H}}$ is a 2-quasi-prime submodule. hence, $\frac{E}{H}$ is a 2-quasi-prime R-module.

Proposition (2.7). Let E_1 and E_2 be two R-modules such that E_2 is 2-quasi-prime R-module. Then $Hom(E_1, E_2)$ is a 2-quasi-prime R-module.

Proof: Let K be any submodule of $Hom(E_1, E_2)$. To show that $ann_R K$ is 2-prime ideal of R. Let $r_1, r_2 \in R$ such that $r_1 r_2 \in ann_R K$. For each $f \in K, r_1 r_2 f = 0$. So $r_1 r_2 f(x) = 0$ for all $x \in E$ since E_2 is a 2-quasi-prime R-module. Then (0) is a 2-quasi-prime submodule of E. Thus either ${r_1}^2 f(x) = 0$ or ${r_2}^2 f(x) = 0$ for all $x \in E$, hence either ${r_1}^2 f = 0$ or ${r_2}^2 f = 0$. Therefore, $ann_R K$ is 2-prime ideal.

Corollary (2.8). The endomorphism of 2-quasi-prime R-module is also a 2-quasi-prime R-module.
Proof: clear.
Recall that an integral domain is a non-zero commutative ring with no non-zero divisors

Proposition (2.9). If E is an R-module and $S = End_R(E)$ such that E is a 2-quasi-prime S-module and S is commutative then S is an integral domain.
Proof: Let $f, g \in S$ such that $f.g = 0$. Thus $(f.g)(x) = 0$ for all $x \in E$. But E is a 2-quasi-prime S-module, so (0) is a 2-quasi-prime S-submodule of E. Thus either $f^2(x) = 0$ or $g^2(x) = 0$ for all $x \in E$. therefore either $f = 0$ or $g = 0$ and this means that S is an integral domain.
The following corollary are immediate consequence of the last theorem.

Corollary (2.10). Let E is an R-module, then E is a 2-quasi-prime R-module if and only if E is a 2-quasi-prime $R/ann_R E$ −module.
Remark (2.11). If E is a 2-quasi-prime R-module, then it is not necessary that R is a 2-quasi-prime ring; for example Z_2 as Z_6-module, Z_2 is a 2-quasi-prime module, since (0) is 2-quasi-prime submodule. But Z_6 is not 2-quasi-prime ring since $\overline{2} \cdot \overline{3} \cdot \overline{1} \in (0)$, but neither $\overline{2}^2 \in (0)$ nor $\overline{3}^2 \in (0)$.
Moreover, if R is a 2-quasi-prime ring and E is an R-module then E is not necessarily 2-quasi-prime module, for example: consider the Z-module Z_6: notice that Z is a 2-quasi-prime ring since (0) is 2-quasi-prime ideal) but Z_6 is not 2-quasi-prime module.

Proposition (2.12). If E is a faithful multiplication R-module, then E is a 2-quasi-prime R-module if and only if R is a 2-quasi-prime ring.
Proof: suppose E is a 2-quasi-prime R-module. To prove that R is 2-quasi-prime ring. Let J be a non-zero ideal of R since E is multiplication R-module so $K = JE$ is a non-zero submodule of E.
Hence $ann_R K$ is a 2-prime ideal of R because E is a 2-quasi-prime module. And since E is a faithful R-module, so $ann_R K = ann_R J$ thus $ann_R J$ is a 2-prime ideal and R is a 2-quasi-prime ring.
Now, for the converse, suppose R is a 2-quasi-prime ring, we want prove that E is 2-quasi prime module: let H be a non-zero submodule of E. Since E is a multiplication R-module, then $H = AE$, for some ideal A of R. But E is faithful so $ann_R H = ann_R AE = ann_R A$, which is 2-prime ideal. Therefore by proposition (2.2) E is 2-quasi-prime R-module.

Recall that an R-module E is uniform if the intersection of any two non-zero submodules of E is not zero [9]

Remark (2.13). We notice that not every 2-quasi-prime R-module is uniform for example the Z-module $Z \oplus Z$ is prime and hence 2-quasi-prime Z-module. But it is not uniform if $U = Z \oplus (0)$ and $K = (0) \oplus Z$ then $U \cap K = (0)$.

Theorem (2.14). Let E be an R-module and let I be an ideal of R, which is contained in $ann_R E$. Then E is a 2-quasi-prime R-module if and only if E is a 2-quasi-prime R/I −module.

Proof: Let E be 2-quasi-prime R-module. To prove that E is 2-quasi-prime R/I −module, we must prove that $ann_{R/I}K$ is 2-prime ideal for every submodule K of E. Now, let $(a+I)=(b+I)\in ann_{\frac{R}{I}}K$ and suppose $a^2+I\notin ann_R K$ so $a^2K\neq 0$, where $(a+I),(b+I)\in R$, then $(ab+I)\in ann_{\frac{R}{I}}K$. Thus $(ab+I)K=0$, so $abK=0$ i.e. $ab\in ann_R K$. But $ann_R K$ is 2-prime ideal since E is a 2-quasi-prime R-module and $a^2K\neq 0$ so$b^2K=0$. Thus $(b^2+I)K=0$ and hence $(b+I)^2\in ann_{\frac{R}{I}}K$. Therefore, $ann_{\frac{R}{I}}K$ is 2-prime ideal and thus E is 2-quasi-prime R/I −module.
The converse is clear.

Remark (2.15). If R is 2-quasi-prime R-module, then I is 2-prime ideal of R, but this not implies that I is 2-quasi-prime ideal of R, as the following example shows consider Z_8 as Z_8-module. One can prove easily that Z_8 asZ_8-module is 2-quasi-prime Z_8 −module. Let $I=\{\overline{0},\overline{4}\}$, I is a 2-prime ideal of Z_8, but it is not a 2-quasi-prime ideal of Z_8 since $\overline{3}\cdot\overline{2}\cdot\overline{2}=\overline{4}\in I$, but $\overline{3}^2.\overline{2}=\overline{2}\notin I$.

Now, we will see the direct sum of 2-quasi-prime R-modules need not to be 2-quasi-prime.

Proposition (2.16). Let $E=E_1\oplus E_2$ be R-module if E_1 and E_2 are 2-quasi-prime module such that $\forall(x,y)\in E, x\neq 0, y\neq 0, ann(x)\subseteq ann(y)$ or $ann(y)\subseteq ann(x)$. Then E is 2-quasi-prime R-module.

Proof: To prove (0_E) is 2-quasi-prime submodule, let $ab(x,y)=(0,0)$ then $abx=0$ or $aby=0$

- $x\neq 0, y=0$

$abx=0$ then $a^2x=0$ or $b^2x=0$ if $a^2x=0$ then $a^2(x,y)=(0,0)$ since $y=0$
If $b^2x=0$ then $b^2(x,y)=(0,0)$

- Case 1: $x=0, y\neq 0$

$aby=0$ so $a^2y=0$ or $b^2y=0$ if $a^2y=0$ then $a^2(x,y)=(0,0)$
If $b^2y=0$ then $b^2(x,y)=(0,0)$

- Case 2: $x=0, y=0$. It is clear that $a^2(x,y)=(0,0)$ and $b^2(x,y)=(0,0)$
- Case 3: $x\neq 0, y=0$. By assumption $ann(x)\subseteq ann(y)$ or $ann(y)\subseteq ann(x)$

Suppose $ann(x)\subseteq ann(y)$
Now, $abx=0$ then $a^2x=0$ or $b^2x=0$ if $a^2x=0$ then $a^2\in ann(x)\subseteq ann(y)$ thus $a^2(x,y)=(0,0)$ if $b^2x=0$ then $b^2\in ann(x)\subseteq ann(y)$ thus $a^2(x,y)=(0,0)$
If $ann(y)\subseteq ann(x)$ since $aby=0$ implies $a^2\in ann(y)\subseteq ann(x)$ thus $a^2(x,y)=(0,0)$
If $b^2y=0$ then $b^2\in ann(y)\subseteq ann(x)$ then $b^2(x,y)=(0,0)$ thus either $a^2(x,y)=(0,0)$ or $b^2(x,y)=(0,0)$ so $(0_E)=(0,0)$ is 2-quasi-prime submodule so E is a 2-quasi-prime module

Proposition (2.17). Each direct summand of 2-quasi-prime R-module is 2-quasi-prime submodule.

Proof: Suppose H_1 and H_2 are two R-submodule of an R-module E such that $E=H_1\oplus H_2$ and E is 2-quasi-prime R-module to show that H_1 is 2-quasi-prime R-submodule let $t_1,t_2\in R$ and $x\in E$ such $t_1t_1\in[H_1\dot{_R}(x)]$ so $x=x_1\oplus x_2$, where $x_1\in H_1, x_2\in H_2$. Thus $t_1t_2(x_1+x_2)\in H_1$and hence that $t_1t_2x_1+t_1t_2x_2\in H_1$ which implies that $t_1t_2x_1\in H_1\cap H_2=(0)$. Thus $t_1t_2\in[(0)\dot{_R}(x_2)]$. Since E is 2-quasi-prime module then (0) is 2-quasi-prime submodule and hence $[(0)\dot{_R}(x_2)]$ is 2-prime ideal thus either ${t_1}^2\in[(0)\dot{_R}(x_2)]$ or ${t_2}^2\in[(0)\dot{_R}(x_2)]$. Thus either ${t_1}^2x_1=0$ or ${t_2}^2x_2=0$. Since ${t_1}^2x={t_1}^2(x_1\oplus x_2)={t_1}^2x_1+{t_1}^2x_2$ and ${t_2}^2x={t_2}^2(x_1+$

$x_2) = t_2{}^2 x_1 + t_2{}^2 x_2$. Hence either $t_1{}^2 \in [H_{1}\dot{{}_R}(x)]$ or $t_2{}^2 \in [H_{1}\dot{{}_R}(x)]$, which means that $[H_{1}\dot{{}_R}(x)]$ is 2-prime ideal of R and H_1 is 2-quasi-prime submodule of E.

Remark (2.18). The converse of Proposition 2.17 is not true the Z-module $E = Z_2 \oplus Z_3$ is not 2-quasi-prime R-module since $ann(Z_2 \oplus Z_3) = 6Z$ is not 2-prime ideal. But Z_2 and Z_3 are 2-quasi-prime R-module.

Conclusion

1. Let E be an R-module, then E is a 2-quasi-prime R-module if and only if $ann(x)$ is 2-prime ideal for every non zero $x \in E$.
2. If E_1 and E_2 be two R-modules such that E_2 is 2-quasi-prime R-module. Then $Hom(E_1, E_2)$ is a 2-quasi-prime R-module.
3. The endomorphism of 2-quasi-prime R-module is also a 2-quasi-prime R-module.
4. If E is an R-module and $S = End_R(E)$ such that E is 2-quasi-prime S-module and S is commutative, the S is an integral domain
5. If E is a faithful multiplication R-module, then E is a 2-quasi-prime R-module if and only if R is a 2-quasi-prime ring.
6. Let E be an R-module and let I be an ideal of R, which is contained in $ann_R E$. Then E is a 2-quasi-prime R-module if and only if E is a 2-quasi-prime R/I −module.

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