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e^*-Extending Modules

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Abstract
This paper aims to introduce the concepts of e^*-closed, e^*-coclosed, and e^*-extending modules as generalizations of the closed, coclossed, and extending modules, respectively. We will prove some properties as when the image of the e*-closed submodule is also e*-closed and when the submodule of the e*-extending module is e*-extending. Under isomorphism, the e*-extending modules are closed. We will study the quotient of e*-closed and e*-extending, the direct sum of e*-closed, and the direct sum of e*-extending.

Keywords: essential submodule, closed submodule, extending modules, e^*-essential submodule, e^*-closed submodule,e^*-coclosed submodule, e^*-extending modules.

المقاسات الموسعة من النمط $-e^*$

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الخلاصة
الهدف من هذه الورقة لتقديم المفاهيم المقاسات الجزئية المغلقة من النمط $-e^*$، المقاسات الجزئية المغلقة المضادة من النمط $-e^*$ والمقاسات الموسعة من النمط $-e^*$ كتعميم للمقاسات الجزئية المغلقة، المقاسات الجزئية المغلقة المضادة و المقاسات الموسوعة. سنقدم في هذه الورقة تعريف لهذه المفاهيم مع برهان خواصهم مثل متى ستكون صورة المقاسات الجزئية المغلقة من النمط $-e^*$ أيضا مغلقة من النمط $-e^*$ ومتى تكون المقاسات الجزئية من المقاسات الموسعة من النمط $-e^*$ هي أيضا موسعة من النمط $-e^*$. وسنرى انه تحت تأثير التشاكل فأن صفة المقاسات الموسعة من النمط $-e^*$ مغلقة. سندرس مقاسات القسمة لكل من المقاسات الجزئية المغلقة من النمط $-e^*$ و المقاسات الموسعة من النمط $-e^*$ و الجمع المباشر لهما.

1. Introduction

In this work M is a right module over a ring R with identity. $E(M)$ is the injective envelope of M. When $S+T=M$ implies $T=M$ for each $T\leq M$, S is called a small submodule of M, symbolized by $S \ll M$. See [1] and [2]. If $S\cap T\neq\{0\}$ for each $0\neq T\leq M$,

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then S is called an essential submodule of M, see[1] and[3]. A submodule S of module M is closed if S has no proper essential extension, see[3]. If every submodule of a module M is essential in the direct summand, then module is said to be extending. M is an extending module if and only if each of its closed submodules is a direct summand, see [4].

In [5], Ozcan introduced a new type of submodules which defined as $Z^*(M) = \{a \in M |\ aR\ is\ small\ in\ E(M)\}$. If $Z^*(M) = M$, then M is called cosingular. Whilst, in [6], Baanoon and Khalid introduced a class of submodules called e^*-essential. If $S \cap T \neq \{0\}$ for each cosingular T where $0 \neq T \leq M$, S is called an e^*-essential submodule of M, symbolized by $S \leq_{e^*} M$. Also, in [7], the same authors used e^*-essential submodules to present a new class of submodules, a generalization of a small submodule, called e^*-essential small. If $S + T = M$ implies $T = M$ for each $T \leq_{e^*} M$, S is called an e^*-essential small submodule of M symbolized by $S \ll_{e^*} M$. The generalization of the radical submodule which is called e^*-radical denoted by, $Rad_{e^*}(M)$ and defined as the intersection of all e^*-essential maximal submodule of a module M. Equivalently, $Rad_{e^*}(M) = \sum_{N \ll_{e^*} M} N$, see [7]. If each proper submodule of M is e^*-essential small, then M is anointed e^*-hollow, where M is a nonzero module, see [7].

As in[8], we will use e^*-essential and e^*-essential small submodules to present a new generalization of closed, coclosed submodules and extending modules. Namely e^*-closed submodules, e^*-coclosed submodules, and e^*-extending modules, respectively. Moreover,we will prove the main properties of these concepts.
Now, let us present the following proposition that is crucial to our work.

Proposition 1.1. Assume that M is a module, $\{L_\alpha\}_{\alpha \in \Lambda}$ is the collection of M's independent submodules, and $L_\alpha \leq_{e^*} L'_\alpha$ for each $\alpha \in \Lambda$, where L'_α is a submodule of M for each $\alpha \in \Lambda$. Then $\oplus_{\alpha \in \Lambda} L_\alpha \leq_{e^*} \oplus_{\alpha \in \Lambda} L'_\alpha$.

Proof. First, consider the case when the index set consists two members $\{L_1, L_2\}$, then by proposition 4 in[6], $L_1 \oplus L_2 \leq_{e^*} L'_1 \oplus L'_2$. Suppose that the result is correct for an index of $m-1$ items. Now, let $\{L_1, L_2, \ldots, L_m\}$ be independent family of submodules of M with $L_i \leq_{e^*} L'_i$ for each $i = 1, \ldots, m$. By the previous case we have $\oplus_i^{m-1} L_i \leq_{e^*} \oplus_i^{m-1} L'_i$. Since $L_m \leq_{e^*} L'_m$, we get $\oplus_i^m L_i \leq_{e^*} \oplus_i^m L'_i$. Finally, let $\{L_\alpha\}_{\alpha \in \Lambda}$ be the independent family of submodules of M and $L_\alpha \leq_{e^*} L'_\alpha$ for each $\alpha \in \Lambda$, let S be a non-zero cosingular of $\oplus_{\alpha \in \Lambda} L'_\alpha$. So S contains a nonzero element which belong to $L'_{\alpha(1)} \oplus \ldots \oplus L'_{\alpha(m)}$ for some $\alpha(i)$. As a result $0 \neq S \cap \left(L'_{\alpha(1)} \oplus \ldots \oplus L'_{\alpha(m)}\right) \leq S$, the submodule of cosingular is cosingular[5], so $S \cap \left(L'_{\alpha(1)} \oplus \ldots \oplus L'_{\alpha(m)}\right)$ is a nonzero cosingular submodule. Since $L_{\alpha(1)} \oplus \ldots \oplus L_{\alpha(m)} \leq_{e^*} L'_{\alpha(1)} \oplus \ldots \oplus L'_{\alpha(m)}$. Hence, $S \cap \left(L'_{\alpha(1)} \oplus \ldots \oplus L'_{\alpha(m)}\right) \cap \left(L_{\alpha(1)} \oplus \ldots \oplus L_{\alpha(m)}\right) \neq 0$ and consequently $S \cap \oplus_{\alpha \in \Lambda} L_\alpha \neq 0$. Therefore, $\oplus_{\alpha \in \Lambda} L_\alpha \leq_{e^*} \oplus_{\alpha \in \Lambda} L'_\alpha$.

2. e*-Closed submodules

In this section, we will prove some properties of e*-closed, as introduce in [5].

Definition 2.1 [6]

A submodule S of a module M is ***e*-closed*** in M, if$S$ has no proper e*-essential extension, (symbolized by $S \leq_{e*c} M$).

Definition 2.2

Suppose that S_1 and S_2 are submodules of a module M. Then S_2 is called ***e*-closure*** of S_1 if S_1 is e*-essential in S_2 and S_2 is e*-closed in M. For example, in the $\mathbb{Z}$-module $\mathbb{Z}_{12}$, we have that $\langle\bar{3}\rangle$ is e*-closure of $\langle\bar{6}\rangle$, since $\langle\bar{6}\rangle$ is e*-essential in $\langle\bar{3}\rangle$ and $\langle\bar{3}\rangle$ is e*-closed in M.

Examples and Remarks 2.3

1- For any cosingular module $M, \{0\}$ is e*-closed, if $\{0\} \leq_{e^*} B \leq M$, then$\{0\} \cap B = \{0\}$ and $B = \{0\}$ (submodule of cosingular module is cosingular [4]). When M is not a cosingular module, that is not generally true. For instance, the $\mathbb{Z}_6$-module $\mathbb{Z}_6$, $\{\bar{0}\}$ is not e*-closed since $\{\bar{0}\} \leq_{e^*} \mathbb{Z}_6$.
2- Every e*-closed submodule is closed. The opposite need not always be true. For instance, $\mathbb{Z}_6$ as a $\mathbb{Z}_6$-module $\langle\bar{2}\rangle$ is closed in $\mathbb{Z}_6$ but not e*-closed, see [6].
3- Assume that M is a cosingular module. Then e*-closed and closed submodules coincide.
4- Let the submodule S of M be e*-closed and e*-essential. Then $S = M$.
5- Every direct summand of a module M is known to be closed in M. However, there is no association with direct summand if e*-closed.For instance, in the $\mathbb{Z}_6$-module$\mathbb{Z}_6, \langle\bar{3}\rangle$ is a direct summand of $\mathbb{Z}_6$ but not an e*-closed submodule.
6- It is not necessary for a module M's intersection of e*-closed submodules to be e*-closed. For instance, in the $\mathbb{Z}$-module $\mathbb{Z}\oplus\mathbb{Z}_2$, let $S_1 = \mathbb{Z}\oplus\{\bar{0}\}$ and $S_2 = \mathbb{Z}(1, \bar{1})$ which are e*-closed submodules in $\mathbb{Z}\oplus\mathbb{Z}_2$, since S_1 and S_2 has no proper e*-essential extension in $\mathbb{Z}\oplus\mathbb{Z}_2$. But $S_1 \cap S_2 = (2, \bar{0})\mathbb{Z} \leq_{e^*} \mathbb{Z}\oplus\{\bar{0}\}$. So $S_1 \cap S_2$ is not e*-closed.
The fundamental characteristics of e*-closed submodules are presented.

Proposition 2.4 [6]
Assume that M is a module, if $S_1 \leq S_2 \leq_{e^*} M$ and $S_1 \leq_{e^*C} M$, then $\frac{S_2}{S_1} \leq_{e^*} \frac{M}{S_1}$.

Proposition 2.5 Assume that $g: M \to W'$ is an epimorphism and $S \leq_{e^*C} M$ such that $\ker(g) \leq S$. Then $g(S) \leq_{e^*C} W'$.

Proof. Suppose that $L' \leq W'$ with $g(S) \leq_{e^*} L'$ .Then $g^{-1}g(S) \leq g^{-1}(L') \leq M$ from proposition 2 in [5] we have $g^{-1}g(S) \leq_{e^*} M$ and from proposition1 in [5] $g^{-1}g(S) \leq_{e^*} g^{-1}(L')$, since $\ker(g) \leq S$, we have $g^{-1}g(S) = Ker(g) + S = S$, so $S \leq_{e^*} g^{-1}(L')$. But S is e*-closed in M; therefore, $S = g^{-1}(L')$ and $g(S) = L'$. Thus, $g(S)$ is e*-closed in W'.

Corollary 2.6 Under isomorphism, the e*-closed submodule is closed.

Corollary 2.7 Suppose that T_1 and T_2 are submodules of M with $T_1 \leq T_2$. If $T_2 \leq_{e^*C} M$, then $\frac{T_2}{T_1}$ is e*-closed in $\frac{M}{T_1}$.

Proposition 2.8 Let $S_1 \leq M$. Then M has an e*-closed submodule S_2 such that $S_1 \leq_{e^*} S_2$.

Proof. Consider $\Lambda = \{S_3 \leq M | S_1 \leq_{e^*} S_3\}$, $\Lambda \neq \emptyset$ since $S_1 \in \Lambda$ and every nonempty chain in Λ has an upper-bounded in Λ, hence Λ has a maximal element, say S_2, according to Zorn's lemma, with $S_1 \leq_{e^*} S_2$. Claim that $S_2 \leq_{e^*C} M$. Assume that there exists $S_2{}' \leq M$ such that $S_2 \leq_{e^*} S_2{}'$. Hence $S_1 \leq_{e^*} S_2{}'$, so $S_2{}' \in \Lambda$. But S_2 is a maximal element in Λ, hence $S_2 = S_2{}'$. Thus S_2 is an e*-closed submodule in M.

Proposition 2.9
Suppose that M is a module, S_1 and S_2 are submodules of M with $S_1 \leq S_2$. If $S_1 \leq_{e^*C} M$, then $S_1 \leq_{e^*C} S_2$.

Proof. Assume that S_1 is an e*-essential submodule of L, where L is a submodule of S_2. Since $S_1 \leq_{e^*C} M$. Hence $S_1 = L$. Thus, S_1 is e*-closed in S_2.

A module M is considered chained if either $S_1 \leq S_2$ or $S_2 \leq S_1$holds true for each of its submodules $S_1$and S_2. See[9].

Proposition 2.10 Assume that M is a chained module, T_1 and T_2 are submodules of M with $T_1 \leq T_2$. If $T_1 \leq_{e^*C} T_2$ and $T_2 \leq_{e^*C} M$, then $T_1 \leq_{e^*C} M$.

Proof. Suppose that $U \leq M$ with T_1 is the e*-essential submodule of U. By the hypothesis has two cases:
Case I: If $U \leq T_2$ since T_1 is e*-closed in T_2. Hence $T_1 = U$. Thus, T_1 is e*-closed in M.
Case II: If $T_2 \leq U$ since T_1 is an e*-essential submodule of U. Hence T_1 is the e*-essential submodule of T_2 and T_2 is the e*-essential submodule of U. But T_1 is e*-closed in T_2 and T_2 is e*-closed in M; therefore, $T_1 = T_2 = U$. Thus, T_1 is e*-closed in M.

The following proposition proves that the direct sum of e*-closed submodules is an e*-closed submodule.

Proposition 2.11 Suppose that W_1 and W_2 are modules with $T_1 \leq W_1$ and $T_2 \leq W_2$. If $T_1 \leq_{e^*C} W_1$ and $T_2 \leq_{e^*C} W_2$, then$T_1 \oplus T_2 \leq_{e^*C} W_1 \oplus W_2$.

Proof. Let $T_1 \oplus T_2 \leq_{e^*} U_1 \oplus U_2$, where $U_1 \leq W_1$ and $U_2 \leq W_2$. Consider the inclusion maps $i_1: U_1 \to U_1 \oplus U_2$ and $i_2: U_2 \to U_1 \oplus U_2$. Since $T_1 \oplus T_2 \leq_{e^*} U_1 \oplus U_2$, then ${i_1}^{-1}(T_1 \oplus T_2) \leq_{e^*} {i_1}^{-1}(U_1 \oplus U_2)$ and ${i_2}^{-1}(T_1 \oplus T_2) \leq_{e^*} {i_2}^{-1}(U_1 \oplus U_2)$. ${i_1}^{-1}(T_1 \oplus T_2) = \{u_1 \in U_1 | \, i_1(u_1) = u_1 \in T_1 \oplus T_2\} = T_1$, ${i_1}^{-1}(U_1 \oplus U_2) = U_1$, ${i_2}^{-1}(T_1 \oplus T_2) = T_2$ and ${i_2}^{-1}(U_1 \oplus U_2) = U_2$. But $T_1 \leq_{e^*C} W_1$ and $T_2 \leq_{e^*C} W_2$. Hence $T_1 = U_1$ and $T_2 = U_2$. Thus, $T_1 \oplus T_2 \leq_{e^*C} W_1 \oplus W_2$.

3. e*-Coclosed submodules

In this section, we will introduce a new concept which is a generalization of coclosed, and prove some properties as in [10] and[11].

Definition 3.1 [7]
Let $T \leq S$ be submodules of M.When $\frac{S}{T} \ll_{e^*} \frac{M}{T}$ implies that $S = T$. S is said to be an ***e**-coclosed submodule***of M (symbolizedby $S \leq_{e^*cc} M$).

Examples and Remarks 3.2
1. Every e^*-coclosed submodule is coclosed.
Let M be a module, S be an e^*-coclosed submodule of M, and T a submodule of S such that $\frac{S}{T} \ll \frac{M}{T}$. Every small is e^*-essential small. As a result, $\frac{S}{T} \ll_{e^*} \frac{M}{T}$, because S is an e^*-coclosed. Thus, $S = T$ and S is a coclosed submodule of M.

2. The opposite of (1) need not always be accurate. For instance, the only proper submodule of $\langle\bar{3}\rangle$ in $\mathbb{Z}_6$ as a $\mathbb{Z}$-module is $\langle\bar{0}\rangle$, $\frac{\langle\bar{3}\rangle}{\langle\bar{0}\rangle} \simeq \langle\bar{3}\rangle$, and $\frac{\mathbb{Z}_6}{\langle\bar{0}\rangle} \simeq \mathbb{Z}_6$. So $\langle\bar{3}\rangle$ is cocolosed in $\mathbb{Z}_6$, but it is not e^*-coclosed in $\mathbb{Z}_6$.
3. In $\mathbb{Z}_6$ as a $\mathbb{Z}_6$-module, $\langle\bar{2}\rangle$ is e^*-coclosed in $\mathbb{Z}_6$. Since the only proper submodule of $\langle\bar{2}\rangle$ is $\langle\bar{0}\rangle$, $\frac{\langle\bar{2}\rangle}{\langle\bar{0}\rangle} \simeq \langle\bar{2}\rangle$, and $\frac{\mathbb{Z}_6}{\langle\bar{0}\rangle} \simeq \mathbb{Z}_6$. $\langle\bar{2}\rangle$ is not an e^*-essential small submodule of $\mathbb{Z}_6$.
4. In $\mathbb{Z}$ as a $\mathbb{Z}$-module, $2\mathbb{Z}$ is not e^*-coclosed submodule of $\mathbb{Z}$. Since there is a proper submodule $4\mathbb{Z}$ of $2\mathbb{Z}$, $\frac{2\mathbb{Z}}{4\mathbb{Z}} \simeq \langle\bar{2}\rangle$, and $\frac{\mathbb{Z}}{4\mathbb{Z}} \simeq \mathbb{Z}_4$. $\langle\bar{2}\rangle$ is an e^*-essential small submodule of $\mathbb{Z}_4$.
5. Direct summand of a module need not be e^*-coclosed. For instance, the submodule $\langle\bar{3}\rangle$ is a direct summand of $\mathbb{Z}_6$ as a $\mathbb{Z}$-module, but it is not e^*-coclosed in $\mathbb{Z}_6$.
6. Let M be an e^*-hollow module. Then M has only one proper e^*-coclosed, which is a zero submodule. Let T be a proper submodule of M. Then $T \ll_{e^*} M$ and so $\frac{T}{\{0\}} \ll_{e^*} \frac{M}{\{0\}}$. Thus, if T is e^*-coclosed in M, then $T = \{0\}$.

The next proposition gives the basic properties of e^*-coclosed submodules.

Proposition 3.3 Let M be a module and let $A_1 \leq A_2 \leq M$.
1) If A_2 is e^*-coclosed in M, then $\frac{A_2}{A_1}$ is e^*-coclosed in $\frac{M}{A_1}$.
2) If $A_1 \ll A_2$ and $\frac{A_2}{A_1}$ is e^*-coclosed in $\frac{M}{A_1}$, then A_2 is e^*-coclosed in M.
3) If A_1 is e^*-coclosed in M, then A_1 is e^*-coclosed in A_2.

Proof.
1) Let $\frac{L}{A_1} \leq \frac{A_2}{A_1}$ such that $\frac{A_2/A_1}{L/A_1} \ll_{e^*} \frac{M/A_1}{L/A_1}$ by (the second isomorphism theorem), $\frac{A_2/A_1}{L/A_1} \simeq \frac{A_2}{L}$ and $\frac{M/A_1}{L/A_1} \simeq \frac{M}{L}$. As a result, $\frac{A_2}{L} \ll_{e^*} \frac{M}{L}$, since A_2 is e^*-coclosed in M. Thus, $A_2 = L$ and $\frac{L}{A_1} = \frac{A_2}{A_1}$. Therefore, $\frac{A_2}{A_1}$ is e^*-coclosed in $\frac{M}{A_1}$.
2) Suppose that $L \leq A_2$ such that $\frac{A_2}{L} \ll_{e^*} \frac{M}{L}$. Define $\lambda: \frac{M}{L} \longrightarrow \frac{M}{L+A_1}$ by $\lambda(m + L) = m + L + A_1$ for each $m \in M$. Easley sees that λ is an epimorphism, so by proposition 3 in [7], $\frac{A_2}{L+A_1} \simeq \frac{A_2/A_1}{L+A_1/A_1} \ll_{e^*} \frac{M/A_1}{L+A_1/A_1} \simeq \frac{M}{L+A_1}$. Since $\frac{A_2}{A_1}$ is e^*-coclosed in $\frac{M}{A_1}$, so $\frac{L+A_1}{A_1} = \frac{A_2}{A_1}$ and $A_2 = L + A_1$. Since $A_1 \ll A_2$; thus $A_2 = L$. Therefore, A_2 is e^*-coclosed in M.
3) Let $L \leq A_1$ such that $\frac{A_1}{L} \ll_{e^*} \frac{A_2}{L} \leq \frac{M}{L}$ So by proposition 1 in [7], $\frac{A_1}{L} \ll_{e^*} \frac{M}{L}$. Since A_1 is e^*-coclosed in M, so $L = A_1$. Therefore, A_1 is e^*-coclosed in A_2.

Proposition 3.4 Let $M = M_1 \oplus M_2$ be a module, and $A \leq_{e^*cc} M_1$. Then $A \leq_{e^*cc} M$.

Proof. Let $A' \leq A$ such that $\frac{A}{A'} \ll_{e^*} \frac{M}{A'} = \frac{M_1 \oplus M_2}{A'}$. Hence $\frac{A}{A'} \ll_{e^*} \frac{M_1}{A'} \oplus \frac{A' \oplus M_2}{A'}$. So $\frac{A}{A'} \ll_{e^*} \frac{M_1}{A'}$ by corollary 1 in [7]. Since $A \leq_{e^*cc} M_1$. Therefore, $A' = A$ and $A \leq_{e^*cc} M$.

Proposition 3.5 Let M be a module and A a nonzero submodule of M. If $A \leq_{e^*cc} M$, then A is not e^*-essential small in M.

Proof. Assume A is e^*-essential small in M and $A \leq_{e^*cc} M$. Because $\{0\} \leq A$ and $A \simeq \frac{A}{\{0\}} \ll_{e^*} \frac{M}{\{0\}} \simeq M$. Then $A = \{0\}$ which is a contradiction. Therefore, A is not an e^*-essential small in M.

4. e*-Extending modules.

We will present a new idea in this part, a generalization of the extending module as in [12], [13] and [14].

Definition 4.1 If every submodule of a module M is e*-essential in a direct summand, the module is said to be ***e*-extending***.

Remarks and Examples 4.2

1. Each extending is an e*-extending module.
2. If M is a cosingular module, then e*-extending and extending modules are coincide.
3. The polynomial ring $R = \mathbb{Z}[x]$ is a commutative Noetherian domain such that $W = \mathbb{Z}[x] \oplus \mathbb{Z}[x]$ as R-module is not extending [4]. Since $\mathbb{Z}[x]$ is a commutative domain which not filed so by Theorem 2.10, [5] R is a right cosingular ring and by Corollary 2.7, [5] any right R-module is cosingular module. Hence W is cosingular R-module, from (2) W is not e*-extending.
4. The direct sum of e*-extendingis not e*-extending. For instance, the $\mathbb{Z}[x]$-module $\mathbb{Z}[x]$is e*-extending because $\mathbb{Z}[x]$ is an integral domain, every non-zero ideal in the integral domain is essential [3], so $\mathbb{Z}[x]$ is extending, hence by (1), $\mathbb{Z}[x]$ is e*-extending. But $\mathbb{Z}[x] \oplus \mathbb{Z}[x]$ as $\mathbb{Z}[x]$-moduleis not e*-extending.
5. Assume P is a prime number. Then the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is e*-extending module.

The fundamental characteristics of e*-extending modules are then presented.

Proposition 4.3 If the module M is an indecomposable, then M is e*-extending if and only if each of its nonzero cyclic submodules is e*-essential in M.

Proof. ($\Longrightarrow$)Clear.
($\Longleftarrow$) Suppose that S is a non-zero submodule and $0 \neq s \in S$. Hence sR is e*-essential in M. Because $sR \leq S \leq M$, hence $S \leq_{e^*} M$. Therefore, M is e*-extending.

The following shows under which condition makes the e*-extending hereditary property.

Proposition 4.4 If M is an e*-extending module and S is a submodule of M such that the intersection of S with any direct summand of M is a direct summand of S, then S is an e*-extending module.

Proof. Let L be a submodule of S ,because M is an e*-extending. There exists a direct summand S_1 of M, with $L \leq_{e^*} S_1$. By the hypothesis, $S \cap S_1$ is a direct summand of S and $L = L \cap S \leq_{e^*} S_1 \cap S$. Thus, S is an e*-extending module.
Recall that a module M is called duo, if every submodule of M is fully invariant, see [15].
Recall that a module M is called distributive if its lattice of submodulesis a distributive lattice, that is, $A \cap (B + C) = (A \cap B) + (A \cap C)$ for any submodules A, B and C of M. See [16].

Proposition 4.5 If M is a duo (or distributive) e*-extending module, then each submodule of M is e*-extending.

Proof. Let S be a submodule of M and S_1 be a submodule of S; because M is an e*-extending. There exists a direct summand L of M, with $S_1 \leq_{e^*} L$, $M = L \oplus L'$. $S = S \cap M = S \cap (L \oplus L')$, but M is a duo (distributive), $S = (S \cap L) \oplus (S \cap L')$. So, $S \cap L$is a direct summand of S and $S_1 = S_1 \cap S \leq_{e^*} S \cap L$. Thus, S is an e*-extending module.

The next proposition gives the characterization of e*-extending modules.

Proposition 4.6 An R-module W is an e*-extending if and only if every e*-closed submodule is a direct summand.

Proof.($\Longrightarrow$) Let S be an e*-closed submodule of W. Since W is e*-extending; there is a direct summand L of W with $S \leq_{e^*} L$. But S is e*-closed. Hence, $S = L$.
($\Longleftarrow$) Let S be a submodule of W. Then, by Proposition 2.8. an e*-closed submodule L exists with $S \leq_{e^*} L$. By the hypothesis, L is a direct summand. Therefore, W is an e*-extending.

Corollary 4.7 Under isomorphism, the e*-extending module is closed.
Proof. Clear using the corollary 2.6.

The direct summand of the e*-extending module is e*-extending, as shown by the following proposition.

Proposition 4.8 A direct summand of e*-extending module is e*-extending.

Proof. Let S be a direct summand of an e*-extending module W. There is a submodule S' of W such that $W = S \oplus S'$. Let L be an e*-closed submodule of S. Hence, $L \oplus S' \leq_{e^*C} S \oplus S' = W$,since W is an e*-extending, so by proposition 4.6. $L \oplus S'$ is a direct summand of W, then $W = L \oplus S' \oplus K$, for some submodule K of W . $S = S \cap W = S \cap (L \oplus S' \oplus K)$ = $(S \cap L) \oplus (S \cap (S' \oplus K))$= $L \oplus (S \cap (S' \oplus K))$. Hence, L is a direct summand of S. Thus, S is e*-extending.

Theorem 4.9 Let W be an R-module. Then the following statements are equivalent.
1.W is e*-extending module.
2. For every submodule S of W, there is a decomposition $W = L \oplus L'$, such that $S \leq L$ and $S + L' \leq_{e^*} W$.
3. For every submodule S of W, there is a decomposition $\frac{W}{S} = \frac{L}{S} \oplus \frac{K}{S}$, such that L is a direct summand of W and $K \leq_{e^*} W$.

Proof.
1 $\Longrightarrow$ 2) Let S be a submodule of W, there is a direct summand L of W such that $S \leq_{e^*} L$, $W = L \oplus L'$ for some $L' \leq W$. By proposition 4 in [6]. $S \oplus L' \leq_{e^*} L \oplus L' = W$. Then $S + L' \leq_{e^*} W$.
2 $\Longrightarrow$ 3) Let S be a submodule of W, there is a decomposition $W = L \oplus L'$, such that $S \leq L$ and $S + L' \leq_{e^*} W$. $\frac{W}{S} = \frac{L \oplus L'}{S} = \frac{L}{S} + \frac{L'+S}{S}$, Since $L \cap (L' + S) = S$. Hence, $\frac{W}{S} = \frac{L}{S} \oplus \frac{L'+S}{S}$. Put $K = L' + S$.
3 $\Longrightarrow$ 1)Let S be a submodule of W, there is a decomposition $\frac{W}{S} = \frac{L}{S} \oplus \frac{K}{S}$, such that L is a direct summand of W and $K \leq_{e^*} W$. To show that $S \leq_{e^*} L$. Since $K \leq_{e^*} W$, then $S = K \cap L \leq_{e^*} W \cap L = L$. Thus, W is e*-extending module.

Proposition 4.10 Let W be an e*-extending module and S be an e*-closed submodule. Then $\frac{W}{S}$ is an e*-extending module.

Proof. Since S is an e*-closed submodule of e*-extending module W. Hence S is a direct summand of W, $W = S \oplus S'$, for some $S' \leq W$. $\frac{W}{S} \simeq S'$ since S' is a direct summand of W. So by Proposition 4.8. S' is e*-extending, and by Corollary 4.7, $\frac{W}{S}$ is an e*-extending module.

Corollary 4.11 Let $f: W \to W'$ be R-homomorphism, and W e*-extending with $kerf$ is e*-closed. Then $f(W)$ is e*-extending.

We present enough requirements for the direct sum of e*-extending modules to be an e*-extending module.

Proposition 4.12 Let $W = W_1 \oplus W_2$ be a distributive module. If W_1 and W_2 are e*-extending modules, then W is e*-extending.

Proof. Let S be a submodule of W. Since W is a distributive module, so $S = S \cap W = S \cap (W_1 \oplus W_2) = (S \cap W_1) \oplus (S \cap W_2)$. Since W_1 and W_2 are e*-extending modules, then there exists a direct summand S_1 of W_1 and S_2 of W_2 such that $S \cap W_1 \leq_{e^*} S_1$ and $S \cap W_2 \leq_{e^*} S_2$. Hence $S \leq_{e^*} S_1 \oplus S_2$, where $S_1 \oplus S_2$ is a direct summand of W. Thus, W is e*-extending.

Proposition 4.13 Let $W = \oplus_{i \in I} W_i$ be an R-module. Where W_i is a submodule of W for each $i \in I = \{1, \dots, n\}$. If W_i is e*-extending for each $i \in I$ and every e*-closed submodule is fully invariant, then W is e*-extending.

Proof. Let S be e*-closed submodule of W. By the hypothesis S is a fully invariant . Hence, $S = S \cap W = S \cap (\oplus_{i \in I} W_i) = \oplus_{i \in I}(S \cap W_i)$. Since W_i is e*-extending with $S \cap W_i \leq W_i$ for each $i \in I$, then there exists a direct summand L_i of W_i for each $i \in I$ such that $S \cap W_i \leq_{e^*} L_i$. Hence, by Proposition 1.1, $= \oplus_{i \in I}(S \cap W_i) \leq_{e^*} \oplus_{i \in I} L_i$. But S is an e*-closed, so $S = \oplus_{i \in I} L_i$ is a direct summand of W. Therefore, W is e*-extending. ■

5. Conclusions.

We Confirm the following outcomes:

1. Under isomorphism, the e*-closed submodule is closed.
2. Every submodule is e*-essential in e*-closed.
3. The direct sum of e*-closed submodules is e*-closed.
4. Every e^*-coclosed submodule is coclosed.
5. The direct sum of e*-extending is not e*-extending.
6. The direct summand of the e*-extending module is e*-extending.

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