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The L^P-spaces of functions from the n-dimensional real space to the N-dimensional quaternionic space

Hussien A. H. Abugirda*, Mohammed Kh. Abdullah, Khaldoun S. Al-Yasiri
Department of Mathematics, College of Science, University of Basrah, Basrah, Iraq

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Abstract

In this paper, we introduce new definitions of the $\boldsymbol{L^P}$- spaces, $\boldsymbol{1 \le P \le \infty}$, namely the $\boldsymbol{L^P}(\mathbb{R}^{\boldsymbol{n}}, \mathbb{H}^{\boldsymbol{N}})$ - spaces, $\boldsymbol{1 \le P \le \infty}$. Here, n and N are natural numbers that are not necessarily equal, such that $n, N \ge 1$. The space $\mathbb{R}^n$ refers to the n-dimensional Euclidean space, $\mathbb{H}$ refers to the quaternions set and $\mathbb{H}^N$ refers to the N-dimensional quaternionic space. Furthermore, we establish and prove some properties of their elements. These elements are quaternion-valued N-vector functions defined on $\mathbb{R}^{\boldsymbol{n}}$, and the $\boldsymbol{L^P}$ spaces have never been introduced in this way before.

Keywords: Hamiltonian skew field of quaternions, Quaternion N-vectors, Quaternion-valued functions, $L^p(\mathbb{R}^n, \mathbb{H}^N)$ spaces, $L^\infty(\mathbb{R}^n, \mathbb{H}^N)$ space.

فضاءات L^P لدوال من الفضاء الحقيقي ذو البعد n الى فضاء المركبات الرباعية ذو البعد N

حسين علي حسين البوكرده*, محمد خلف عبد الله, خلدون سعد غالب
قسم الرياضيات, كلية العلوم, جامعة البصرة, البصرة, العراق

الخلاصة

في هذا البحث نقدم تعريف جديد للفضاءات $\boldsymbol{L^P}, \boldsymbol{1 \le P \le \infty}$, وهو تعريف الفضاءات $\boldsymbol{L^P}(\mathbb{R}^{\boldsymbol{n}}, \mathbb{H}^{\boldsymbol{N}})$, $\boldsymbol{1 \le P \le \infty}$, هنا n و N هما عددان طبيعيان ليس من الضروري أن يتساويان وهما يحققان $n, N \ge 1$.الفضاء $\mathbb{R}^n$ يشير الى الفضاء الإقليدي ذات البعد n و $\mathbb{H}$ تشير الى مجموعة الرباعيات و $\mathbb{H}^N$ تشير الى فضاء المتجهات ذات البعد N بمركبات رباعية. علاوة على ذلك نقدم مع البرهان بعض خصائص العناصر لهذه الفضاءات. هذه العناصر هي عباره عن دوال إتجاهية ذات بعد N بمركبات رباعية وهي معرفة على $\mathbb{R}^n$, وإن الفضاءات $\boldsymbol{L^P}$ لم يسبق أن قدمت بهذا الشكل من قبل.

1. Introduction

There have been a vast number of studies on the real quaternions set $\mathbb{H}$ (which is sometimes called the Hamiltonian skew field of quaternions) since it has been introduced for the first time in 1843 by W. R. Hamilton [1] and [2]. Technological improvements have been provided when the real quaternions are put into practice and it has been studied in different areas like geometry, algebra, computer-aided design, and physics.

*Email: ameer.abdulameer@sc.uobaghdad.edu.iq

One of the recent papers in this regard is the paper M. J. Saadan et al. [3] in which among other interesting results, the spaces $L^{\mathrm{p}}(\mathrm{G}^2, \mathbb{H})$ and $L^{\infty}(\mathrm{G}^2, \mathbb{H})$ have been defined for the quaternion-valued functions $u: \mathrm{G}^2 \longrightarrow \mathbb{H}$, where G is a locally compact abelian group. Also, in [4] quaternion-valued positive definite functions have been studied on the countable real Hilbertian nuclear spaces, locally compact abelian groups and on the linear space of all real numerical sequences $\mathbb{R}^M = \{(x_1, \cdots, x_d, \cdots): x_d \in \mathbb{R}\}$ endowed with the Tychonoff topology. On the other hand, in [5] the module structure of the N-dimensional quaternionic space $\mathbb{H}^N$ has been created with other interesting results. The idea of this paper is inspired by the results of all the interesting papers mentioned above.

Although the majority of scientific studies are on real space, when the concept of raising the dimension of the space is considered, the approaches of this work would be different. To explain this statement, consider a limit problem, then the offered solution by the N-vectors approximations is more rational in this case. Hence, the need arises to study N-dimensional spaces like the space $\mathbb{H}^N$ and the N-dimensional vector valued functions like the quaternion-valued N-vector functions.

The main object of the study in this paper is the quaternion-valued N-vector functions $u: \mathbb{R}^n \longrightarrow \mathbb{H}^N$, where the natural numbers $n, N \geq 1$ are not necessarily equal. The crucial goal of the study is to define the $\boldsymbol{L^P}(\mathbb{R}^{\boldsymbol{n}}, \mathbb{H}^{\boldsymbol{N}})$ – spaces, $\mathbf{1} \leq \boldsymbol{P} \leq \infty$, and to analyze some of the most important concepts of these spaces. Here, $\mathbb{R}^n$ refers to the real n-vector space $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n-times}$, $\mathbb{H}$ refers to the real quaternions set, and $\mathbb{H}^N$ refers to the N-dimensional quaternionic space (quaternionic N-vector space) $\mathbb{H}^N = \underbrace{\mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}}_{N-times}$, the details will be included in the preliminaries in the next section.

The authors in this paper followed the same strategy of [6] and [7] for referring to the modulus of a space $\mathbb{x}$ (as an example) by $\|\cdot\|_{\mathbb{x}}$ due to the different algebraic systems that are used in this work and to make the clearness to the reader for which modulus and for which space. The absolute value for a real number will be denoted by $|\cdot|_{\mathbb{R}}$.

2. Preliminaries

This section will be devoted to summaries of the background of the topic under consideration. We also present the most important basics that are needed throughout this work. References will be given to direct the reader to more details.

2.1 The real quaternion space $\mathbb{H}$

The most popular definition of the real quaternions set $\mathbb{H}$ is given in the following form:
$\mathbb{H} = \{q \,|q = a + bi + cj + dk, a, b, c, d \in \mathbb{R}\}$,
with the following properties for i, j and k which are called Hamilton's multiplication rules:
$i \cdot j = -j \cdot i = k, j \cdot k = -k \cdot j = i, k \cdot i = -i \cdot k = j$ and $j^2 = k^2 = i^2 = ijk = -1$.

Actually, each quaternion element, for example, $q = a + bi + cj + dk$ can be written as a sum of two parts, one of them is called the scalar part or the real part and is denoted by $Sc(q)$ or simply S_q, in this case, $S_q = a$. Other elements are called the vector, spatial, or pure part which are denoted by $Vec(q)$ or simply V_q, in this case, $V_q = bi + cj + dk$. Therefore, $q = S_q + V_q$. The conjugate of the quaternion q is $\bar{q} = S_q - V_q$, and the modulus $\|q\|_{\mathbb{H}}$ of a quaternion q is defined by $\|q\|_{\mathbb{H}} = \sqrt{q\bar{q}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. We refer the reader to [5], [8], [9], [10], [11], [12], and [13] for more interesting details , properties and results regarding the quaternions and the algebraic operations such as the addition and the multiplication of two quaternion elements, and also the multiplication of a quaternion element by a real scalar. The above typical representation of the quaternions will be used in this paper, However, the quaternion numbers can also be represented in a matrix form. Assume that I is 4×4 identity matrix, and assume that H, J, K are 4×4 real matrices. The typical quaternion can be rewritten in the following matrix form:

$$Q = aI + bH + cJ + dK,$$

where the matrices $I, H, J,$ and K satisfy the Hamiltonian conditions. For more details about the matrix form, we refer the reader to [14].

2.2 The N-dimensional quaternionic space $\mathbb{H}^N$

Let $\mathbb{H}^N$ be the N-dimensional quaternions set which is the set of all real quaternion N-vectors. It is defined as follows:

$$\mathbb{H}^N = \underbrace{\mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}}_{N-times},$$

and is represented by the following set:

$$\mathbb{H}^N = \{\vec{q} = (q_1, q_2, \cdots, q_N) | q_1, q_2, \cdots, q_N \in \mathbb{H}\}.$$

Since $q_l \in \mathbb{H}$ for all $l = 1,2, \cdots, N$ this means it can be written as $q_l = S_{q_l} + V_{q_l}$. Therefore, the quaternionic N-vector $\vec{q}$ itself can be written as a sum of two N-vectors as in the following form:

$$\vec{q} = S_{\vec{q}} + V_{\vec{q}} \, ,$$

where the real N-vector $S_{\vec{q}} = \left(S_{q_1}, S_{q_2}, \cdots, S_{q_N}\right)$ represents the scalar (real) part of $\vec{q}$ and the pure (spatial) quaternion N-vector $V_{\vec{q}} = \left(V_{q_1}, V_{q_2}, \cdots, V_{q_N}\right)$ represents the vector part of $\vec{q}$. Consequently, the conjugate of the quaternionic n-vector $\vec{q}$ is:

$$\bar{\vec{q}} = S_{\vec{q}} - V_{\vec{q}}.$$

The set of all pure (spatial) quaternionic N-vectors $\mathbb{H}_p^N$ (where p comes from the word ‘‘pure’’) can be defined as the set of all vector parts of the quaternionic N-vectors of the set $\mathbb{H}^N$:

$$\mathbb{H}_p^N = \left\{V_{\vec{q}} = \left(V_{q_1}, V_{q_2}, \cdots, V_{q_N}\right) \middle| \vec{q} \in \mathbb{H}^N\right\}.$$

The modulus of N-dimensional quaternion vector $\vec{q} = (q_1, q_2, \cdots, q_N)$ in the vector space $\mathbb{H}^N$ defined in [5] by $\|\vec{q}\|_{\mathbb{H}^N} = \sqrt{\sum_{l=1}^{N} \|q_l\|_{\mathbb{H}}^2}$. To understand the vector space structure of N-dimensional quaternionic space over real space. We recommend the reader to see[5]. Among other interesting results, an inner product function and a norm function have been defined on the N-dimensional quaternionic space. The analysis concepts of the quaternionic N-vector valued functions such as limits, continuity, and the derivatives have been discussed and described in there considering the component-wise metric on $\mathbb{H}^N$.

3. The L^P and L^∞- spaces of quaternion-valued N-vector functions on $\mathbb{R}^n$

This section contains the main results of this paper. We start with introducing the definitions of the quaternion-valued N-vector functions, $L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{H}^N)$ - spaces and $L^\infty(\mathbb{R}^n, \mathbb{H}^N)$ - space. To the best of our knowledge, these definitions are not introduced before in the sense of $\mathbb{H}^N$. After that, some properties of the functions that belong to these spaces will be discussed clearly.

3.1 Definition (Quaternion-valued N-vector functions on $\mathbb{R}^n$)

Let n and N be the natural numbers that are not necessarily equal, such that $n, N \geq 1$. Consider the Euclidean real n-vector space $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n-times}$, and the quaternionic N-vector space $\mathbb{H}^N = \underbrace{\mathbb{H} \times \mathbb{H} \times \cdots \times \mathbb{H}}_{N-times}$. If u is a function from $\mathbb{R}^n$ to $\mathbb{H}^N$, i.e., $u: \mathbb{R}^n \longrightarrow \mathbb{H}^N$. Then u is called a quaternion-valued N-vector function on $\mathbb{R}^n$.

3.2 Definition ($L^p(\mathbb{R}^n, \mathbb{H}^N)$ - spaces, $1 \leq p < \infty$)

Let n and N be the natural numbers that are not necessarily equal, such that $n, N \geq 1$. Consider the n-dimensional Euclidean space $\mathbb{R}^n$, and the N-dimensional quaternionic space $\mathbb{H}^N$. We define the spaces $L^p(\mathbb{R}^n, \mathbb{H}^N)$ for $1 \leq p < \infty$ as follows:

$$L^p(\mathbb{R}^n, \mathbb{H}^N) = \left\{u : \mathbb{R}^n \longrightarrow \mathbb{H}^N \middle| u \text{ is measurable on } \mathbb{R}^n, \text{and } \|u\|_{L^p(\mathbb{R}^n,\mathbb{H}^N)} < \infty\right\},$$

where:

$$\|u\|_{L^p(\mathbb{R}^n,\mathbb{H}^N)} = \left(\int_{\mathbb{R}^n} \|u(x)\|_{\mathbb{H}^N}^p dx\right)^{\frac{1}{p}}.$$

3.3 Definition ($L^\infty(\mathbb{R}^n, \mathbb{H}^N)$ - space)

Let n and N be the natural numbers that are not necessarily equal, such that $n, N \geq 1$. Consider the n-dimensional Euclidean space $\mathbb{R}^n$, and the N-dimensional quaternionic space $\mathbb{H}^N$. We define the space $\boldsymbol{L}^\infty(\mathbb{R}^n, \mathbb{H}^N)$ in the following form:

$$L^\infty(\mathbb{R}^n, \mathbb{H}^N) = \left\{u : \mathbb{R}^n \longrightarrow \mathbb{H}^N \middle| u \text{ is measurable on } \mathbb{R}^n, \text{ and } \|u\|_{L^\infty(\mathbb{R}^n,\mathbb{H}^N)} < \infty\right\},$$

where:

$$\|u\|_{L^\infty(\mathbb{R}^n,\mathbb{H}^N)} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} \|u(x)\|_{\mathbb{H}^N}.$$

After defining the $L^p(\mathbb{R}^n, \mathbb{H}^N)$ - spaces, $1 \leq p \leq \infty$, we will state and prove the following theorem.

3.4 Theorem

Let $u : \mathbb{R}^n \longrightarrow \mathbb{H}^N$ be a quaternion-valued N-vector function, which means for all $x \in \mathbb{R}^n$ there exist quaternions $u_t \in \mathbb{H}$, $t = 1,2,3, \cdots, N$ such that $u(x) = (u_1, u_2, \cdots, u_N)$, which also means for every fixed t there exist $u_{t,s} \in \mathbb{R}$, $s = 0,1,2,3$ such that $u_t = u_{t,0} + u_{t,1}i + u_{t,2}j + u_{t,3}k$. If u is a measurable function, then for $1 \leq p \leq \infty$ the following statements are equivalent:

I. The quaternion-valued N-vector function $u : \mathbb{R}^n \longrightarrow \mathbb{H}^N$ is an element of the space $L^p(\mathbb{R}^n, \mathbb{H}^N)$, i.e., $u \in L^p(\mathbb{R}^n, \mathbb{H}^N)$.

II. The components $u_t \in L^p(\mathbb{R}^n, \mathbb{H})$, $\forall t = 1,2,3, \cdots, N$.

III. For all $t = 1,2,3, \cdots, N$, the components $u_{t,s} \in L^p(\mathbb{R}^n, \mathbb{R})$, $\forall s = 0,1,2,3$.

Proof

First, we prove for $1 \leq p < \infty$,

We assume that I is satisfied and prove II. Suppose that $u \in L^p(\mathbb{R}^n, \mathbb{H}^N)$. This means $\|u\|_{L^p(\mathbb{R}^n,\mathbb{H}^N)} < \infty$. Note that:

$\|u_t(x)\|_{\mathbb{H}} \leq \|u(x)\|_{\mathbb{H}^N}$, $\forall t = 1,2,3, \cdots, N$. Hence,

$$\|u_t\|_{L^p(\mathbb{R}^n,\mathbb{H})}^p = \int_{\mathbb{R}^n} \|u_t(x)\|_{\mathbb{H}}^p dx \leq \int_{\mathbb{R}^n} \|u(x)\|_{\mathbb{H}^N}^p dx = \|u\|_{L^p(\mathbb{R}^n,\mathbb{H}^N)}^p < \infty.$$

Therefore, $u_t \in L^p(\mathbb{R}^n, \mathbb{H})$, $\forall t = 1,2,3, \cdots, N$.

Now, we assume that I is satisfied and proveII. Suppose that $u_t \in L^p(\mathbb{R}^n, \mathbb{H})$, $\forall t = 1,2,3, \cdots, N$. Therefore, $\|u_t\|_{L^p(\mathbb{R}^n,\mathbb{H})} < \infty$, $\forall t = 1,2,3, \cdots, N$. Then:

$$
\begin{aligned}
\|u\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H}^N)} &= \int_{\mathbb{R}^n} \|u(\mathrm{x})\|^p_{\mathbb{H}^N} d\mathrm{x},\\
&= \int_{\mathbb{R}^n} \left(\sqrt{\|u_1(\mathrm{x})\|^2_{\mathbb{H}} + \|u_2(\mathrm{x})\|^2_{\mathbb{H}} + \cdots + \|u_N(\mathrm{x})\|^2_{\mathbb{H}}}\right)^p d\mathrm{x},\\
&\le \int_{\mathbb{R}^n} (\|u_1(\mathrm{x})\|_{\mathbb{H}} + \|u_2(\mathrm{x})\|_{\mathbb{H}} + \cdots + \|u_N(\mathrm{x})\|_{\mathbb{H}})^p d\mathrm{x},\\
&= N^p \int_{\mathbb{R}^n} \left(\frac{1}{N}\|u_1(\mathrm{x})\|_{\mathbb{H}} + \frac{1}{N}\|u_2(\mathrm{x})\|_{\mathbb{H}} + \cdots + \frac{1}{N}\|u_N(\mathrm{x})\|_{\mathbb{H}}\right)^p d\mathrm{x},\\
&\le N^p \int_{\mathbb{R}^n} \left(\left(\frac{1}{N}\|u_1(\mathrm{x})\|^p_{\mathbb{H}} + \frac{1}{N}\|u_2(\mathrm{x})\|^p_{\mathbb{H}} + \cdots + \frac{1}{N}\|u_N(\mathrm{x})\|^p_{\mathbb{H}}\right)^{\frac{1}{p}}\right)^p d\mathrm{x},\\
&= N^p \int_{\mathbb{R}^n} \left(\frac{1}{N}\|u_1(\mathrm{x})\|^p_{\mathbb{H}} + \frac{1}{N}\|u_2(\mathrm{x})\|^p_{\mathbb{H}} + \cdots + \frac{1}{N}\|u_N(\mathrm{x})\|^p_{\mathbb{H}}\right) d\mathrm{x},\\
&= N^{p-1} \int_{\mathbb{R}^n} \left(\|u_1(\mathrm{x})\|^p_{\mathbb{H}} + \|u_2(\mathrm{x})\|^p_{\mathbb{H}} + \cdots + \|u_N(\mathrm{x})\|^p_{\mathbb{H}}\right) d\mathrm{x},\\
&= N^{p-1} \left(\int_{\mathbb{R}^n} \|u_1(\mathrm{x})\|^p_{\mathbb{H}} d\mathrm{x} + \int_{\mathbb{R}^n} \|u_2(\mathrm{x})\|^p_{\mathbb{H}} d\mathrm{x} + \cdots + \int_{\mathbb{R}^n} \|u_N(\mathrm{x})\|^p_{\mathbb{H}} d\mathrm{x}\right),\\
&= N^{p-1} \left(\|u_1\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})} + \|u_2\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})} + \cdots + \|u_N\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})}\right),\\
&< \infty.
\end{aligned}
$$

Hence, I if and only if II is proved for $1 \le p < \infty$.

Now, we assume that II is satisfied and prove III. Suppose that $u_t \in L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{H})$, $\forall t = 1,2,3,\cdots,N$. Therefore, $\|u_t\|_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})} < \infty$, $\forall t = 1,2,3,\cdots,N$. Note that for every fixed $t \in \{1,2,3,\cdots,N\}$ the following holds:

$\left|u_{t,s}(\mathrm{x})\right|_{\mathbb{R}} \le \|u_t(\mathrm{x})\|_{\mathbb{H}}$, $\forall s = 0,1,2,3$. Hence,

$$\|u_t\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{R})} = \int_{\mathbb{R}^n} \left|u_{t,s}(\mathrm{x})\right|^p_{\mathbb{R}} d\mathrm{x} \le \int_{\mathbb{R}^n} \|u_t(\mathrm{x})\|^p_{\mathbb{H}} d\mathrm{x} = \|u_t\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})} < \infty.$$

Therefore, $\forall t = 1,2,3,\cdots,N$, the components $u_{t,s} \in L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{R})$, $\forall s = 0,1,2,3$.

Now, we assume that III is satisfied and prove II. For every fixed $t \in \{1,2,3,\cdots,N\}$, suppose that $u_{t,s} \in L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{R})$, $\forall s = 0,1,2,3$. Therefore, $\left\|u_{t,s}\right\|_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{R})} < \infty$, $\forall s = 0,1,2,3$. then:

$$
\begin{aligned}
\|u_t\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})} &= \int_{\mathbb{R}^n} \|u_t(\mathrm{x})\|^p_{\mathbb{H}} d\mathrm{x},\\
&= \int_{\mathbb{R}^n} \left(\sqrt{\left(u_{t,0}(\mathrm{x})\right)^2 + \left(u_{t,1}(\mathrm{x})\right)^2 + \left(u_{t,2}(\mathrm{x})\right)^2 + \left(u_{t,3}(\mathrm{x})\right)^2}\right)^p d\mathrm{x},\\
&\le \int_{\mathbb{R}^n} \left(\left|u_{t,0}(\mathrm{x})\right|_{\mathbb{R}} + \left|u_{t,1}(\mathrm{x})\right|_{\mathbb{R}} + \left|u_{t,2}(\mathrm{x})\right|_{\mathbb{R}} + \left|u_{t,3}(\mathrm{x})\right|_{\mathbb{R}}\right)^p d\mathrm{x},\\
&= 4^p \int_{\mathbb{R}^n} \left(\frac{1}{4}\left|u_{t,0}(\mathrm{x})\right|_{\mathbb{R}} + \frac{1}{4}\left|u_{t,1}(\mathrm{x})\right|_{\mathbb{R}} + \frac{1}{4}\left|u_{t,2}(\mathrm{x})\right|_{\mathbb{R}} + \frac{1}{4}\left|u_{t,3}(\mathrm{x})\right|_{\mathbb{R}}\right)^p d\mathrm{x},\\
&\le 4^p \int_{\mathbb{R}^n} \left(\left(\frac{1}{4}\left|u_{t,0}(\mathrm{x})\right|^p_{\mathbb{R}} + \frac{1}{4}\left|u_{t,1}(\mathrm{x})\right|^p_{\mathbb{R}} + \frac{1}{4}\left|u_{t,2}(\mathrm{x})\right|^p_{\mathbb{R}} + \frac{1}{4}\left|u_{t,3}(\mathrm{x})\right|^p_{\mathbb{R}}\right)^{\frac{1}{p}}\right)^p d\mathrm{x},\\
&= 4^p \int_{\mathbb{R}^n} \left(\frac{1}{4}\left|u_{t,0}(\mathrm{x})\right|^p_{\mathbb{R}} + \frac{1}{4}\left|u_{t,1}(\mathrm{x})\right|^p_{\mathbb{R}} + \frac{1}{4}\left|u_{t,2}(\mathrm{x})\right|^p_{\mathbb{R}} + \frac{1}{4}\left|u_{t,3}(\mathrm{x})\right|^p_{\mathbb{R}}\right) d\mathrm{x},\\
&= 4^{p-1} \int_{\mathbb{R}^n} \left(\left|u_{t,0}(\mathrm{x})\right|^p_{\mathbb{R}} + \left|u_{t,1}(\mathrm{x})\right|^p_{\mathbb{R}} + \left|u_{t,2}(\mathrm{x})\right|^p_{\mathbb{R}} + \left|u_{t,3}(\mathrm{x})\right|^p_{\mathbb{R}}\right) d\mathrm{x},\\
&= 4^{p-1} \left(\int_{\mathbb{R}^n} \left|u_{t,0}(\mathrm{x})\right|^p_{\mathbb{R}} d\mathrm{x} + \int_{\mathbb{R}^n} \left|u_{t,1}(\mathrm{x})\right|^p_{\mathbb{R}} d\mathrm{x} + \int_{\mathbb{R}^n} \left|u_{t,2}(\mathrm{x})\right|^p_{\mathbb{R}} d\mathrm{x} + \right.\\
&\qquad \left. \int_{\mathbb{R}^n} \left|u_{t,3}(\mathrm{x})\right|^p_{\mathbb{R}} d\mathrm{x}\right),\\
&= 4^{p-1} \left(\left\|u_{t,0}\right\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{R})} + \left\|u_{t,1}\right\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{R})} + \left\|u_{t,2}\right\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{R})} + \left\|u_{t,3}\right\|^p_{L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{R})}\right),\\
&< \infty.
\end{aligned}
$$

Hence, II if and only if III is proved for $1 \le p < \infty$.

Therefore, I $\Leftrightarrow$ II $\Leftrightarrow$ III proved for $1 \le p < \infty$.

Now, we prove for $p = \infty$,

We assume that I is satisfied and prove III. Suppose that $u \in L^{\infty}(\mathbb{R}^n, \mathbb{H}^N)$. This means $\|u\|_{L^{\infty}(\mathbb{R}^n,\mathbb{H}^N)} < \infty$. Note that for every fixed $s \in \{0,1,2,3\}$, the definition of $\|u_t(\mathrm{x})\|_{\mathbb{H}}$ and the definition of supremum imply:
$|u_{t,s}(\mathrm{x})|_{\mathbb{R}} \leq \|u_t(\mathrm{x})\|_{\mathbb{H}} \leq \|u(\mathrm{x})\|_{\mathbb{H}^N} < \infty$, $\forall t = 1,2,3,\cdots,N$.
Hence,
$\|u_{t,s}\|_{L^{\infty}(\mathbb{R}^n,\mathbb{R})} \leq \|u\|_{L^{\infty}(\mathbb{R}^n,\mathbb{H}^N)} < \infty$.
Therefore, for all $t = 1,2,3,\cdots,N$, the components $u_{t,s} \in L^{\infty}(\mathbb{R}^n,\mathbb{R})$, $\forall s = 0,1,2,3$.

Now, we assume that III is satisfied and prove II. For every fixed $t \in \{1,2,3,\cdots,N\}$, suppose that $u_{t,s} \in L^{\infty}(\mathbb{R}^n,\mathbb{R})$, $\forall s = 0,1,2,3$. In another way, $\|u_{t,s}\|_{L^{\infty}(\mathbb{R}^n,\mathbb{R})} < \infty$, $\forall s = 0,1,2,3$.

On the other hand, for every fixed $t \in \{1,2,3,\cdots,N\}$, we have $\|u_t\|_{\mathbb{H}}^2 = \sum_{s=0}^{3}|u_{t,s}(\mathrm{x})|_{\mathbb{R}}^2$.
This implies:

$$\begin{aligned}\|u_t\|_{\mathbb{H}} &\leq 4\max\left\{|u_{t,s}(\mathrm{x})|_{\mathbb{R}} : s = 0,1,2,3\right\},\\ &\leq 4\sum_{s=0}^{3}|u_{t,s}(\mathrm{x})|_{\mathbb{R}}.\end{aligned}$$

Thus,

$$\begin{aligned}\|u_t\|_{L^{\infty}(\mathbb{R}^n,\mathbb{H})} &= \operatorname*{ess\,sup}_{\mathrm{x}\in\mathbb{R}^n}\|u_t(\mathrm{x})\|_{\mathbb{H}},\\ &\leq 4\sum_{s=0}^{3}\operatorname*{ess\,sup}_{\mathrm{x}\in\mathbb{R}^n}|u_{t,s}(\mathrm{x})|_{\mathbb{R}},\\ &= 4\sum_{s=0}^{3}\|u_{t,s}\|_{L^{\infty}(\mathbb{R}^n,\mathbb{R})},\\ &< \infty.\end{aligned}$$

Therefore, the components $u_t \in L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H})$, $\forall t = 1,2,3,\cdots,N$.

Now, we assume that II is satisfied and prove I. Suppose that $u_t \in L^{\infty}(\mathbb{R}^n,\mathbb{H})$, $\forall t = 1,2,3,\cdots,N$. Therefore, $\|u_t\|_{L^{\infty}(\mathbb{R}^n,\mathbb{H})} < \infty$, $\forall t = 1,2,3,\cdots,N$.

On the other hand, we have $\|u\|_{\mathbb{H}^N}^2 = \sum_{t=1}^{\mathrm{N}}\|u_t\|_{\mathbb{H}}^2$. Which implies:

$$\begin{aligned}\|u\|_{\mathbb{H}^N} &\leq \mathrm{N}\max\{\|u_t\|_{\mathbb{H}} : t = 1,2,\cdots,\mathrm{N}\},\\ &\leq \mathrm{N}\sum_{t=1}^{\mathrm{N}}\|u_t\|_{\mathbb{H}}.\end{aligned}$$

Thus,

$$\begin{aligned}\|u\|_{L^{\infty}(\mathbb{R}^n,\mathbb{H}^N)} &= \operatorname*{ess\,sup}_{\mathrm{x}\in\mathbb{R}^n}\|u(\mathrm{x})\|_{\mathbb{H}^N},\\ &\leq \mathrm{N}\sum_{t=1}^{\mathrm{N}}\operatorname*{ess\,sup}_{\mathrm{x}\in\mathbb{R}^n}\|u_t\|_{\mathbb{H}},\\ &= \mathrm{N}\sum_{t=1}^{\mathrm{N}}\|u_t\|_{L^{\infty}(\mathbb{R}^n,\mathbb{H})},\\ &< \infty.\end{aligned}$$

Therefore, $u \in L^{\mathrm{p}}(\mathbb{R}^n,\mathbb{H}^N)$.
Hence, I ⇒ III ⇒ II ⇒ I is proved for $p = \infty$.
This means I ⇔ II ⇔ III is proved for $p = \infty$.
□

3.5 Lemma

Every quaternion-valued N-vector $u = (u_1, u_2, \cdots, u_N)$ can be written in a complex number form as $u = U_1 + U_2 j$, where $U_1 = A + Bi$, $U_2 = C + Di$ and $A, B, C, D \in \mathbb{R}^N$.

Proof

Let $u \in \mathbb{H}^N$. This means it is written as $u = (u_1, u_2, \cdots, u_N)$, where $u_t \in \mathbb{H}$, $\forall t = 1,2,3,\cdots,N$. Consequently, for every fixed t there exist $u_{t,s} \in \mathbb{R}$, $s = 0,1,2,3$ such that $u_t = u_{t,0} + u_{t,1}i + u_{t,2}j + u_{t,3}k$. Hence, u can be written in the following form:

$$\begin{aligned}u = (&u_{1,0} + u_{1,1}i + u_{1,2}j + u_{1,3}k,\\ &u_{2,0} + u_{2,1}i + u_{2,2}j + u_{2,3}k,\end{aligned}$$

$\vdots$

$u_{N,0} + u_{N,1}i + u_{N,2}j + u_{N,3}k),$

This can be also rearranged to the following form:

$u = (u_{1,0}, u_{2,0}, \cdots, u_{N,0}) + (u_{1,1}, u_{2,1}, \cdots, u_{N,1})i + (u_{1,2}, u_{2,2}, \cdots, u_{N,2})j +$
$(u_{1,3}, u_{2,3}, \cdots, u_{N,3})k.$

To simplify the last form, assume $A = (u_{1,0}, u_{2,0}, \cdots, u_{N,0})$, $B = (u_{1,1}, u_{2,1}, \cdots, u_{N,1})$, $C = (u_{1,2}, u_{2,2}, \cdots, u_{N,2})$ and $D = (u_{1,3}, u_{2,3}, \cdots, u_{N,3})$. Hence, u is written as follows:

$u = A + Bi + Cj + Dk,$

where $A, B, C, D \in \mathbb{R}^N$.

Now, by Hamilton's multiplication rules, specifically, $ij = k$, u can be written in a complex number form:

$$\begin{aligned} u &= A + Bi + Cj + Dk \\ &= A + Bi + Cj + Dij \\ &= (A + Bi) + (C + Di)j \\ &= U_1 + U_2 j, \end{aligned}$$

where $U_1 = A + Bi$ and $U_2 = C + Di$. Note that $U_1, U_2 \in \mathbb{C}^N$.

□

3.6 Corollary

For any quaternion-valued N-vector function $u(\mathrm{x}) = (u_1, u_2, \cdots, u_N)$, where $u_t \in \mathbb{H}$, $\forall t = 1,2,3, \cdots, N$ (Which means for every fixed t there exist $u_{t,s} \in \mathbb{R}$, $s = 0,1,2,3$ such that $u_t = u_{t,0} + u_{t,1}i + u_{t,2}j + u_{t,3}k$). If $u \in L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{H}^N)$, for $1 \le p \le \infty$. Then, u can be written as $u = U_1 + U_2 j$, where $U_1, U_2 \in L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{C}^N)$.

Proof

The proof of this corollary is a consequence of Lemma (3.5) and Theorem (3.4).

□

3.7 Remark

An interesting conclusion of Corollary (3.6), is that the space $C_c(\mathbb{R}^n, \mathbb{H}^N)$ is a dense subspace of the space $L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{H}^N)$, and the space $C_0(\mathbb{R}^n, \mathbb{H}^N)$ is the closure of the space $C_c(\mathbb{R}^n, \mathbb{H}^N)$ in the uniform metric.

4. Conclusion

In this paper, we introduced the definitions of the $L^{\mathrm{p}}(\mathbb{R}^n, \mathbb{H}^N), 1 \le \mathrm{p} \le \infty$, spaces of functions from the n-dimensional real space $\mathbb{R}^n$ to the N-dimensional quaternionic space $\mathbb{H}^N$, where the natural numbers $n, N \ge 1$ are not necessarily equal. To the best of our knowledge, the L^p- spaces for this kind of function have never been introduced before. Also, some important properties of the functions in these spaces are established and proved. In future studies, one may try to consider these new definitions instead of the previous studies that include the L^{p}- spaces to see and get more important results. For example, it may be used to extend the results of [15], and [16].

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