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Boubaker Scaling Operational Matrices for Solving Calculus of Variation Problems

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Abstract
In this paper, a general expression formula for the Boubaker scaling (BS) operational matrix of the derivative is constructed. Then it is used to study a new parameterization direct technique for treating calculus of the variation problems approximately. The calculus of variation problems describe several important phenomena in mathematical science. The first step in our suggested method is to express the unknown variables in terms of Boubaker scaling basis functions with unknown coefficients. Secondly, the operational matrix of the derivative together with some important properties of the BS are utilized to achieve a non-linear programming problem in terms of the unknown coefficients. Finally, the unknown parameters are obtained using the quadratic programming technique. Some numerical examples are included to confirm the accuracy and applicability of the suggested direct parameterization method.

Keywords: Operational matrix, Boubaker Scaling Functions, Calculus of variation, Lagrange multiplier technique, Non-linear programming problem.

مصفوفات عمليات بوبكر القياسية لحل مسائل حسبان التغاير

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الخلاصة
في هذا البحث تم بناء صيغة تعبير عامة لمصفوفة عمليات مشتقة بوبكر القياسية "BS" . ثم تم استخدامها لاقتراح طريقة مباشرة جديدة هي إيجاد المعاملات من خلال تقريب حساب مسألة التغاير. تصف مسائل حسبان التغاير العديد من الظواهر المهمة في علم الرياضيات. الخطوة الأولى في طريقتنا المقترحة هي التعبير عن المتغيرات المجهولة بعدد من حدود بوبكر القياسية كأساس للدالة ذات المعاملات غير المعروفة. والخطوة الثانية هي إستخدام مصفوفة عمليات المشتقة مع بعض خصائص بوبكر القياسية لتحقيق مسألة البرمجة غير الخطية من حيث المعاملات غير المعروفة. وأخيرا، تم الحصول على المعلمات المجهولة باستخدام تقنية البرمجة التربيعية. وأيضا تم إضافة بعض الأمثلة العددية لاثبات دقة وإمكانية تطبيق الطريقة المباشرة المقترحة في إيجاد المعلمات.

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1. Introduction

The calculus of variation problems is attracted by numerous researchers in astrophysics [1] and engineering [2], they needed to find the minimum or maximum of a special function. The important role of such subject in engineering and science leads to certain attention for this kind of problem. In recent years, various methods and algorithms have been employed to obtain approximate solutions to the calculus of variation problems. Direct methods are a popular techniques for solving such problems. They are based on direct minimizing the functional by transforming the original variational problem into a mathematical programming problem using either the discretization or parameterization techniques. In [3], the authors used the parameterization technique based on Laguerre and Hermite polynomials to reduce the original variational problem into a quadratic programming problem. In [4], a variational iteration method was employed for solving some problems in the calculus of variations. A numerical approach based on the multi quadratic radial basis functions was presented in [5] in order to solve problems in the calculus of variation. The authors in [6] illustrated an approximated method based on the direct minimizing of the functional in the constrained and unconstrained variational problems with fixed or free endpoint conditions calculus of variation problems. In the parameterization technique, the variables of the problem are approximated with a finite length series of Legendre polynomials, Chebyshev polynomials, etc. with unknown parameters. In the work that is presented in [7], the direct method of differential transform method was employed for solving certain problems in the calculus of variations. The authors of [8] solved the variational problems of fixed or moving boundary conditions by an approximation method based on the operational matrix of integration for Muntz wavelets. An iterative technique for the approximate solution is applied to the calculus of variation in [9]. In [10], an efficient numerical technique was investigated for treating a kind of fractional variational problems using the generalized hat operational matrix direct method. In addition, the class of approximated methods is based on orthogonal polynomials named spectral methods in [11]. They are implemented in different techniques such as the collocation method and the Galerkin method. The authors of [12, 13] extended spectral techniques based on wavelet bases and a special orthogonal polynomials in order to solve certain calculus of variation problems. They utilized the second kind of Chebyshev wavelets and the generalized Veita- Pell polynomials, respectively. Many references are based on interesting polynomial expansions to approximate the solution of different problems. For example, new modified Chebyshev polynomials [14], B-spline Polynomials [15], Boubaker Hybrid Functions [16], Boubaker polynomials [17,18], and Hermite cubic spline functions [19]. They are applied to solve optimal control problems, fractional Emden-Fowler problems and fractional calculus of variation. In addition, the Boubaker wavelets were applied in [20] for the numerical solution of important problems that describe phenomena in mathematical science and astrophysics, namely the thermal explosions and the stellar structure. The Haar wavelets, finite difference method and Crank-Nicolson finite difference method are used in [21-24] for solving some direct and indirect problems. Further, in [25], the Rayleigh-Ritz method was extended together with operational matrices of different orthogonal polynomials such as Gegenbauer polynomials, shifted Legendre polynomials and shifted Chebyshev polynomials of the first, the third and the fourth kinds to solve a certain class for variational problems. An analytical algorithm based on the Adomian decomposition algorithm utilized in [26] for solving problems in the calculus of variations. Some other numerical techniques for the approximate solution of variational problems are found in [27-30]. Recently, the operational matrix methods have been seen to be useful for approximate treating problems in variational calculus.

In this paper, some new properties of the Boubaker scaling functions are first constructed and then the direct parameterizations method together with operational matrices of the Boubaker scaling basis functions is proposed to determine an approximate solution to the calculus of variation problems. The application of the presented method to the calculus of variation will lead to a nonlinear quadratic programming problem. Then we use the Lagrange multiplier technique to obtain an algebraic system. In fact, the proposed method is an improvement of the parameterization technique. The advantage of the proposed method is that the number of unknown parameters to be determined is less than the other parameterization techniques.

The organization of the present article is as follows: The following section is to list the Boubaker scaling functions and their important properties. In Section 3, we describe the basic formulation of the Boubaker scaling direct method which is needed for the development and the present a clear overview of this method. We illustrate how the proposed method can be used to transform the original problem into a nonlinear quadratic programming problem based on the Lagrange multiplier technique. In section 4, the obtained numerical results are reported and demonstrated the efficiency of the suggested numerical technique is by solving some test examples. Section 5 ends this work with some conclusions.

2. Preliminaries

2.1 Boubaker Scaling Functions

Scaling functions have been successfully used in many scientific and engineering fields. Boubaker scaling functions can be defined as follows[31,32]:

$$BS_{nm}(t)=\begin{cases}2^{\frac{k}{2}}B_m(2^{k+1}t-2n+1) & \frac{2n-1}{2^{k+1}}\le t\le\frac{2n}{2^{k+1}}\\ 0 & \text{otherwise},\end{cases} \tag{1}$$

where $n=1,2,\dots,;$ k can be assumed to be any positive integer, m is the degree of the Boubaker polynomials and t denotes the time for $n=0,1,\dots,M$.

Note that a recursive relation that yields the Boubaker polynomials is:

$$B_m(t)=t\,B_{m-1}(t)-B_{m-2}(t),\;\; m>2, \tag{2}$$

with $B_0(t)=1,\; B_1(t)=t,\; B_2(t)=t^2+2.$

Important properties of the Boubaker polynomials are:

$$B_m(0)=2\,cos\left(\frac{m+2}{2}\,\pi\right), n\ge 1,$$

$$B_m(-t)=(-1)^m B_m(t).$$

2.2 Function Approximation

A function $f(t)\in l^2\,[0,1]$ may be expanded as:

$$f(t)=\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}c_{nm}\;BS_{nm}(t), \tag{3}$$

where $c_{nm}=\langle f(t),BS_{nm}(t)\rangle$, in which $\langle .,.\rangle$ represents the inner product in $l^2\,[0,1]$. Consider the truncated series in Eq. (3), to obtain:

$$f(t)=\sum_{n=1}^{2^k}\sum_{m=0}^{M}c_{nm}\,BS_{nm}(t)=c^T\,BS(t), \tag{4}$$

where c and $BS(t)$ are $2^{k-1}(M+1)\times 1$ matrices that are given by:

$$c=\left[c_{00}\,,c_{01},\dots,c_{0M}\,,c_{10},c_{11}\,,\dots,c_{1M},c_{2^k-1,0}\,,c_{2^k-1,1}\,,\dots,c_{2^k,M}\right]^T, \tag{5}$$

$$BS(t)=[BS_{00},BS_{01},\dots,BS_{0M},BS_{10},BS_{11},\dots,BS_{1M},BS_{2^k0},BS_{2^k1}\dots,BS_{2^kM}]^T. \tag{6}$$

2.3 Operational Matrix of Derivative

The operational matrix of the derivative for Boubaker scaling functions is derived in this section.

The derivative of the vector $BS(t)$ can be obtained as follows:

$$\frac{d}{dt}BS(t) = DBS(t). \quad (7)$$

First, we choose $k = 1$ and $M = 5$, then the following scaling Boubaker function can be evaluated:

$$BS_1 = BS_{10} = \begin{cases} 2^{\frac{1}{2}} & 0 \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$BS_2 = BS_{11} = \begin{cases} 2^{\frac{1}{2}}\,(4t-1) & 0 \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$BS_3 = BS_{12} = \begin{cases} 2^{\frac{1}{2}}\,(16\,t^2 - 8t + 3) & 0 \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$BS_4 = BS_{13} = \begin{cases} 2^{\frac{1}{2}}\,(64t^3 - 48\,t^2 + 16t - 2) & 0 \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$BS_5 = BS_{14} = \begin{cases} 2^{\frac{1}{2}}(256t^4 - 256t^3 + 96\,t^2 - 16t - 1) & 0 \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$BS_6 = BS_{15} = \begin{cases} 2^{\frac{1}{2}}(1025t^5 - 1280t^4 + 576t^3 - 112\,t^2 - 4t + 3) & 0 \leq t < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

$$BS_7 = BS_{20} = \begin{cases} 2^{\frac{1}{2}}\,(1) & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$BS_8 = BS_{21} = \begin{cases} 2^{\frac{1}{2}}\,(4t-3) & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$BS_9 = BS_{22} = \begin{cases} 2^{\frac{1}{2}}\,(16t^2 - 24t + 11) & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$BS_{10} = BS_{23} = \begin{cases} 2^{\frac{1}{2}}\,(64t^3 - 144t^2 + 112t - 30) & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$BS_{11} = BS_{24} = \begin{cases} 2^{\frac{1}{2}}(256t^4 - 768t^3 + 864t^2 - 432t + 79) & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$BS_{12} = BS_{25} = \begin{cases} 2^{\frac{1}{2}}(1024\,t^5 - 3840t^4 + 5696t^3 - 4176t^2 + 1500t - 207) & \frac{1}{2} \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

So, after differentiate the above equations, one can get for $0 \leq t < \frac{1}{2}$:

$\frac{dBS_1}{dt} = 0$, $\frac{dBS_2}{dt} = 4BS_1$, $\frac{dBS_3}{dt} = 8BS_2$, $\frac{dBS_4}{dt} = -20BS_1 + 12BS_3$,
$\frac{dBS_5}{dt} = -16BS_2 + 16BS_4$, $\frac{dBS_6}{dt} = 52BS_1 - 12BS_3 + 20BS_5$.

and for $\frac{1}{2} \leq t < 1$, one can obtain:

$$\frac{dBS_7}{dt} = \frac{dBS_1}{dt}, \quad \frac{dBS_8}{dt} = \frac{dBS_2}{dt}, \quad \frac{dBS_9}{dt} = \frac{dBS_3}{dt}, \frac{dBS_{10}}{dt} = \frac{dBS_4}{dt},$$

$\frac{dBS_{11}}{dt} = \frac{dBS_5}{dt}, \frac{dBS_{12}}{dt} = \frac{dBS_6}{dt}$.
Due to the support of $BS_i, i = 1, 2, \ldots, 12$.
The matrix D can be formulated as follows:
$D = \begin{bmatrix} F & O \\ O & F \end{bmatrix}, m = 0, 1, \ldots, M, \ n = 1, 2, \ldots, 2^k, i = (n+1)(M+1) + (m+1)$, where

$$F = 4\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ -5 & 0 & 3 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 & 0 \\ 13 & 0 & -3 & 0 & 5 & 0 \end{bmatrix}.$$

The new scalar Boubaker functions operational matrix of the derivative is introduced in Theorem (1).

Theorem (1):
Suppose that $BS(t)$ is the scalar Boubaker vector defined in Eq.(7). The derivative of $BS(t)$ can be represented by

$$\frac{dBS(t)}{dt} = D\ BS(t),$$

Where the matrix D is the operational matrix of the derivative with dimension $2^k(M + 1)$defined as below:

$$D = \begin{bmatrix} F & O & \cdots & O \\ O & F & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & F \end{bmatrix}.$$

Note that the matrix F is of dimension $(M+1) \times (M+1)$, its elements are defined as below:

$$f_{i,j} = 2^{k+1}\begin{cases} i & j = i \\ (-1)^{\frac{(i-j)}{2}}(2f_{i-1,2} + f_{i-2,1}) & j = 1 \\ (f_{i-1,j+1} + f_{i-1,j-1} - f_{i-2,j}) & i > j \text{ and } i, j > 1 \\ 0 & j > i \end{cases} \qquad (8)$$

Proof:
Using the Boubaker polynomials into [0, 1], then the r^th element of vector $BS(t)$ can be written as:

$$BS_r(t) = BS_{n,m}(t) = 2^{\frac{k}{2}}B_m(2^{k+1}t - 2n + 1), \qquad (9)$$

for $\frac{2n-1}{2^{k+1}} \le t \le \frac{2n}{2^{k+1}}$ and $BS_r(t) = 0$ outside the interval $t \in \left[\frac{2n-1}{2^{k+1}}, \frac{2^n}{2^{k+1}}\right]$,
where $r = (n+1)(M+1) + (m+1), m = 0, 1, \ldots, M,\ n = 1, 2, \ldots, 2^k$. Differentiate Eq.(9) yields:

$$\frac{dBS_r(t)}{dt} = 2^{\frac{k}{2}}.2^k\dot{B}_m(2^{k+1}t - 2n + 1). \qquad (10)$$

for $\frac{2n-1}{2^{k+1}} \le t \le \frac{2n}{2^{k+1}}$.
The scalar Boubaker function expansion becomes:

$$\frac{dBS_r(t)}{dt} = \sum_{i=(n+1)(M+1)+1}^{n(M+1)} a_i BS_i(t).$$

Since we have $\frac{dBS_0(t)}{dt} = 0$, this implies that $\frac{dBS_r(t)}{dt} = 0$ for:
$r = 1, (M+1), 2(M+1) + 1, \ldots, (2^k - 1)(M+1) + 1$.

This means that the first row in matrix F will be zero. By substituting the following derivative of Boubaker polynomials,

$$\frac{dB_m(t)}{dt} = \sum_{\substack{j=1 \\ (i-j)even}}^{i} e_{ij}B_j(t),$$

where: $$e_{ij} = \begin{cases} i & i=j \\ (-1)^{\frac{(i-j)}{2}}\left(2e_{i-1,2}+e_{i-2,1}\right) & j=1 \\ \left(e_{i-1,j-1}+e_{i-1,j+1}-e_{i-2,j}\right) & i>j \text{ and } i,j>1 \\ 0 & j>i \end{cases} \tag{11}$$

Now, for $\frac{2^n-1}{2^{k+1}} \le t \le \frac{2^n}{2^{k+1}}$,

Expanding Eq.(11) in scalar Boubaker basis, yields:

$$\frac{dBS_r(t)}{dt} = 2^{k+1} \sum_{\substack{j=1 \\ (i-j)even}}^{i} e_{ij}BS_j(t).$$

This is the desired result.

Corollary: The operational matrix for n^{th} derivatives can be defined as $\frac{d^n}{dt^n}BS(t) = D^nBS(t)$.

2.4 The Product Operation Matrix for Scalar BoubakerFunction

The product property of two scalar Boubaker bases may be applied to find the approximate solution of differential and integral equations.

The following property of the product of two Boubaker scaling function vectors can be written as below:

$$BS(t)\, BS^T(t)C = \underline{C}BS(t), \tag{12}$$

where C is defined in Eq. (5) and $\underline{C}$ is a $m \times m$ matrix.

Since the entries of vector $BS(t)$ are the interval $\left[\frac{2n-1}{2^{k+1}}, \frac{2^n}{2^{k+1}}\right]$ and zero outside this interval. Therefore; $BS_{nm}BS_{lk} = 0, n \neq l$, we also have:

$$BS_{1i}(t)BS_{1i}(t) = BS_{1,2i}(t) - 2BS_{1,0}(t) + 4\sum_{k=1}^{i-1} BS_{1,2(i-k)}(t).$$

$$BS_{1i}(t)BS_{1j}(t) = BS_{1,i+j}(t) + BS_{1,|i-j|}(t) + 4\sum_{k=1}^{min(i,j)-1} BS_{1,i+j-2k}(t).$$

In general,

$$BS_{ni}(t)BS_{nj}(t) = BS_{n,i+j}(t) + BS_{n,|i-j|}(t) + 4\sum_{k=1}^{min(i,j)-1} BS_{n,i+j-2k}(t),$$

and

$$BS_{ni}(t)BS_{ni}(t) = BS_{n,2i}(t) - 2BS_{n,0}(t) + 4\sum_{k=1}^{i-1} BS_{n,2(i-k)}(t).$$

The product $BS(t)\, BS^T(t)$ with M=2, k=2 can be defined as

$$BS(t)\, BS^T(t) = \begin{pmatrix} BS_0 & 0 \\ 0 & BS_1 \end{pmatrix},$$

where:

$$BS_i = \begin{bmatrix} BS_{i2} - 2BS_{i0} & BS_{i3} + BS_{i1} & BS_{i4} + BS_{i2} \\ BS_{i3} + BS_{i1} & BS_{i4} + 4BS_{i2} - 2BS_{i0} & BS_{i3} + BS_{i1} \\ BS_{i4} + BS_{i2} & BS_{i3} + BS_{i1} & BS_{i6} + 4BS_{i4} + 4BS_{i2} - 2BS_{i0} \end{bmatrix}$$

for i=1, 2.

Therefore; the 6×6 matrix $\underline{C}$ in Eq.(12) can be defined as:

$$\underline{C} = \begin{pmatrix} c_0 & 0 \\ 0 & c_1 \end{pmatrix},$$

where c_i, i=0,1, are 3×3 matrices given by:

$$C_i = \begin{bmatrix} c_{i2} - 2c_{i0} & c_{i3} + c_{i1} & c_{i4} + c_{i2} \\ c_{i3} + c_{i1} & c_{i4} + 4c_{i2} - 2c_{i0} & c_{i3} + c_{i1} \\ c_{i4} + c_{i2} & c_{i3} + c_{i1} & c_{i6} + 4c_{i4} + 4c_{i2} - 2c_{i0} \end{bmatrix}.$$

3. Direct Algorithm for Solving Calculus of Variation Problem Using $BS(t)$

Consider the following variation problem:

$$J\big(x(t)\big) = \int_0^1 \Big(F\big(x^2(t), \dot{x}^2(t)\big), g(x, \dot{x}, t)\Big)\, dt, \tag{13}$$

together with the points:

$$x(0) = \alpha_1, x(1) = \alpha_2. \tag{14}$$

Approximating the variable $x(t)$ using $BS(t)$, gives:

$$x(t) = a^T BS(t), \tag{15}$$

where $a = [a_1, a_2, \dots, a_N]^T$, is $(N+1) \times 1$ vector of unknown parameters, then $\dot{x}(t)$ can be expressed as:

$$\dot{x}(t) = a^T \dot{BS}(t), \tag{16}$$

where $\dot{BS}(t)$ is the derivative vector of $BS(t)$.

The obtained functional J is a nonlinear mathematical programming problem of unknown parameters $a_1, a_2, \dots, a_N$ after substituting Eqs. (15-16) into Eq. (13) as follows

$$J\big(x(t)\big) = \int_0^1 \Big(F\big(a^T BS(t) BS(t)^T a, a^T \dot{BS}(t) \dot{BS}(t)^T a\big), g\big(a^T BS(t), a^T \dot{BS}(t), t\big)\Big) dt \tag{17}$$

Equation (17) can be simplified to:

$$J\big(x(t)\big) = \frac{1}{2} a^T H a + c^T a,$$

where $H = 2\int_0^1 F\big(BS(t)BS(t)^T, \dot{BS}(t)\dot{BS}(t)^T\big)\, dt$,

$$c^T = \int_0^1 g\big(a^T BS(t), a^T \dot{BS}(t), t\big)\, dt.$$

Both Eq. (5) and Eq. (14) give

$$x(0) = a^T BS(0) = \alpha_1, x(1) = a^T BS(1) = \alpha_2.$$

Finally, the obtained quadratic programming problem may be rewritten as below:

$$J(x) = \frac{1}{2} a^T H a + c^T a,$$

subject to $Fa - b = 0$.

where $F = \begin{bmatrix} BS^T(0) \\ BS^T(1) \end{bmatrix}$, $b = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$.

Note that: $BS(0) = \big[\sqrt{2} \ \ -\sqrt{2} \ \ 3\sqrt{2} \ \ -2\sqrt{2} \ \ -\sqrt{2} \ \ \dots\big]$,

and $BS(1) = \big[\sqrt{2} \ \ 3\sqrt{2} \ \ 11\sqrt{2} \ \ 30\sqrt{2} \ \ 79\sqrt{2} \ \ \dots\big]$.

Using the Lagrange multiplier technique, one can obtain the optimal values of the unknown parameters a^*:

$$a^* = -H^{-1}c + H^{-1}F^T(FH^{-1}F^T)^{-1}(FH^{-1}c + b),$$

4. Numerical Examples

Example 1: Consider the time-varying functional as below:

$$Min\, J(x(t)) = \int_0^1 (\dot{x}^2(t) + t\dot{x}(t) + x^2(t))dt, \tag{18}$$

together with the conditions

$$x(0) = 0 \text{ and } x(1) = 0.25, \tag{19}$$

where $x(t) = 0.5 + \alpha_1 e^t + \alpha_2 e^t$, $\alpha_1 = \frac{2-e}{4(e^2-1)}$, $\alpha_2 = \frac{e-2e^2}{4(e^2-1)}$ represents the exact solution.

Eqs (18-19) can be simplified to the following nonlinear mathematical programming problem

$$J(x(t)) = \frac{1}{2} a^T H a + c^T a,$$

subject to $Fa - b = 0$.
where:

$$H = 2\int_0^1 [(\dot{BS}(t)\dot{BS}(t)^T) + BS(t)BS(t)^T]dt,$$

$$c^T = \int_0^1 t\dot{BS}(t)^T dt, \quad F = \begin{bmatrix} BS(0)^T \\ BS(1)^T \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}.$$

The optimal unknown vector is $a = [0.08590739 \;\; 0.05835338 \;\; -0.00888302 \;\; 0.00045286 \;\; -0.00000080]$.
The numerical results are illustrated in Table 1 and Figure 1 with $k = 1$ and $M = 5$ compared with the exact solution.

Table 1: The numerical results with $k = 1$ and $M = 5$ for Example 1.

t	x_{exact}	$x_{appr}(t)$	error
0	0.00000000	0.00000000	0.00000000
0.1	0.04195072	0.04180894	0.00014178
0.2	0.07931714	0.07922813	0.00008900
0.3	0.11247322	0.11250422	0.00003099
0.4	0.14175081	0.14188312	0.00013231
0.5	0.16744291	0.16761009	0.00016717
0.6	0.18980668	0.18992964	0.00012296
0.7	0.20906592	0.20908562	0.00001969
0.8	0.22541340	0.22532116	0.00009223
0.9	0.23901272	0.23887871	0.00013401
1.0	0.25000000	0.25000000	0.00000000

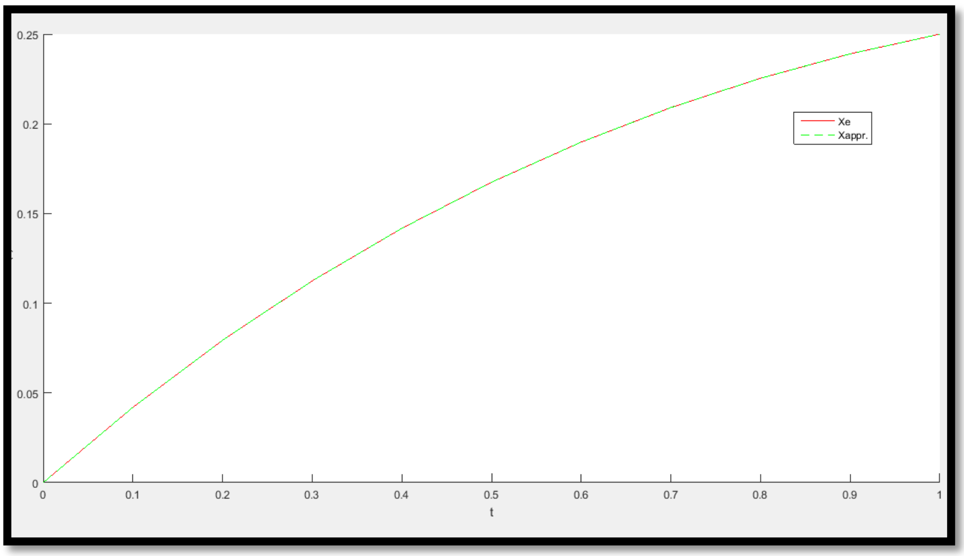

Figure 1: Exact and approximate solutions for Example 1.

Example 2:Consider the following time-varying functional:

$$Min\,J(x(t)) = \int_0^1 (\dot{x}^2(t) + x^2(t))dt,$$

together with the boundary conditions $x(0) = 0, \quad x(1) = 1$.

Here, $x(t) = \frac{e^t - e^{-t}}{e^1 - e^{-1}}$ is the exact solution.

for this problem we have $H = \int_0^1 \left(B\dot{S}(t) B\dot{S}(t)^T + BS(t)BS(t)^T \right) dt$,

$c^T = 0, \; F = \begin{bmatrix} BS(0)^T \\ BS(1)^T \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The optimal unknown vector is: a_0=0.14274223, a_1=0.15310148, a_2=0.00464964, a_3=0.00179430, a_4=0.00000107.

The numerical results are listed in Table 2 and plotted in Figure 2 with $k = 1$ and $M = 5$ compared with the exact solution.

Table 2: The numerical results with $k = 1$ and $M = 5$ for Example 2.

t	x_{exact}	$x_{appr}(t)$	error
0	0	0.00000000	0.00000000
0.1	0.08523370	0.08540202	0.00016831
0.2	0.171320454	0.17144774	0.00012728
0.3	0.259121838	0.25911063	0.00001120
0.4	0.349516600	0.34936512	0.00015147
0.5	0.443409441	0.44318654	0.00022289
0.6	0.541740074	0.54155117	0.00018889
0.7	0.645492623	0.64543623	0.00005638
0.8	0.755705480	0.75581985	0.00011437
0.9	0.873481690	0.87368110	0.00019941
1.0	1.000000000	1.00000000	0.00000000

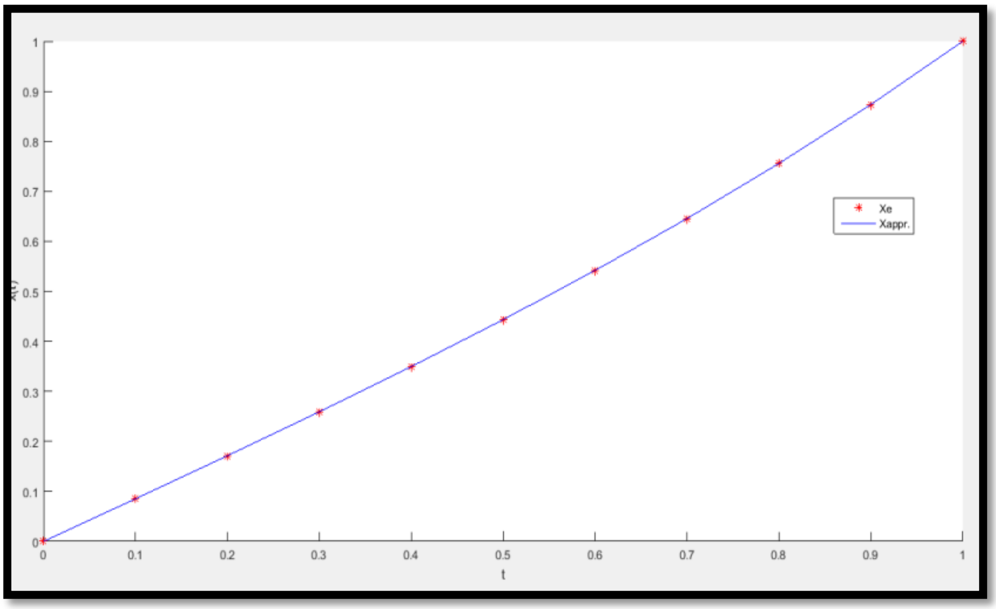

Figure 2: Exact and approximate solutions for Example 2.

Example 3: $Min\, J(x(t)) = \int_0^1 (\dot{x}^2(t) - x^2(t))dt$,
together with the conditions: $x(0) = 0$ and $x(1) = 1$.
For this problem, $x(t) = \frac{\sin t}{\sin 1}$ represents the exact solution.
In this case, $H = \int_0^1 \left(B\dot{S}(t)B\dot{S}(t)^T - BS(t)BS(t)^T\right) dt$,
$c^T = 0, \; F = \begin{bmatrix} BS(0)^T \\ BS(1)^T \end{bmatrix}, \; b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
The optimal unknown vector is:
$a = [0.22112400 \;\; 0.20572597 \;\;\; 0.00654117 \;\;\; -0.00214508 \;\;\; 0.00006468]$.
Table 3 and Figure 3 show the numerical results with $k = 1$ and $M = 5$ compared with the exact solution.

Table 3: The numerical results with $k = 1$ and $M = 5$ for Example 3.

t	x_{exact}	$x_{appr.}$	error
0	0	0.00000000	0.00000000
0.1	0.11864154	0.11862515	0.00001638
0.2	0.23609766	0.23610539	0.00000773
0.3	0.35119476	0.35121959	0.00002483
0.4	0.46278285	0.46280286	0.00002001
0.5	0.56974696	0.56974648	0.00000047
0.6	0.67101835	0.67099797	0.00002037
0.7	0.76558514	0.76556105	0.00002409
0.8	0.85250246	0.85249562	0.00000683
0.9	0.93090186	0.93091783	0.00001596
1.0	1.00000000	1.00000000	0.00000000

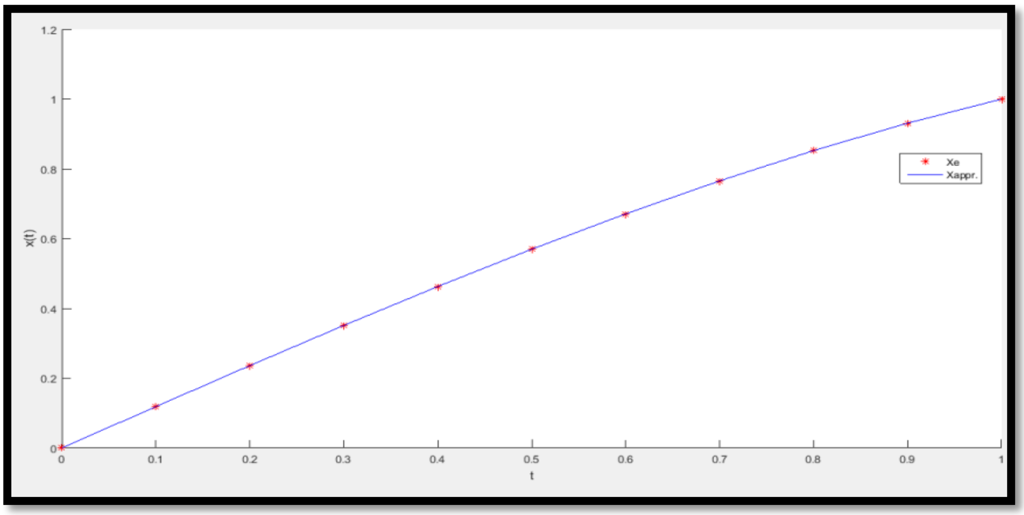

Figure 3: Exact and approximate solutions for Example 3.

5. Conclusion

A new direct parameterization method is developed for solving problems in the calculus of variations based on Boubaker scaling basis functions. Some interesting properties are given concerning the Boubaker scaling function required for the suggested parameterization technique, and they are used to transfer the solution computation of calculus of variations problems to the quadratic programming technique. Consequently, the numerical solution concurs with the exact solution even with a small number of the Boubaker scaling used in estimation. The proposed method is applied to some test examples to show the accuracy and the implementation of the suggested technique.

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