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Existence and Qualitative Property of Differential Equation with Delayed Arguments

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Abstract

In this paper, some conditions to guarantee the existence of bounded solution to the second order multi delayed arguments differential equation are given. The Krasnoselskii theorem used to the Lebesgue's dominated convergence and fixed point to obtain some new sufficient conditions for existence of solutions. Some important lemmas are established that are useful to prove the main results for oscillatory property. We also submitted some sufficient conditions to ensure the oscillation criteria of bounded solutions to the same equation.

Keywords: Existence of Nonocillatory Bounded Solutions, Oscillation Criteria, Second Order Multiple Delay Differential Equation, Banach Space.

الوجود وخاصية نوعية لمعادلة تفاضلية ذات متغيرات تباطؤية.

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الخلاصة

في هذا البحث يجب ان نعطي بعض الشروط لضمان وجود حل مقيد للمعادلة التفاضلية ذات متغيرات تباطؤية متعددة من الرتبة الثانية. تم استخدام نظرية Krasnoselskii لتقارب ليبيك المهيمن والنقطة الصامدة للحصول على بعض الشروط الجديدة الكافية لوجود الحلول. تم إنشاء بعض التمهيديات المهمة التي تعتبر مهمة في إثبات النتائج الرئيسية لخاصية التذبذب. لقد اقترحنا أيضًا بعض الشروط الكافية لضمان معايير التذبذب للحلول المقيدة لنفس المعادلة.

1. Introduction

The field of differential equations is centered on the study of many other research fields, namely the study of analytical and numerical methods [1,2]. The existence of the solutions plays an important role in field of differential equations [3,4]. Furthermore, the considerations of the stability of the solution, the theory of oscillation and the asymptotic behavior of solutions are

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also significant issue in this field [5-7]. There are variety applications of differential equations in different science fields, such as physics, engineering, biology and many other scientific disciplines. [8,9].

Y. Liu J. Zhang J. Yan [10] investigated the existence of oscillatory solutions for a forced second order nonlinear delay differential equations:

$$\frac{d}{d\xi}\left(a_\zeta(\xi)\psi\left(\frac{d}{dt}\chi(\xi)\right)\right)+\sum_{\varsigma=1}^{\lambda}\Gamma_\varsigma\left(\xi,\chi\left(g_\varsigma(\xi)\right)\right)= r(\xi)$$

Where $\Gamma_\varsigma \in \mathrm{C}([\xi_0,\infty)\times\Re,\Re), g_\varsigma(\xi)\leq\xi, \lim_{\xi\to\infty} g_\varsigma(\xi)=\infty, \varsigma=1,\dots,\lambda, \psi\in\mathrm{C}^1(\Re,\Re)$

In [11] Y. Liu, J. Zhang and J. Yan considered the existence of oscillatory solutions for the nonlinear second order delay differential equations with the perturbed term:

$$\frac{d}{d\xi}\left(a(\xi)\psi\left(\frac{d}{dt}\chi(\xi)\right)\right)+\int_l^m\varphi(\xi,\tau)\,\Gamma\big(\chi(\xi-\tau)\big)d\tau = r(\xi), \qquad \xi\geqslant\xi_0$$

Where $a(\xi)\in\mathrm{C}^1([\xi_0,\infty),\Re^+), \varphi\in\mathrm{C}([\xi_0,\infty)\times[l,m],\Re),\ r\in\mathrm{C}([\xi_0,\infty),\Re)$, such that $\Gamma\in\mathrm{C}([\xi_0,\infty),\Re), \psi\in\mathrm{C}^1(\Re,\Re)$, where $\psi(u)$ is increasing function for all $u\in\Re$, ψ^{-1} satisfies the local Lipischitz condition.

The researchers J. Džurina and I. Jadlovská in [12] studied the second-order half-linear delay differential equation:

$\frac{d}{d\xi}\left(a(\xi)\left(\frac{d}{dt}\chi(\xi)\right)^{\omega}\right)+ r(\xi)\chi^{\omega}\big(\tau(\xi)\big)=0$, where $\omega>0$ is a quotient of positive odd integer numbers, $a,\tau\in\mathrm{C}^1\big([\xi_0,\infty),(0,\infty)\big)$ and $r\in\mathrm{C}\big([\xi_0,\infty),(0,\infty)\big)$ with $\tau(\xi)\leq\xi,\ \pi(\xi)\leq\xi,$ $\tau'(\xi)\geq 0$.

B. Baculíková, B. Sudha, K. Thangavelu and E. Thandapani in [13] dealt with oscillation of a second order delay differential equations with a nonlinear nonpositive neutral term:

$$\frac{d}{d\xi}\left(r(\xi)\frac{d}{d\xi}\Big(\chi(\xi)-a(\xi)\chi^{\omega}\big(\tau(\xi)\big)\Big)\right)+b(\xi)\chi^{\varepsilon}\big(\pi(\xi)\big)=0, \xi\geqslant\xi_0>0$$

subject to the following conditions:

1) $0<\omega\leq 1$, and ε are ratio of odd positive integers;
2) $r\in\mathrm{C}^1\big([\xi_0,\infty),(0,\infty)\big), a,b\in\mathrm{C}\big([\xi_0,\infty),(0,\infty)\big), 0< a(\xi)\leq p<1, \forall\xi\geq\xi_0$
3) $\tau\in\mathrm{C}^1([\xi_0,\infty),\Re), \pi\in\mathrm{C}^1([\xi_0,\infty),\Re), \tau(\xi)\leq\xi,\ \pi(\xi)\leq\xi,\ \tau'(\xi)>0, \pi'(\xi)>0, \lim_{\xi\to\infty}\tau(\xi)=\lim_{\xi\to\infty}\pi(\xi)=\infty$

In this paper, we focus on existence and oscillatory solution to the second order non-linear DDEs with Multiple delays:

$$\frac{d^2}{d\xi^2}\chi(\xi)=-\sum_{\varsigma=1}^{\lambda}\alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right)+\frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda}b_\varsigma(\xi)\Gamma_\varsigma\left(\xi,\chi\left(\tau_\varsigma(\xi)\right)\right) \qquad (1.1)$$

During this work we will impose the following hypotheses

(i) $C(H_1,H_2)$ denotes to the set for all functions that are continuous; $f\colon H_1\to H_2$ with the supremum norm $\|.\|$.

(ii) We suppose that α_ς, $b_\varsigma \in C(\Re^+,\Re^+), (\varsigma = 1,2,\dots,\eta)$ and the functions $\tau_\varsigma : \Re^+ \to \Re^+$ are differentiable with $\tau_\varsigma(t) \longrightarrow \infty$ as $\xi \longrightarrow \infty$.
(iii) The functions $\gamma_\varsigma(\chi)$ and $\Gamma_\varsigma(t,\chi)$ are continuous and the second function satisfies Lipschitz condition in χ. That is, there are positive constants M_ς $(\xi = 1,2,\dots,\lambda)$, such that
$\left|\Gamma_\varsigma(\xi,\chi) - \Gamma_\varsigma(\xi,\gamma)\right| \preccurlyeq M_\varsigma|\chi-\gamma| \quad \varsigma = 1,2,\dots,\eta,$

The solution $\chi(\xi)$ satisfies Eq.(1.1) for $\xi \succcurlyeq \xi_1$. We say that solution $\chi(\xi)$ is a nonoscillatory solution if it is eventually negative or eventually positive, so there exists $\xi_* \succcurlyeq \xi_0$, such that $\chi(\xi) \succ 0$ or $\chi(\xi) \prec 0$ for all $\xi \succcurlyeq \xi_*$, otherwise the solution is said to be oscillatory [6].
We need the following lemma and theorem in the main results of second section.

Lemma 1.1: [14] (Krasnoselskii Fixed Point Theorem).
In Banach space X with Ʊ is closed convex and bounded subset in X, if S_1, S_2: Ʊ $\longrightarrow$ X, ∋ $S_1\chi + S_2\gamma \in$ Ʊ, $\forall \chi,\gamma \in$ Ʊ. If S_1 is mapping with contractive feature and S_2 is a completely continuous mapping, then $S_1\chi + S_2\gamma = \chi$ is a solution on Ʊ.

Theorem 1.2 [15] (The Lebesgue Dominated Convergence Theorem)
Let $\{\wp_n\}$ be sequence of measurable functions on E and q be integrable function on E with dominates $\{\wp_n\}$ on E such that $|\wp_n(\chi)| \preccurlyeq q(\chi)$ on E, for all n.If $\{\wp_n\} \to \{\wp\}$ is pointwise a.e. on E, then $\wp$ is integrable on E with $\lim_{n\to\infty} \int_E \wp_n = \int_E \wp$, E is a measurable finite set.

Lemma 1.3 [16]
Let $g(t) \in C[R,R^+], R^+ = [0,\infty), \sigma(t), \alpha(t)$ be continuous strictly increasing functions with $\lim_{t\to\infty}\sigma(t) = \infty, \lim_{t\to\infty}\alpha(t) = \infty$ and $\sigma(t) < \alpha(t)$, for $\sigma(t) \geq t_0$,
if $\int_{t_0}^{\infty} g(t)dt < \infty$ then $\lim_{t\to\infty}\int_{\sigma(t)}^{\alpha(t)} g(s)ds = 0$.

2. Existence and Oscillatory Bounded Solutions of differential equation with Delayed Arguments:

In this section, we introduce new sufficient conditions to ensure that the solution exists and bounded by two positive functions u and v on $[\xi_1,\infty)$ of Eq.(1.1), $\xi_1 \succcurlyeq \xi_0$. The existence of positive bounded solution is studied, while existence of eventually negative solution can be found similarly.
Suppose the following conditions to be hold in the included results in this section:
A1. $\sigma_2 \prec a_\varsigma(\xi), b_\varsigma(\xi) \preccurlyeq \sigma_1$, $\sigma_1,\sigma_2 \neq 0$, are constants, $\varsigma = 1,2,3,\dots,\lambda$
A2. $\rho_1\chi(\xi) \preccurlyeq \gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right) \preccurlyeq \rho_2\chi(\xi)$, $\rho_1,\rho_2 \neq 0$, are constants, $\varsigma = 1,2,3,\dots,\lambda$.
A3. $\mu_1\chi(\xi) \preccurlyeq \Gamma_\varsigma\left(\xi,\chi\left(\tau_\xi(\xi)\right)\right) \preccurlyeq \mu_2\chi(\xi)$, $\mu_1,\mu_2 \neq 0$, are constants, $\varsigma = 1,2,3,\dots,\lambda$.

Theorem 2.1
Assume that A1- A3 hold, and the bounded functions $u, v \in C^1(\mathbb{N},[0,\infty))$, and $\xi_1 \succcurlyeq \xi_0 + \rho$:

$$u(\xi) \preccurlyeq u(\xi_1) \text{ and } v(\xi) \succcurlyeq v(\xi_1)\,,\ \xi_0 \preccurlyeq \xi \preccurlyeq \xi_1 \qquad (2.1)$$

$$\frac{1}{\sigma_2\rho_1}\left(\sigma_1\mu_2\int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(t)\right)dt - \frac{1}{\sigma_2}v(t)\right) \preccurlyeq \int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(t)\right)dt\,ds \preccurlyeq K \prec \infty \qquad (2.2)$$

$$\int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda}u\left(\tau_{\varsigma}(\mathrm{t})\right)dt\,ds \leqslant \frac{1}{\sigma_1\rho_2}\left(-\sigma_2\mu_1\int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda}u\left(\tau_{\varsigma}(\mathrm{t})\right)dt+\sigma_2 u(\xi)\right)\leqslant -\mathrm{K} \prec -\infty, \xi \geqslant \xi_1, \qquad (2.3)$$

Then Eq.(1.1) has a bounded solution by positive functions u and v.

Proof

Let $I(\mathrm{t}) = \int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_{\varsigma}(\mathrm{t})\right)dt\,ds$ and then the condition (2.2) implies that $\lim_{t\to\infty} I(\mathrm{t}) = \lim_{t\to\infty}\int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_{\varsigma}(\mathrm{t})\right)dt\,ds = 0.$ (2.4)

Let $(C([\xi_0,\infty),\mathfrak{R}), \|.\|)$ such that $\|\chi\| = \sup_{\xi\geq\xi_0}|\chi(\xi)|$, then $C([\xi_0,\infty),\mathfrak{R})$ is a Banach space.

Let $\Psi \subset C([\xi_0,\infty),\mathfrak{R})$ as:

$$\Psi = \{\chi(\xi): \chi(\xi) \in \mathrm{C}([\xi_0,\infty),\mathfrak{R}) \text{ with } u(\xi) \leqslant \chi(\xi) \leqslant v(\xi), \xi \geqslant \xi_0\}, \qquad (2.5)$$

such that Ψ is closed and convex.

The mappings φ_1 and φ_2: $\Psi \to C([\xi_0,\infty),\mathfrak{R})$ are defined as:

$$(\varphi_1\chi)(\xi) = \begin{cases} \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} b_{\varsigma}(\mathrm{t})\Gamma_{\varsigma}\left(\mathrm{t},\chi\left(\tau_{\varsigma}(\mathrm{t})\right)\right)\mathrm{dt}, & \xi \geqslant \xi_1, \\ (\varphi_1\chi)(\xi_1), & \xi_0 \leqslant \xi \leqslant \xi_1, \end{cases}$$

$$(\varphi_2\chi)(\xi) = \begin{cases} -\int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} a_{\varsigma}(\mathrm{t})\gamma_{\varsigma}\left(\chi\left(\tau_{\varsigma}(\mathrm{t})\right)\right)dt\,ds, & \xi \geqslant \xi_1, \\ (\varphi_2 x)(\xi_1) & \xi_0 \leqslant \xi \leqslant \xi_1, \end{cases} \qquad (2.6)$$

φ_1 and φ_2 satisfy eq (1.1) for all $\chi, \mathcal{Y} \in \Psi$ and $\xi \geqslant \xi_1$, then:

By using conditions A1 and A2, we have

$$(\varphi_1\chi)(\xi) + (\varphi_2\mathcal{Y})(\xi) \leqslant \sigma_1\mu_2\int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda}\chi\left(\tau_{\varsigma}(\mathrm{t})\right)\mathrm{dt} - \sigma_2\rho_1\int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda}\left(\mathcal{Y}\left(\tau_{\varsigma}(\mathrm{t})\right)\right)dt\,ds$$

$$\leqslant \sigma_1\mu_2\int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_{\varsigma}(\mathrm{t})\right)\mathrm{dt} - \sigma_2\rho_1\int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda}\left(v\left(\tau_{\varsigma}(\mathrm{t})\right)\right)dt\,ds$$

$$\leqslant \sigma_1\mu_2\int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_{\varsigma}(\mathrm{t})\right)\mathrm{dt} - \sigma_2\rho_1\frac{1}{\sigma_2\rho_1}\left(\sigma_1\mu_2\int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda}(v\left(\tau_{\varsigma}(\mathrm{t})\right)dt - \frac{1}{\sigma_2}v(\xi))\right) = \frac{1}{\sigma_2}v(\xi) \leqslant v(\xi)$$

$\forall\xi \in [\xi_0,\xi_1]$, by using eq.(2.1) and eq.(2.6) we have

$$(\varphi_1\chi)(\xi) + (\varphi_2\mathcal{Y})(\xi) = (\varphi_1\chi)(\xi_1) + (\varphi_2\mathcal{Y})(\xi_1) \leqslant v(\xi_1) \leqslant v(\xi).$$

So, $\forall\,\xi \geqslant \xi_1$, this implies to :

$$(\varphi_1\chi)(\xi) + (\varphi_2\mathcal{Y})(\xi)$$
$$= \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) dt - \int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} a_\varsigma(\mathrm{t})\gamma_\varsigma\left(\mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right)\right) dt\, ds$$
$$\geqslant \sigma_2\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(\mathrm{t})\right) \mathrm{dt} - \sigma_1\rho_2 \int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds$$

$$\geqslant \sigma_2\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) \mathrm{dt} - \sigma_1\rho_2 \int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} \left(v\left(\tau_\varsigma(\mathrm{t})\right)\right) dt\, ds$$
$$\geqslant \sigma_2\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) \mathrm{dt} - \sigma_1\rho_2 \mathrm{K}$$
$$\geqslant \sigma_2\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) \mathrm{dt} + \sigma_1\rho_2 \frac{1}{\sigma_1\rho_2}\left(-\sigma_2\mu_1 \int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) dt + \sigma_2 u(\xi)\right) \geqslant u(\xi)$$
$\forall\, \xi \in [\xi_0, \xi_1]$, by using eq.(2.1) and eq.(2.6) we get:
$$(\varphi_1\chi)(\xi) + (\varphi_2\mathcal{Y})(\xi) = (\varphi_1\chi)(\xi_1) + (\varphi_2\mathcal{Y})(\xi_1)$$
$$\geqslant u(\xi_1) \geqslant u(\xi) \qquad (2.7)$$
So, $\varphi_1\chi + \varphi_2\mathcal{Y} \in \Psi, \forall\, \chi, \mathcal{Y} \in \Psi, \chi > \mathcal{Y}$. Now, we have to prove that φ_1 is contraction mapping on Ψ. $\forall\, \chi, \mathcal{Y} \in \Psi$ for $\xi \geqslant \xi_1$:
$$\|\varphi_1 x - \varphi_1\mathcal{Y}\| = \sup_{t \geq t_1} |(\varphi_1 x)(\xi) - (\varphi_1\mathcal{Y})(\xi)|$$
$$= \sup_{\xi \geqslant \xi_1}\left|\int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) dt - \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\, \Gamma_\varsigma\left(\mathrm{t}, \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right)\right) dt\right|$$
.
$$\|\varphi_1 x - \varphi_1\gamma\| \leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\,\mu_2 \int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(\mathrm{t})\right) dt - \sigma_2\,\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right) dt\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\,\mu_2 \int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) dt - \sigma_2\,\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) dt\right|$$
By equation (2.5) and A1, we have

$$\sup_{\xi \geqslant \xi_1}\left|\sigma_1\,\mu_2 \int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) dt - \sigma_2\,\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) dt\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\,\mu_2 \int_{\xi=1}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) dt - \sigma_2\,\mu_1 \int_{\xi}^{\infty}\sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) dt\right|$$
$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_2\,\mathrm{K}\rho_1 + \frac{1}{\sigma_2} v(\xi) - \sigma_1\,\mathrm{K}\rho_2 - \sigma_2 u(\xi)\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1} \left|\sigma_1 \,\mathrm{K}\rho_2 + \frac{1}{\sigma_2} v(\xi) - \sigma_1 \,\mathrm{K}\rho_2 - \sigma_2 u(\xi)\right|$$

$$\leqslant \sup_{t\xi \geqslant \xi_1} \left|\frac{1}{\sigma_2} v(\xi) - \sigma_2 u(\xi)\right|$$

$$\leqslant \sup_{t\xi \geqslant \xi_1} \left|\frac{1}{\sigma_2} v(\xi) - \frac{1}{\sigma_2} u(\xi)\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1} \frac{1}{\sigma_2} |v(\xi) - u(\xi)| \leqslant \sup_{t \geqslant t_1} \frac{1}{\sigma_2} |\chi(\xi) - \mathcal{Y}(\xi)|)$$

$$\leqslant M\|\chi - \mathcal{Y}\| \tag{2.8}$$

Where , $M = \frac{1}{\sigma_2}$

Also, for $\xi \in [\xi_0, \ \xi_1]$.

$$\|\varphi_1\chi - \varphi_1\mathcal{Y}\| = \sup_{\xi_0 \leqslant \xi \leqslant \xi_1} |(\varphi_1\chi)(\mathrm{t}) - (\varphi_1\mathcal{Y})(\mathrm{t})|$$

$$= \sup_{t_0 \leqslant t \leqslant t_1} |(\varphi_1\chi)(\xi_1) - (\varphi_1\mathcal{Y})(\xi_1)|$$

$$= \sup_{\xi \geqslant \xi_1} \left| \int_{\xi_1}^{\infty} \sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) dt - \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\, \Gamma_\varsigma\left(\mathrm{t}, \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right)\right) dt \right|$$

$$\leqslant \left|\sigma_1 \,\mu_2 \int_{\xi_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t} - \sigma_2 \,\mu_1 \int_{\xi_1}^{\infty} \sum_{\varsigma=1}^{\lambda} \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right) dt\right|$$

$$\leqslant \left|\sigma_1 \,\mu_2 \int_{\xi_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t} - \sigma_2 \,\mu_1 \int_{\xi\_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t}\right|$$

By equation (2.5) and A1, we have

$$\left|\sigma_1 \,\mu_2 \int_{\xi_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t} - \sigma_2 \,\mu_1 \int_{\xi\_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t}\right|$$

$$\leqslant \left|\sigma_1 \,\mu_2 \int_{\xi_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t} - \sigma_2 \,\mu_1 \int_{\xi\_1=1}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t}\right|$$

By condition (2.1) we have:

$$\leqslant \left|\sigma_2 \,\mathrm{K}\rho_1 + \frac{1}{\sigma_2} v(\xi_1) - \sigma_1 \,\mathrm{K}\rho_2 - \sigma_2 u(\xi_1)\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1} \left|\sigma_1 \,\mathrm{K}\rho_2 + \frac{1}{\sigma_2} v(\xi) - \sigma_1 \,\mathrm{K}\rho_2 - \sigma_2 u(\xi)\right|$$

$$\leqslant \left|\frac{1}{\sigma_2} v(\xi) - \sigma_2 u(\xi)\right|$$

$$= \left|\frac{1}{\sigma_2} v(\xi) - \frac{1}{\sigma_2} u(\xi)\right|$$

$$= \sup_{\xi \geqslant \xi_1} \frac{1}{\sigma_2} |v(\xi) - u(\xi)| \leqslant \sup_{t \geqslant t_1} \frac{1}{\sigma_2} |\chi(\xi) - \mathcal{Y}(\xi)|)$$

$$\leqslant M\|\chi - \mathcal{Y}\| \tag{2.9}$$

Where , $M = \frac{1}{\sigma_2}$. This implies that

$$\|\varphi_1\chi - \varphi_1\mathcal{Y}\| \leqslant M\|\chi - \mathcal{Y}\| \tag{2.10}$$

Thus, φ_1 is mapping with contractive property on Ψ. Now, we have to prove that φ_2 has completely property to continuous mapping. First of all, we need to show that φ_2 is continuous mapping.

Let $\chi_k = \chi_k(\xi) \in \Psi$. Since Ψis closed, thus $\chi_k(\xi)$ tend to $\chi(\xi)$ as $k \to \infty$, $\chi(\xi) \in \Psi$. For $\xi \geqslant \xi_1$, yield:

$$\|(\varphi_2\chi_k)(\xi) - (\varphi_2\chi)(\xi)\| = \sup_{t \geqslant t_1}|(\varphi_2\chi_k)(\xi) - (\varphi_2\chi)(\xi)|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|-\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda} a_\varsigma(t)\gamma_\varsigma\left(\chi_k\left(\tau_\varsigma(t)\right)\right)dt\,ds + \int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda} a_\varsigma(t)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|-\sigma_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\gamma_\varsigma\left(\chi_k\left(\tau_\varsigma(t)\right)\right)dt\,ds + \sigma_1\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\gamma_\varsigma\left(\chi_k\left(\tau_\varsigma(t)\right)\right)dt\,ds - \sigma_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\rho_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(\chi_k\left(\tau_\varsigma(t)\right)\right)dt\,ds - \sigma_2\rho_1\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(\chi\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\rho_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(v_k\left(\tau_\varsigma(t)\right)\right)dt\,ds - \sigma_2\rho_1\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(u\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$= \sup_{\xi \geqslant \xi_1}\left|\sigma_2\rho_1\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(u\left(\tau_\varsigma(t)\right)\right)dt\,ds - \sigma_1\rho_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(v_k\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\left|\sigma_1\rho_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(v\left(\tau_\varsigma(t)\right)\right)dt\,ds - \sigma_1\rho_2\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(v_k\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\sigma_1\rho_2\left|\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(v_k\left(\tau_\varsigma(t)\right)\right)dt\,ds - \int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}\left(v\left(\tau_\varsigma(t)\right)\right)dt\,ds\right|$$

$$\leqslant \sup_{\xi \geqslant \xi_1}\sigma_2\rho_1\left|\int_\xi^\infty\int_s^\infty\sum_{\varsigma=1}^{\lambda}[\left(v_k\left(\tau_\varsigma(t)\right)\right) - \left(v\left(\tau_\varsigma(t)\right)\right)]dt\,ds\right|$$

$$\preccurlyeq \sup_{\xi \succcurlyeq \xi_1} \sigma_2\rho_1\Bigg(\left|\int_{\xi}^{\infty}\int_{s}^{\infty}[\left(v_k(\tau_1(\mathrm{t}))\right)-\left(v(\tau_1(\mathrm{t}))\right)]dt\,ds\right| + \left|\int_{\xi}^{\infty}\int_{s}^{\infty}[\left(v_k(\tau_2(\mathrm{t}))\right)-\left(v(\tau_2(\mathrm{t}))\right)]dt\,ds\right| + \cdots + \left|\int_{\xi}^{\infty}\int_{s}^{\infty}[\left(v_k(\tau_\lambda(\mathrm{t}))\right)-\left(v(\tau_\lambda(\mathrm{t}))\right)]dt\,ds\right|\Bigg) \tag{2.11}$$

According to (2.3), and the bounded property of $\left(v\left(\tau_\varsigma(\xi)\right)\right)$, we get

$$\int_{\xi}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right)dt\,ds \prec \infty \quad , \tag{2.12}$$

Since $\left|v_k\left(\tau_\varsigma(s)\right)-v\left(\tau_\varsigma(s)\right)\right| \to 0$, as k tend to ∞, $\varsigma = 1,2,3,\dots,\lambda$. By dominant convergence theorem to Lebesgue, it yields:

$$\lim_{k\to\infty}\|(\varphi_2\chi_k)(\xi)-(\varphi_2\chi)(\xi)\| = 0 \tag{2.13}$$

It reduces that φ_2 is continuous mapping.

To prove that $\varphi_2\Psi$ is relatively compact, we must accentual that $\{\varphi_2\chi : \chi \in \Psi\}$ is uniformly bounded and equicontinuous on $[\xi_0,\infty]$, by theorem of Arzelã-Ascoli [17]. From (2.5), yield $\{\varphi_2\chi : \chi\in\Psi\}$ is uniformly bounded.

To secure that $\{\varphi_2\chi : \chi\in\Psi\}$ is equicontinuous on $[\xi_0,\infty)$, let $\chi\in\Psi$ and any $\varepsilon \succ 0$, by (2.12), so $\exists\ \xi_* \succcurlyeq \xi_1$ large enough:

$$\int_{\xi_*}^{\infty}\int_{s}^{\infty} v\left(\tau_\varsigma(\mathrm{t})\right)dt\,ds \prec \frac{\sqrt{\varepsilon}}{2\rho_2\sigma_1}, \qquad ,\xi \succcurlyeq \xi_* \succcurlyeq \xi_1 \quad , \tag{2.14}$$

Then, for any given $\varepsilon \succ 0$ and $\chi \in \mho$, $T_2 \succ T_1 \succcurlyeq \xi_*$, we have

$$\|(\varphi_2\chi_k)(T_2)-(\varphi_2\chi)(T_1)\| = \sup_{T_2\succ T_1\succcurlyeq \mathrm{t}_*}|(\varphi_2\chi_k)(T_2)-(\varphi_2\chi)(T_1)|$$

$$\preccurlyeq |(\varphi_2\chi_k)(T_2)| + |(\varphi_2\chi)(T_1)|$$

$$\preccurlyeq \int_{T_2}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} a_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi_k\left(\tau_\varsigma(\mathrm{t})\right)\right)dt\,ds + \int_{T_1}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} a_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right)dt\,ds$$

$$\preccurlyeq \sigma_1\int_{T_2}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi_k\left(\tau_\varsigma(\mathrm{t})\right)\right)dt\,ds + \sigma_1\int_{T_1}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right)dt\,ds$$

$$\preccurlyeq \sigma_1\rho_2\int_{T_2}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} \chi_k\left(\tau_\varsigma(\mathrm{t})\right)dt\,ds + \sigma_1\rho_2\int_{T_1}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(\mathrm{t})\right)dt\,ds$$

$$\preccurlyeq \sigma_1\rho_2\int_{T_2}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} v_k\left(\tau_\varsigma(\mathrm{t})\right)dt\,ds + \sigma_1\rho_2\int_{T_1}^{\infty}\int_{s}^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right)dt\,ds$$

$$\prec \rho_2\sigma_1 \frac{\sqrt{\varepsilon}}{2\rho_2\sigma_1} + \rho_2\sigma_1 \frac{\sqrt{\varepsilon}}{2\rho_2\sigma_1} = \sqrt{\varepsilon}, \tag{2.15}$$

Where $\sqrt{\varepsilon} = \varepsilon_1$

For $\chi \in \Psi$ and $\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*$, we get

$$\|(\varphi_2\chi)(T_2) - (\varphi_2\chi)(T_1)\| = \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} |(S_2\chi)(T_2) - (S_2\chi)(T_1)|$$

$$= \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} \left| \int_{T_1}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} a_\varsigma(t)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds - \int_{T_2}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} a_\varsigma(t)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds \right|$$

$$\preccurlyeq \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} \left| \sigma_1 \int_{T_1}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds - \sigma_2 \int_{T_2}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds \right|$$

$$\preccurlyeq \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} \left| \sigma_2\rho_2 \int_{T_2}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds - \sigma_1\rho_1 \int_{T_1}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds \right|$$

$$\preccurlyeq \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} \left| \sigma_1\rho_1 \int_{T_2}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds - \sigma_1\rho_1 \int_{T_1}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds \right|$$

$$\preccurlyeq \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} \left| \sigma_1\rho_1 \int_{T_1}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds - \sigma_1\rho_1 \int_{T_2}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds \right|$$

$$\preccurlyeq \sup_{\xi_1 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_*} \sigma_1\rho_1 \left| \int_{T_1}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds - \int_{T_2}^{t_*}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \gamma_\varsigma\left(\chi\left(\tau_\varsigma(t)\right)\right) dt\, ds \right|$$

$$= \sigma_1\rho_1 \int_{T_1}^{T_2}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(t)\right) dt\, ds$$

$$\preccurlyeq \sigma_1\rho_1 \int_{T_1}^{T_2}\int_s^{\infty}\sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(t)\right) dt\, ds$$

$$\preccurlyeq \sigma_1\rho_1 \frac{\sqrt{\varepsilon}}{2\rho_2\sigma_1}(T_2 - T_1).$$

Thus there exists $\delta_1 = \frac{2\rho_2\sqrt{\varepsilon}}{\rho_1}$, such that

$$|(\varphi_2\chi)(T_2) - (\varphi_2\chi)(T_1)| \prec \varepsilon, \text{if } \ 0 \prec T_2 - T_1 \prec \delta_1 \tag{2.16}$$

Finally, let $V(\xi) = \frac{v(\xi)}{a(\xi)}$, then for any $\chi \in \Psi$, $\xi_0 \preccurlyeq T_1 \prec T_2 \preccurlyeq \xi_1$, by mean value theorem there exists $k_1 \in (T_1, T_2)$ and $\delta_2 = \frac{\varepsilon}{V'(k_1)} \succ 0$ such that

$$|(\varphi_2 x)(T_2) - (\varphi_2 x)(T_1)| = \left|\left(\frac{v}{a}\right)(T_2) - \left(\frac{v}{a}\right)(T_1)\right|$$
$$= |V(T_2) - V(T_1)|$$
$$= |V'(k_1)(T_2 - T_1)|$$
$$= |V'(k_1)|(T_2 - T_1) \prec \varepsilon,$$

if $0 \prec T_2 - T_1 \prec \delta_2 \prec \delta_1$. (2.17)

Hence, $\varphi_2\Psi$ is a relatively compact set. By using lemma (1.1), it follows that Eq. (1.1) has solution that is relatively bounded from below.

Next theorem is generalizing of theorem (2.1). We will show that the solution to Eq. (1.1) exists and bounded by convergent series $\sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi)$ and $\sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi)$.

Theorem 2.2

Suppose that A1- A3, Eq.(2,3) hold, and there are convergent series $\sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi), \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi) \in (\mathbb{N},[0,\infty))$, $\xi_1 \geqslant \xi_0+\rho$ such that

$$\sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi) \leqslant \sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi_1) \text{ and } \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi) \geqslant \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi_1),\ \xi_0 \leqslant \xi \leqslant \xi_1 \tag{2.18}$$

$$-\int_\xi^\infty \int_s^\infty \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds \leqslant \frac{1}{\sigma_2\rho_1}\left(-\sigma_1\mu_2 \int_\xi^\infty \sum_{\varsigma=1}^{\lambda} v\left(\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} + \sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi)\right), \xi \geqslant \xi_1 \tag{2.19}$$

$$-\int_\xi^\infty \int_s^\infty \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds \geqslant \frac{1}{\sigma_1\rho_2}\left(-\sigma_2\mu_1 \int_\xi^\infty \sum_{\varsigma=1}^{\lambda} u\left(\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} + \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi)\right), \xi \geqslant \xi_1$$

Then the Eq.(1.1) has a bounded solution by convergent series $\sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi), \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi) \in C^1$.

Proof

Let $(C([\xi_0,\infty),\Re), \|.\|)$ such that $\|\chi\| = \sup_{\mathrm{t}\geqslant \mathrm{t}_0}|\chi(\xi)|$, then $\mathrm{C}([\xi_0,\infty),\Re)$ is a Banach space, let $\Psi \subset C([\xi_0,\infty),\Re)$ we define Ψ as:

$$\Psi = \{\chi(\xi) : \chi(\xi) \in \mathrm{C}([\xi_0,\infty),\Re) : u(\xi) \leqslant \chi(\xi) \leqslant v(\xi),\quad \xi \geqslant \xi_0\} \tag{2.20}$$

Such that Ψ is a closed and convex. The mappings φ_1 and $\varphi_2 : \Psi \to \mathrm{C}([\xi_0,\infty),\Re)$ is defined as:

$$(\varphi_1\chi)(\xi) = \begin{cases} \int_\xi^\infty \sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) \mathrm{dt}, & \xi \geqslant \xi_1, \\ (\varphi_1\chi)(\xi_1), & \xi_0 \leqslant \xi \leqslant \xi_1, \end{cases}$$

$$(\varphi_2\chi)(\xi) = \begin{cases} -\int_\xi^\infty \int_s^\infty \sum_{\varsigma=1}^{\lambda} a_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right) dt\, ds, & \xi \geqslant \xi_1, \\ (\varphi_2\chi)(\xi_1) & \xi_0 \leqslant \xi \leqslant \xi_1, \end{cases} \tag{2.21}$$

We are going to prove for any $\chi, \mathcal{Y} \in \Psi$, but $\varphi_1\chi + \varphi_2\mathcal{Y} \in \Psi$ and $\forall \chi, \mathcal{Y} \in \Psi, \xi \geqslant \xi_1$:

$$\begin{aligned}(\varphi_1\chi)(\mathrm{t}) + (\varphi_2\mathcal{Y})(\xi) &= \int_\xi^\infty \sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) \mathrm{dt} - \int_\xi^\infty \int_s^\infty \sum_{\varsigma=1}^{\lambda} a_\varsigma(\mathrm{t})\gamma_\varsigma\left(\mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right)\right) dt\, ds \\ &\leqslant \sigma_1\mu_2 \int_\xi^\infty \sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(\mathrm{t})\right) \mathrm{dt} - \sigma_2\rho_1 \int_\xi^\infty \int_s^\infty \sum_{\varsigma=1}^{\lambda} \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds\end{aligned}$$

$$\preccurlyeq \sigma_1\,\mu_2 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right)\mathrm{dt} - \sigma_2\rho_1 \int_{\xi}^{\infty}\int_{s}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds$$

$$\preccurlyeq \sigma_1\,\mu_2 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right)\mathrm{dt} + \sigma_2\rho_1 \frac{1}{\sigma_2\rho_1}\left(-\sigma_1\mu_2 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} + \sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi)\right)$$

$$= \sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi) \preccurlyeq \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi) \tag{2.22}$$

Let $\xi \in [\xi_0, \xi_1]$, using (2.22) and (2.18) we get:

$$(\varphi_1 x)(\xi) + (\varphi_2 \mathcal{Y})(\xi) = (\varphi_1\chi)(\xi_1) + (\varphi_2\mathcal{Y})(\xi_1)$$
$$\preccurlyeq \sum_{\varsigma=1}^{\lambda} v_\varsigma\,(\xi_1) \preccurlyeq \sum_{\varsigma=1}^{\lambda} v_\varsigma\,(\xi)$$

Moreover, for all$\xi \succcurlyeq \xi_1$, it yields:

$$(\varphi_1\chi)(\xi) + (\varphi_2\mathcal{Y})(\xi)$$
$$= \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right)\mathrm{dt} - \int_{\xi}^{\infty}\int_{s}^{\infty} \sum_{\varsigma=1}^{\lambda} a_\varsigma\,(\mathrm{t})\gamma_\varsigma\left(\mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right)\right) dt\, ds$$
$$\succcurlyeq \sigma_2\,\mu_1 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} \chi\left(\tau_\varsigma(\mathrm{t})\right)\mathrm{dt} - \sigma_1\rho_2 \int_{\xi}^{\infty}\int_{s}^{\infty} \sum_{\varsigma=1}^{\lambda} \mathcal{Y}\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds$$
$$\succcurlyeq \sigma_2\,\mu_1 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right)\mathrm{dt} - \sigma_1\rho_2 \int_{\xi}^{\infty}\int_{s}^{\infty} \sum_{\varsigma=1}^{\lambda} v\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds$$

$$\succcurlyeq \sigma_2\,\mu_1 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right)\mathrm{dt} + \sigma_1\rho_2 \frac{1}{\sigma_1\rho_2}\left(-\sigma_2\mu_1 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} + \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi)\right)$$

$$\succcurlyeq \sigma_2\,\mu_1 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\tau_\varsigma(\mathrm{t})\right)\mathrm{dt} - \sigma_2\mu_1 \int_{\xi}^{\infty} \sum_{\varsigma=1}^{\lambda} u\left(\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} + \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi)$$
$$= \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi) \succcurlyeq \sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi) \tag{2.23}$$

Then for $\xi \in [\xi_0, \xi_1]$, using (2.18) and (2.23), we obtain:

$$(\varphi_1\chi)(\xi) + (\varphi_2\mathcal{Y})(\xi) = (\varphi_1\chi)(\xi_1) + (\varphi_2\mathcal{Y})(\xi_1)$$
$$\succcurlyeq \sum_{\varsigma=1}^{\lambda} u_\varsigma\,(\xi_1) \succcurlyeq \sum_{\varsigma=1}^{\lambda} u_\varsigma\,(\xi)$$

Thus, $\varphi_1\chi + \varphi_2\mathcal{Y} \in \Psi, \forall \chi, \mathcal{Y} \in \Psi$. By using similarly steps in theorem (2.1), we conclude result. By lemma (1.1) there exists $\chi_0 \in \Psi, \ni \varphi_1\chi_0 + \varphi_2\chi_0 = \chi_0$. We realize that $\chi_0(t)$ is a bounded solution by convergent series $\sum_{\varsigma=1}^{\lambda} u_\varsigma(\xi), \sum_{\varsigma=1}^{\lambda} v_\varsigma(\xi) \in C^1$ of the Eq. (1.1).

3. Oscillatory of differential equation with Delayed Arguments:
In the present section, we investigate the for oscillatory criteria to Eq. (1.1) and we use some basic lemmas:
Lemma 3.1 [18]:
Let $\chi \in C^{\lambda}[\Re, \Re]$ and $\chi^{(\lambda)}(\xi)\chi^{(\lambda-1)}(\xi) > 0,\ \xi \geq \xi_0, \xi \in (-\infty, \infty)$
Then the following statements hold
1. If $\chi^{(\lambda)}(\xi)$is positive for $\xi \geq \xi_0$ then $\chi^{(\varsigma)}(\xi)$ is increasing for $\xi \geq \xi_0$ and $\lim\limits_{\xi\to\infty} \chi^{(\varsigma)}(\xi) = \infty$ for $\varsigma = \lambda - 1, \lambda - 2, \dots, 0$
2. If $\chi^{(\lambda)}(\xi)$is negative for $\xi \geq \xi_0$ then $\chi^{(\varsigma)}(\xi)$ is decreasing for $\xi \geq \xi_0$ and $\lim\limits_{\xi\to\infty} \chi^{(\varsigma)}(\xi) = -\infty$ for $\varsigma = \lambda - 1, \lambda - 2, \dots, 0$

Lemma 3.2 [19]:
Assume that
I. *If* $\varphi, \vartheta, \chi, \tau, \varrho \in C\big[[\xi_0, \infty), \Re\big],\ \varphi(\xi) \prec 0, \lim\limits_{\xi\to\infty} \varphi(\xi)$ exist, $0 \prec \vartheta_1(\xi) \preccurlyeq 1$, $\tau(\xi) \prec \xi$, $\varrho(\xi) \succcurlyeq \xi,\ \xi \succcurlyeq \xi_0,\ \lim\limits_{\xi\to\infty} \tau(\xi) = \infty$ and

$$\chi(\xi) \preccurlyeq \varphi(\xi) + \vartheta_1(\xi) \max\{\chi(s): \tau(\xi) \preccurlyeq s \preccurlyeq \varrho(\xi)\}, \qquad \xi \succcurlyeq \xi_0 \ . \tag{3.1}$$

Then $\chi(\xi)$ cannot be positive for $\xi \succcurlyeq \xi_1 \succcurlyeq \xi_0$.
II. If $\varphi, \vartheta, \chi, \tau, \varrho \in C\big[[\xi_0, \infty); \Re\big],\ \varphi(\xi) \succ 0, \lim\limits_{\xi\to\infty} \varphi(\xi)$ exist, $0 \prec \vartheta_2(\xi) \preccurlyeq 1, \tau(\xi) \prec \xi, \varrho(\xi) \succcurlyeq \xi,\ \xi \succcurlyeq \xi_0,\ \lim\limits_{\xi\to\infty} \tau(\xi) = \infty$ and

$$\chi(\xi) \succcurlyeq \varphi(\xi) + \vartheta_2(\xi) \min\{\chi(s): \tau(\xi) \preccurlyeq s \preccurlyeq \varrho(\xi)\}, \qquad \xi \succcurlyeq \xi_0. \tag{3.2}$$

Then $\chi(\xi)$ cannot be negative for $\xi \succcurlyeq \xi_1 \succcurlyeq \xi_0$

Lemma 3.3 [20]:
Assume that $\curlyvee, \mu \in C[\Re^+, \Re^+]$ are continuous functions such that $\curlyvee(\xi) \prec \xi, \curlyvee'(\xi) \succcurlyeq 0$ for $\xi \succcurlyeq \xi_0$ with $\lim\limits_{\xi\to\infty} \curlyvee(\xi) = \infty$.
If $\lim\limits_{\xi\to\infty} \inf \int_{\curlyvee(\xi)}^{\xi} \mu(s)\, ds \succ \frac{1}{e}$, (3.3)
then the inequality $x'(\xi) + \mu(\xi)x\big(\curlyvee(\xi)\big) \preccurlyeq 0$ has no eventually positive solution.
Lemma 3.4: Assume that:

$$\varpi(\xi) = \chi(\xi) - \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} \int_s^{\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(s)\right)} \alpha_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t}\, ds - \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} \tag{3.4}$$

And the following assumptions hold:

H1: $\vartheta_2(\xi) \leq \dfrac{\Gamma_\varsigma\left(\xi, \chi\left(\tau_\varsigma(\xi)\right)\right)}{\chi\left(\tau_\varsigma(\xi)\right)} \leq \dfrac{\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right)}{\chi\left(\tau_\varsigma(\xi)\right)} \leq \vartheta_1(\xi), \qquad \rho(\xi) = \max\{\tau_\varsigma(\xi)\}$

H2: $\lim\limits_{\xi\to\infty} \inf \sum\limits_{\varsigma=1}^{\lambda} \left[\int_T^{\xi} \int_s^{\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(s)\right)} \alpha_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t}) d\mathrm{t}\, ds + \int_T^{\xi} b_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t}) d\mathrm{t}\right] \geq 1$

If $\chi(\xi)$ is eventually positive bounded solution of Eq. (1.1) with$\left(\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(\xi)\right)\right)' \geq 0$ then: $\varpi(\xi)$ is negative non-decreasing function.

Proof. Assume that a solution $\chi(\xi)$ is a non-oscillatory bounded solution of the Eq.(1.1). So, suppose that $\chi(\xi)$ is eventually positive bounded solution, there is $\xi_1 \geq \xi_0 + \rho$ such that $\chi(\xi) > 0$ for $\xi \geq \xi_1$.

$$\frac{d}{d\xi}\varpi(\xi) = \frac{d}{d\xi}\chi(\xi) - \sum_{\varsigma=1}^{\lambda} \int_{\xi}^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi))} \alpha_\varsigma(t)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right)d\mathrm{t} - \sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\xi)\right)\right)$$

$$\frac{d^2}{d\xi^2}\varpi(\xi) = \frac{d^2}{d\xi^2}\chi(\xi) - \sum_{\varsigma=1}^{\lambda}\left[\alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\right)\right)(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' - \alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right)\right] - \frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\xi)\right)\right)$$

From equation (1.1), we obtain that:

$$\frac{d^2}{d\xi^2}\varpi(\xi) = -\sum_{\varsigma=1}^{\lambda} \alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right) + \frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\xi, \chi\left(\tau_\varsigma(\xi)\right)\right) - \sum_{\varsigma=1}^{\lambda}\left[\alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\gamma_\varsigma\left(\chi(\curlyvee_\varsigma(\xi))\right)(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' - \alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right)\right] - \frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\xi)\right)\right)$$

$$\frac{d^2}{d\xi^2}\varpi(\xi) = -\sum_{\varsigma=1}^{\lambda} \alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\gamma_\varsigma\left(\chi(\curlyvee_\varsigma(\xi))\right)(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' \leq 0 \qquad (3.5)$$

So, we conclude that $\frac{d^2}{d\xi^2}\varpi(\xi) \leq 0$ and $\frac{d}{d\xi}\varpi(\xi)$ is monotone (nonincreasing) function.
And $\varpi(\xi)$ is monotone function. Two cases can be considered:

Case1:
If $\frac{d^2}{d\xi^2}\varpi(\xi) \leq 0$ and $\frac{d}{d\xi}\varpi(\xi) \leq 0$ for $\xi \geq \xi_1$, then by lemma 3.1 it follows that $\lim_{\xi\to\infty}\varpi(\xi) = -\infty$ and with (3.4) we imply that $\lim_{\xi\to\infty}\chi(\xi) = -\infty$, which is a contradiction.

Case 2:
$\frac{d^2}{d\xi^2}\varpi(\xi) \leq 0$ and $\frac{d}{d\xi}\varpi(\xi) \geq 0$, we claim that $\varpi(\xi) \leq 0,\ \xi \geq \xi_1$.
Otherwise, $\varpi(\xi) \geq 0$, so there exists $\psi > 0$ such that $\varpi(\xi) \geq \psi,\ \xi \geq \xi_2 \geq \xi_1$
Then from (3.4):

$$\chi(\xi) \geq \psi + \sum_{\varsigma=1}^{\lambda}\int_T^{\xi}\int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right)d\mathrm{t}\,ds + \sum_{\varsigma=1}^{\lambda}\int_T^{\xi} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right)d\mathrm{t}$$

Since $\chi(\xi)$ is bounded then $\lim_{\xi\to\infty}\inf\chi(\xi) = \varphi,\ 0 \leq \varphi < \infty$
So there is a sequence $\{\boldsymbol{\varrho_\upsilon}\}$, such that $\lim_{\boldsymbol{\upsilon}\to\infty}\boldsymbol{\varrho_\upsilon} = \infty$ and $\lim_{\boldsymbol{\upsilon}\to\infty}\chi(\boldsymbol{\varrho_\upsilon}) = \varphi$

$\mathfrak{h}_1(\xi) = \min\{\tau_\varsigma(\xi)\}$ and $\mathfrak{h}_2(\xi) = \max\{\tau_\varsigma(\xi)\}, \xi \geq \xi_2$

$$\chi(\eta_{\boldsymbol{\upsilon}}) = \min\{\chi(\xi), \mathfrak{h}_1(\boldsymbol{\varrho_\upsilon}) \leq \xi \leq \mathfrak{h}_2(\boldsymbol{\varrho_\upsilon})\}$$

So $\chi(\eta_{\boldsymbol{\upsilon}}) \leq \chi(\tau_\varsigma(\xi))$

$\lim_{\boldsymbol{\upsilon}\to\infty} \eta_{\boldsymbol{\upsilon}} = \infty$ and $\lim_{\boldsymbol{\upsilon}\to\infty}\inf \chi(\eta_{\boldsymbol{\upsilon}}) \geq \varphi$

$$\chi(\boldsymbol{\varrho_\upsilon}) \geq \psi + \sum_{\varsigma=1}^{\lambda} \int_T^{\boldsymbol{\varrho_\upsilon}} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t}\, ds + \sum_{\varsigma=1}^{\lambda} \int_T^{\boldsymbol{\varrho_\upsilon}} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t}$$

$$\chi(\boldsymbol{\varrho_\upsilon}) \geq \psi + \sum_{\varsigma=1}^{\lambda} \int_T^{\boldsymbol{\varrho_\upsilon}} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(t)\vartheta_2(\mathrm{t})\chi\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t}\, ds + \sum_{\varsigma=1}^{\lambda} \int_T^{\boldsymbol{\varrho_\upsilon}} b_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t})\chi\left(\tau_\varsigma(\mathrm{t})\right) d\mathrm{t}$$

$$\chi(\boldsymbol{\varrho_\upsilon}) \geq \psi + \sum_{\varsigma=1}^{\lambda} \chi(\eta_{\boldsymbol{\upsilon}}) \left\{ \int_T^{\boldsymbol{\varrho_\upsilon}} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t}) dt\, ds + \int_T^{\boldsymbol{\varrho_\upsilon}} b_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t}) dt \right\}$$

By taking limit inferior to the both sides of the last inequality as $\boldsymbol{\upsilon} \to \infty$, it follows that:
$\varphi \geq \psi + \varphi$ which is a contradiction.

Lemma 3.5: Assume that $\varpi(\xi)$ is defined as in (3.4) and H1 hold with:
H2: $\lim_{\xi\to\infty} \inf \sum_{\varsigma=1}^{\lambda} \left[\int_T^{\xi} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t}) dt\, ds + \int_T^{\xi} b_\varsigma(\mathrm{t})\vartheta_2(\mathrm{t}) dt\right] \leq 1$
If $\chi(\xi)$ is eventually positive bounded solution of Eq. (1.1) with $(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' \leq 0$ then:
$\varpi(\xi)$ is positive non-increasing function.

Proof. Assume that a solution $\chi(\xi)$ is a non-oscillatory bounded solution of the Eq.(1.1). So, suppose that $\chi(\xi)$ is eventually positive bounded solution, there is $\xi_1 \geq \xi_0 + \rho$ such that $\chi(\xi) > 0$ for $\xi \geq \xi_1$.

$$\frac{d}{d\xi}\varpi(\xi) = \frac{d}{d\xi}\chi(\xi) - \sum_{\varsigma=1}^{\lambda} \int_{\xi}^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi))} \alpha_\varsigma(t)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t} - \sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\xi)\right)\right)$$

$$\frac{d^2}{d\xi^2}\varpi(\xi) = \frac{d^2}{d\xi^2}\chi(\xi) - \sum_{\varsigma=1}^{\lambda} \left[\alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\right)\right)(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' - \alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right)\right] - \frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\xi)\right)\right)$$

From equation (1.1), we obtain that:

$$\frac{d^2}{d\xi^2}\varpi(\xi) = -\sum_{\varsigma=1}^{\lambda} \alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right) + \frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\xi, \chi\left(\tau_\varsigma(\xi)\right)\right)$$
$$-\sum_{\varsigma=1}^{\lambda}\left[\alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\gamma_\varsigma\left(\chi(\curlyvee_\varsigma(\xi))\right)(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' - \alpha_\varsigma(\xi)\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\xi)\right)\right)\right]$$
$$-\frac{d}{d\xi}\sum_{\varsigma=1}^{\lambda} b_\varsigma(\xi)\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\xi)\right)\right)$$
$$\frac{d^2}{d\xi^2}\varpi(\xi) = -\sum_{\varsigma=1}^{\lambda} \alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))\gamma_\varsigma\left(\chi(\curlyvee_\varsigma(\xi))\right)(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' \geq 0 \qquad (3.6)$$

So, we conclude that $\frac{d^2}{d\xi^2}\varpi(\xi) \geq 0$ and $\frac{d}{d\xi}\varpi(\xi)$ is monotone (nonincreasing) function.
And $\varpi(\xi)$ is monotone function. Two cases can be considered:

Case1:

If $\frac{d^2}{d\xi^2}\varpi(\xi) \geq 0$ and $\frac{d}{d\xi}\varpi(\xi) \geq 0$ for $\xi \geq \xi_1$ by lemma 3.1 it follows that $\lim_{\xi\to\infty}\varpi(\xi) = \infty$ and with (3.4) we imply that $\lim_{\xi\to\infty}\chi(\xi) = \infty$, which is a contradiction.

Case 2:

$\frac{d^2}{d\xi^2}\varpi(\xi) \geq 0$ and $\frac{d}{d\xi}\varpi(\xi) \leq 0$, we claim that $\varpi(\xi) \geq 0$, $\xi \geq \xi_1$.
Otherwise, $\varpi(\xi) \geq 0$, so there exist $\psi < 0$ such that $\varpi(\xi) \leq \psi$, $\xi \geq \xi_2 \geq \xi_1$
Then from (3.4):

$$\chi(\xi) \leq \psi + \sum_{\varsigma=1}^{\lambda}\int_T^{\xi}\int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right)dt\,ds + \sum_{\varsigma=1}^{\lambda}\int_T^{\xi} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right)dt$$

Since $\chi(\xi)$ is bounded then $\lim_{\xi\to\infty}\inf\chi(\xi) = \varphi$, $0 \leq \varphi < \infty$
So there is a sequence $\{\boldsymbol{\varrho_\upsilon}\}$, such that $\lim_{\boldsymbol{\upsilon}\to\infty}\boldsymbol{\varrho_\upsilon} = \infty$ and $\lim_{\boldsymbol{\upsilon}\to\infty}\chi(\boldsymbol{\varrho_\upsilon}) = \varphi$
$\mathfrak{h}_1(\xi) = \min\{\tau_\varsigma(\xi)\}$ and $\mathfrak{h}_2(\xi) = \max\{\tau_\varsigma(\xi)\}, \xi \geq \xi_2$

$$\chi(\eta_{\boldsymbol{\upsilon}}) = \max\{\chi(\xi), \mathfrak{h}_1(\boldsymbol{\varrho_\upsilon}) \leq \xi \leq \mathfrak{h}_2(\boldsymbol{\varrho_\upsilon})\}$$

So $\chi(\eta_{\boldsymbol{\upsilon}}) \geq \chi(\tau_\varsigma(\xi))$
$\lim_{\boldsymbol{\upsilon}\to\infty}\eta_{\boldsymbol{\upsilon}} = \infty$ and $\lim_{\boldsymbol{\upsilon}\to\infty}\inf\chi(\eta_{\boldsymbol{\upsilon}}) \geq \varphi$

$$\chi(\boldsymbol{\varrho_\upsilon}) \leq \psi + \sum_{\varsigma=1}^{\lambda}\int_T^{\boldsymbol{\varrho_\upsilon}}\int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right)d\mathrm{t}\,ds$$
$$+ \sum_{\varsigma=1}^{\lambda}\int_T^{\boldsymbol{\varrho_\upsilon}} b_\varsigma(\mathrm{t})\Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right)d\mathrm{t}$$

$$\chi(\boldsymbol{\varrho_\upsilon}) \leq \psi + \sum_{\varsigma=1}^{\lambda}\int_T^{\boldsymbol{\varrho_\upsilon}}\int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\vartheta_1(\mathrm{t})\chi\left(\tau_\varsigma(\mathrm{t})\right)d\mathrm{t}\,ds + \sum_{\varsigma=1}^{\lambda}\int_T^{\boldsymbol{\varrho_\upsilon}} b_\varsigma(\mathrm{t})\vartheta_1(\mathrm{t})\chi\left(\tau_\varsigma(\mathrm{t})\right)d\mathrm{t}$$

$$\chi(\boldsymbol{\varrho_\upsilon}) \leq \psi + \sum_{\varsigma=1}^{\lambda}\chi(\eta_{\boldsymbol{\upsilon}})\left\{\int_T^{\boldsymbol{\varrho_\upsilon}}\int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t})\vartheta_1(\mathrm{t})dt\,ds + \int_T^{\boldsymbol{\varrho_\upsilon}} b_\varsigma(\mathrm{t})\vartheta_1(\mathrm{t})dt\right\}$$

By taking limit inferior to the both sides of the last inequality as $\boldsymbol{\upsilon} \to \infty$, it follows that:
$\varphi \leq \psi + \varphi$ which is a contradiction.

Theorem 3.1
Assume that all conditions of lemma 3.4 hold and $\Upsilon(\mathrm{t})$ is defined as in (3.4) in addition to the condition:

$$\lim_{\xi\to\infty} \sup \sum_{\varsigma=1}^{\lambda} \left[\left\{ \left(\int_T^{\xi} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t}) d\mathrm{t}\, ds \right. + \sum_{\varsigma=1}^{\lambda} \vartheta_1(\xi) \int_T^{\xi} b_\varsigma(\mathrm{t}) d\mathrm{t} \right\} \right] \preccurlyeq 1 \qquad (3.7)$$

Then every solution to Eq. (1.1) oscillates.

Proof
Assume that a solution $\chi(\xi)$ is a non-oscillatory of the Eq. (1.1). So, let $\chi(\xi)$ is eventually positive solution, there is $\xi_1 \succcurlyeq \xi_0 + \mathfrak{h}_2(\xi), \ni \chi(\xi) \succ 0, \ \xi \succcurlyeq \xi_1$.

$$\chi(\xi) = \varpi(\xi) + \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(t) \gamma_\varsigma\left(\chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t}\, ds$$
$$+ \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} b_\varsigma(\mathrm{t}) \Gamma_\varsigma\left(\mathrm{t}, \chi\left(\tau_\varsigma(\mathrm{t})\right)\right) d\mathrm{t}$$

$$\chi(\xi) \leq \varpi(\xi) + \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t}) \vartheta_1(\mathrm{t}) \chi\left(\tau_\varsigma(\mathrm{t})\right) dt\, ds + \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} b_\varsigma(\mathrm{t}) \vartheta_1(\mathrm{t}) \chi\left(\tau_\varsigma(\mathrm{t})\right) dt$$

$$\chi(\xi) \leq \varpi(\xi) + \sum_{\varsigma=1}^{\lambda} \vartheta_1(\xi) \max_{\mathfrak{h}_1(\xi) \preccurlyeq \xi \preccurlyeq \mathfrak{h}_2(\xi)} \chi(\xi) \left\{ \int_T^{\xi} \int_s^{\tau_\varsigma^{-1}(\curlyvee_\varsigma(s))} \alpha_\varsigma(\mathrm{t}) d\mathrm{t}\, ds + \sum_{\varsigma=1}^{\lambda} \int_T^{\xi} b_\varsigma(\mathrm{t}) d\mathrm{t} \right\}$$

By using lemma (3.2-I) then $\chi(\xi)$ cannot be positive function on $[\xi_3, \infty)$ which contradicts to $\chi(\xi) \succ 0$.

Theorem 3.2
Assume that all conditions of lemma 3.5 hold and $\varpi(\xi)$ is defined as in (3.4) with $\curlyvee_\varsigma(\xi) < \xi, \tau_\varsigma(\xi) < \xi$ and $(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\xi)))' = -\alpha(\xi)$ in addition to the condition:

$$\lim_{\xi\to\infty} \inf \sum_{\varsigma=1}^{\lambda} \left[\left\{ \int_{\curlyvee_\varsigma(\delta(\xi))}^{\xi} \int_s^{\delta(s)} \alpha_\varsigma\left(\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(\mathrm{t})\right)\right) \vartheta_2\left(\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(\mathrm{t})\right)\right) \alpha(\mathrm{t}) dt\, ds \right\} \right] \geq \frac{1}{e} \qquad (3.8)$$

Then every solution of Eq. (1.1) oscillates.

Proof
Assume that a solution $\chi(\xi)$ is a non-oscillatory of the Eq. (1.1). So, let $\chi(\xi)$ be eventually positive solution, there is $\xi_1 \succcurlyeq \xi_0 + \mathfrak{h}_2(\xi), \ni \chi(\xi) \succ 0, \ \xi \succcurlyeq \xi_1$.
Integrating (3.5) from ξ to $\delta(\xi), \delta(\xi) > \xi, \curlyvee_\varsigma(\delta(\xi)) < \xi, \lim_{\xi\to\infty} \curlyvee_\varsigma(\delta(\xi)) = \infty, \varsigma = 1,2, \ldots, \lambda$

$$\frac{d}{d\xi}\varpi(\delta(\xi)) - \frac{d}{d\xi}\varpi(\xi) = \sum_{\varsigma=1}^{\lambda} \int_{\xi}^{\delta(\xi)} \alpha_\varsigma\left(\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(\mathrm{t})\right)\right) \gamma_\varsigma\left(\chi\left(\curlyvee_\varsigma(\mathrm{t})\right)\right) \alpha(\mathrm{t}) dt$$

$$\frac{d}{d\xi}\varpi(\delta(\xi)) - \frac{d}{d\xi}\varpi(\xi) \geq \sum_{\varsigma=1}^{\lambda} \int_{\xi}^{\delta(\xi)} \alpha_\varsigma\left(\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(\mathrm{t})\right)\right) \vartheta_2\left(\tau_\varsigma^{-1}\left(\curlyvee_\varsigma(\mathrm{t})\right)\right) \chi\left(\curlyvee_\varsigma(\mathrm{t})\right) \alpha(\mathrm{t}) dt$$

but from (3.4) $\chi(\xi) \geq \varpi(\xi)$:

$$\frac{d}{d\xi}\varpi(\delta(\xi)) - \frac{d}{d\xi}\varpi(\xi) \geq \sum_{\varsigma=1}^{\lambda} \int_{\xi}^{\delta(\xi)} \alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\vartheta_2(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\varpi(\curlyvee_\varsigma(\mathrm{t}))\alpha(\mathrm{t})d\mathrm{t}$$

$$\frac{d}{d\xi}\varpi(\delta(\xi)) - \frac{d}{d\xi}\varpi(\xi) \geq \sum_{\varsigma=1}^{\lambda} \varpi(\curlyvee_\varsigma(\delta(\xi))) \int_{\xi}^{\delta(\xi)} \alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\vartheta_2(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\alpha(\mathrm{t})d\mathrm{t}$$

$$-\frac{d}{d\xi}\varpi(\xi) \geq \sum_{\varsigma=1}^{\lambda} \varpi(\curlyvee_\varsigma(\delta(\xi))) \int_{\xi}^{\delta(\xi)} \alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\vartheta_2(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\alpha(\mathrm{t})d\mathrm{t}$$

$$\frac{d}{d\xi}\varpi(\xi) + \sum_{\varsigma=1}^{\lambda} \varpi(\curlyvee_\varsigma(\delta(\xi))) \int_{\xi}^{\delta(\xi)} \alpha_\varsigma(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\vartheta_2(\tau_\varsigma^{-1}(\curlyvee_\varsigma(\mathrm{t})))\alpha(\mathrm{t})d\mathrm{t} \leq 0$$

By lemma 3.3 then the last inequality has no eventually positive solution.

4. Conclusions

In the main results, we formulated some effective conditions to ensure the existence of bounded solutions by convergent sequences also by convergent series. Moreover, sufficient conditions to ensure the oscillation of bounded solution to Eq. (1.1). The obtained conditions are efficient and perfect to conclude the oscillatory property.

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