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On $(k, m) - n$ −Paranormal Operators

Salim Dawood Mohsen
Mathematics Department, College of Education, Mustansiriyah University, Baghdad, Iraq

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Abstract
The new type of paranormal operators that have been defined in this study on the Hilbert space, is $(k, m) - n$ −paranormal operators. In this paper we introduce and discuss some properties of this concept such as: the sum and product of two $(k, m) - n$ −paranormal, the power of $(k, m) - n$ −paranormal. Further, the relationships between the $(K, m) - n$ −paranormal operators and other kinds of paranormal operators have been studied.

Keywords: (k, m)-n-paranormal operator, paranormal operator.

حول المؤثرات فوق السوية $(k, m) - n -$

سالم داود محسن
قسم الرياضيات ، كلية التربية، الجامعة المستنصرية، بغداد ، العراق

الخلاصة
نوع جديد من المؤثرات فوق السوية سوف يعرف في هذه الدراسة على فضاء هلبرت وهوة المؤثرات فوق السوية $(k, m) - n -$ والمعرفة على فضاءات هلبرت' في هذا للبحث قدمنا وناقشنا بعض خواص هذا المفهوم ومنها : جمع وضرب اثنين من المؤثرات فوق السوية $(k, m) - n -$ ، قوى فوق السوية $(k, m) - n -$ كذلك ناقشنا العلاقات بين المؤثر فوق السوي $(k, m) - n -$ مع الصنوف الاخرى من المؤثرات.

1. Introduction

Many authors in the current study in recent years focus own studied in special type of bounded linear operator named paranormal operators and begging give some generalized of this type of operator, such as, Arora and Thukral [1]. In 1987 also defined the class M*-paranormal and introduced topological properties with characterization of this concept. In 2012 the class K-quasi-paranormal with some basic properties and characterization has been introduced by Mecher, Salah [2]. Zuo, Fei in 2015 introduced another class which named quasi-*-n-paranormal and studied spectrum properties [3]. In add, k-quasi-*-paranormal was presented with it's properties by Rashid in 2016 [4] . In this paper, we shall introduce a new class allegedly operates as a (k, m)-n-paranormal, which is generalization of paranormal operator, also giving some theorems and illustrate some examples for its properties such as: the sum and product of two (k, m)-n-paranormal, the power of (k, m)-n-paranormal , also the tensor product

*Email: dr_salim2015@yahoo.com

between (k, m)-n-paranormal and the identity operator, alongside that we explain relationships between the (K, n)-n-paranormal operators and other kinds of paranormal operators.

2. Preliminaries

Definition 2.1: [5-7]

If $T \in B(H)$, then the follows as:

$T^*T = TT^*$, $T^*T = I$ and $\|h_x\|^2 \leq \|{h_x}^2\|$, to every $x \in H$, is called normal, isometry and paranormal operator, respectively, where I is the identity operator.

In this definition that follows, we will now review the notion of classes paranormal operators

Definition 2.2: [8-11]

i. The operator $T \in B(H)$ in [10]is *-n-paranormal if $\|{T^*}_x\|^{n+1} \leq \|{T^{n+1}}_x\|\|x\|^n$, $\forall x \in H$.

ii. The operator $T \in B(H)$ is n-paranormal operator if $\|T_x\|^{n+1} \leq \|{T^{n+1}}_x\|\|x\|^n$, $\forall x \in H, n \geq 1$

iii. The operator $T \in B(H)$ is *-n-paranormal if $\|{T^*}_x\|^{n+1} \leq \|{T^{n+1}}_x\|\|x\|^n$, $\forall x \in H$.

vi. The operator $T \in B(H)$ is (m,n)-paranormal operator if $\|T_x\|^{n+1} \leq m\|{T_x}^{n+1}\| \|x\|^n$, $\forall x \in H$.

3. Main results

Definition 3.1:

For any positive real number K and any positive integers n and m, the bounded linear operator $T: H \to H$ is called (k,m)-n-paranormal in that case: $\|{T^m}_x\|^{1+n} \leq k\|T^n({T^m}_x)\|\|x\|^n$ for all $x \in H$.

Theorem 3.1:

For the positive integer numbers n, m, and real number φ the operator $T \in B(H)$ is (k,m)-n-paranormal if and only if:

$$k^{\frac{2}{1+n}}{T^*}^m\, {T^*}^n\, T^n\, T^m - (1+n)\varphi^n {T^*}^m\, T^m + k^{\frac{2}{1+n}} n\varphi^{1+n} I \geq 0,\ \forall\varphi > 0 \qquad \text{(a)}$$

Proof:

By using Definition 3.1 " $\|{T^m}_x\|^{1+n} \leq k\|T^n({T^m}_x)\|\|x\|^n$ for all $x \in H$ " we may obtain, $\|{T^m}_x\|^2 \leq k^{\frac{2}{1+n}}\|T^n({T^m}_x)\|^{\frac{2}{1+n}}\|x\|^{\frac{2n}{n+1}}$,

And by the generalized arithmetic-geometric mean inequality:

$$\frac{k^{\frac{2}{1+n}}\varphi^{-n}}{1+n}\|T^n({T^m}_x)\|^2 + \frac{k^{\frac{2}{1+n}}\varphi\, n}{1+n}\|x\|^2 \geq k^{\frac{2}{1+n}}(\varphi^{-n}\|T^n({T^m}_x)\|^2)^{\frac{1}{1+n}}(\varphi\|x\|^2)^{\frac{n}{1+n}}$$
$$= k^{\frac{2}{1+n}}\|T^n({T^m}_x)\|^{\frac{2}{1+n}}\|x\|^{\frac{2n}{1+n}}$$
$$\geq \|{T^m}_x\|^2$$

So: $k^{\frac{2}{1+n}}{T^*}^m\, {T^*}^n\, T^n T^m - (1+n)\varphi^n {T^*}^m T^m + k^{\frac{2}{1+n}} n\varphi^{1+n} I \geq 0$, for all $\varphi > 0$.

By the generalized arithmetic-geometric mean inequality, we get

$\frac{\varphi^{-n}}{1+n} + \frac{n\varphi}{1+n} = 1$, this means, $\varphi^n = \frac{1}{(1-\varphi)^{n+1}}$.

Conversely, suppose that the equation (a) is holds, when $x \in H$, with $\|T^n({T^m}_x)\| = 0$, multiple (a) by φ^{-n}, we get, $\|{T^m}_x\|^{1+n} = 0$, because that we have

$$\|{T^m}_x\|^{1+n} \leq k\|T^n({T^m}_x)\|\|x\|^n$$

when $x \in H$, with $\|T^n({T^m}_x)\| > 0$, and $\varphi = \left(\frac{\|T^n({T^m}_x)\|}{\|x\|}\right)^{\frac{2}{1+n}}$ in equation (a):

$$\|{T^m}_x\|^{1+n} \leq k\|T^n({T^m}_x)\|\|x\|^n$$

So T is (k, m)-n-paranormal operator.

Now, we illustrate the relation among (k, m)-n-paranormal with another classes that appeared in definition 2.

Remarks 3.2:
i. When $m = n = k = 1$, with T is self-adjoint, the (k, m)-n-paranormal is become *-paranormal
ii. When $m = k = 1$, the (k,m)-n-paranormal is become n-paranormal.
iii. When $m = k = 1$, with T is self-adjoint, the (k, m)-n-paranormal is become n-*-paranormal.
vi. The (k, m)-n-paranormal becomes (m, n)-paranormal when m equals 1.

Proposition 3.3:
An operator $T \in B(H)$, was (k,m)-n-paranormal, then:
i. δT is (k, m)-n-paranormal, δ is real number
ii. T^a is (k, m)-n-paranormal, where a is positive integer number.

Proof:
i. Let $\delta \neq 0$ with $k, n \geq 1$, by Theorem 3.2:

$$k^{\frac{2}{1+n}}(\delta T)^{*m}(\delta T)^{*n}(\delta T)^{n}(\delta T)^{m} - (1+n)\varphi^{n}(\delta T)^{*m}(\delta T)^{m}$$
$$= k^{\frac{2}{1+n}}\delta^{*m}\delta^{*n}\delta^{\ n}\delta^{\ m}T^{*m}T^{*n}T^{\ n}T^{\ m} - (1+n)\varphi^{n}\delta^{*m}\delta^{\ m}T^{*m}T^{\ m}$$
$$= k^{\frac{2}{1+n}}(|\delta|^2)^{m+n}(T^{*m}T^{*n}T^{\ n}T^{\ m}) - (1+n)\varphi^{n}(|\delta|^2)^{m}$$
$$= (|\delta|^2)^{m+n}\left(k^{\frac{2}{1+n}}\,T^{*\ m}\,T^{*\ n}\,T^{\ n}\,T^{\ m} - (1+n)\left(\frac{\varphi}{|\delta|^2}\right)^{n}T^{*\ m}\,T^{\ m}\right)$$
$$\geq -k^{\frac{2}{1+n}}n\varphi^{1+n}I$$

For that, δT is an (k, m)-n-paranormal.
ii. By using the mathematical induction, we shill prove T^a is (k,m)-n-paranormal
Suppose the result is true for
1) $a = 1$, so $\|T^{m}{}_{x}\|^{1+n} \leq k\|T^{n}(T^{m}{}_{x})\|\|x\|^{n}$.
2) $a = l$, $(\|T^{m}{}_{x}\|^{1+n})^{l} \leq (k\|T^{n}(T^{m}{}_{x})\|\|x\|^{n})^{l}$.
3) Now, prove for$a = l + 1$,

$$(\|T^{m}{}_{x}\|^{1+n})^{l+1} = (\|T^{m}{}_{x}\|^{1+n})^{l}(\|T^{m}{}_{x}\|^{1+n})^{1}$$
$$\leq (k\|T^{n}(T^{m}{}_{x})\|\|x\|^{n})^{l}(k\|T^{n}(T^{m}{}_{x})\|\|x\|^{n})^{1}$$

By 1) and 2), we get,

$$= (k\|T^{n}(T^{m}{}_{x})\|\|x\|^{n})^{l+1}$$

$$(\|T^{m}{}_{x}\|^{1+n})^{l+1} \leq= (k\|T^{n}(T^{m}{}_{x})\|\|x\|^{n})^{l+1}$$

and by result, T^a is (k, m)-n-paranormal.

Remark 3.4:
The operators T and S should be two (k, m)-n-paranormal operators, however, $T + S$ is not always a (k,m)-n-paranormal operator, consider the example that follows

Example 3.5:
Let $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then, for each $\varphi > 0$. Consider the follows as:

$$T^{*3}T^{3} - 2\varphi T^{*2}T^{2} + \varphi^{2}T^{*}T = \begin{pmatrix} 5 - 2\varphi + 5\varphi^{2} & 0 \\ 0 & 1280 - 128\varphi + 5\varphi^{2} \end{pmatrix},$$

Which is implies that, for each $\varphi > 0$, therefore, T is (5,3)-1- -paranormal operator.

$5S^{*3}S^3 - 2\varphi S^{*2}S^2 + 5\varphi^2 S^*S = \begin{pmatrix} 0 & 5-2\varphi+5\varphi^2 \\ 5\varphi^2 & 0 \end{pmatrix}$,
Which is implies that for each $\varphi > 0$, so S is (5,3)-1-paranormal operator.
Since the operator $M = T + S = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, we get
$5M^{*3}M^3 - 2\lambda M^{*2}M^2 + 5\lambda^2 I = \begin{pmatrix} 5-4\lambda+5\varphi^2 & 75-14\varphi \\ 75-14\varphi & 2405-226\varphi+5\varphi^2 \end{pmatrix}$,
Which is not positive for $\varphi = 7$, so, M is not (5,3)-1 -paranormal.

Theorem 3.6:
If $T, S \in B(H)$ are two (k, m)-n-paranormal on Hilbert space H. If $TS = ST = T^*S = S^*T = 0$, then $T + S$ is (k, m)-n-paranormal.

Proof: Supposing that T, S are (k, m)-n-paranormal, by Theorem 3.1, we see that

$$k^{\frac{2}{1+n}}(T+S)^{*m}(T+S)^{n*}(T+S)^{n}(T+S)^{m}$$
$$= k^{\frac{2}{1+n}}(T^*+S^*)^{m}(T^*+S^*)^{n}(T+S)^{n}(T+S)^{m}$$
$$= k^{\frac{2}{1+n}}\left((T^{*m}+S^{*m})(T^{*n}+S^{*n})\right)\left((T^{n}+S^{n})(T^{m}+S^{m})\right)$$
$$= k^{\frac{2}{1+n}}(T^{*m}T^{*n}+T^{*m}S^{*n}+S^{*m}T^{*n}+S^{*m}S^{*n})(T^{n}T^{m}+T^{n}S^{m}+S^{n}T^{m}+S^{n}S^{m})$$
$$= k^{\frac{2}{1+n}}(T^{*m}T^{*n}T^{n}T^{m}+S^{*m}S^{*n}S^{n}S^{m})$$
$$\geq \left((1+n)\varphi^n T^{*m}T^{m} - k^{\frac{2}{1+n}}n\varphi^{1+n}I\right) + \left((1+n)\varphi^n S^{*m}S^{m} - k^{\frac{2}{1+n}}n\varphi^{1+n}I\right)$$
$$= (1+n)\varphi^n (T+S)^{*m}(T+S)^{m} - k^{\frac{2}{1+n}}n\varphi^{1+n}(I+I)$$

So: $k^{\frac{2}{1+n}}(T+S)^{*m}(T+S)^{*n}(T+S)^{n}(T+S)^{m} - (1+n)\varphi^n(T+S)^{*m}(T+S)^{m} + k^{\frac{2}{1+n}}n\varphi^{1+n}I \geq 0$, hence, $T+S$ is (k, m)-n-paranormal.

Theorem 3.7:
If $T, S \in B(H)$ are two (k,m)-n-paranormal and S is isometry operator, TS should be (k,m)-n-paranormal, when the following conditions are satisfy:
i. $TS^* = S^*T$
ii. $TS = ST$

Proof: Supposing that T, S are (k, m)-n-paranormal, by Theorem 3.2, we see that

$$k^{\frac{2}{1+n}}(TS)^{*m}(TS)^{*n}(TS)^{m}(TS)^{n} - (1+n)\varphi^n(TS)^{*m}(TS)^{m}$$
$= k^{\frac{2}{1+n}}(ST)^{*m}(ST)^{*n}(TS)^{m}(TS)^{n} - (1+n)\varphi^n(ST)^{*m}(TS)^{m}$, which implies that
$$k^{\frac{2}{1+n}}(T^{*m}S^{*m})(T^{*n}S^{*n})(T^{n}S^{n})(T^{m}S^{m}) - (1+n)\varphi^n(T^{*m}S^{*m})(T^{m}S^{m})$$
$$= k^{\frac{2}{1+n}}(T^{*m}(S^{*m}T^{*n})(S^{*n}T^{n})(S^{n}T^{m})S^{m}) - (1+n)\varphi^n(T^{*m}(S^{*m}T^{m})S^{m})$$
$$= k^{\frac{2}{1+n}}\left(T^{*m}(T^{*n}S^{*m})(T^{n}S^{*n})(S^{n}T^{m})S^{m}\right) - (1+n)\varphi^n(T^{*m}(T^{m}S^{*m})S^{m})$$

$= k^{\frac{2}{1+n}} \left((T^{*m}T^{*n}) \ (S^{*\,m}\,T^{\,n}) \ \ (S^{*\,n}S^{\,n})(T^{\,m}S^{\,m}) \right) -$
$(1+n)\varphi^n \left(T^{*m}\,T^{\,m}\left(S^{*m}S^{\,m}\right)\right)$

$= k^{\frac{2}{1+n}}\left((T^{*m}T^{*\,n}) \ (T^{\,n}\,S^{*m}) \ (S^{\,m}\,T^{\,m})\right) - (1+n)\varphi^n T^{*\,m}\,T^{\,m}$, which implies that

$k^{\frac{2}{1+n}}((T^{*m}T^{*n}) \ T^{\,n} \,(S^{*m}\,S^{\,m})\,T^{\,m}) - (1+n)\,\varphi^n T^{*\,m}\,T^{\,m}$

$= k^{\frac{2}{1+n}}\,T^{*m}T^{*n}T^{\,n}T^{\,m} - (1+n)\,\varphi^n T^{*m}T^{\,m} \geq -k^{\frac{2}{1+n}}n\varphi^{1+n}I$

So,

$$k^{\frac{2}{1+n}}(T\,S)^{*\,m}\,(T\,S)^{*\,n}\,(T\,S)^{\,m}(T\,S)^{\,n} - (1+n)\varphi^n (T\,S)^{*\,m}\,(T\,S)^{\,m} + k^{\frac{2}{1+n}}n\varphi^{1+n}I \geq 0$$

Therefore, TS is (k, m)-n-paranormal.

Now, we will point the adjoint of (k, m)-n-paranormal and explain that by example

Remark and example 3.8:

If T is (k,m)-n-paranormal, then T^* need not likewise be (k,m)-n-paranormal. To assist illustrate that, consider the following example:

Let $T = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$, for every $\varphi > 0$, consider that follows:

$4T^{*3}\,T^{\,3} - 2\varphi T^{*2}T^2 + 4\varphi^2 I = \begin{pmatrix} 40 - 20\varphi + 4\varphi^2 & 0 \\ 0 & 4\varphi^2 \end{pmatrix}$,

for each $\varphi > 0$, i.e., T is (4,3)-1-paranormal.

Now, considering the operator $M = T^* = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, for each $\varphi > 0$:

$$4M^{*3}M^3 - 2\varphi M^{*2}M^2 + 4\varphi^2 I = \begin{pmatrix} 4 - 2\varphi + 4\varphi^2 & 12 - 6\varphi \\ 12 - 6\varphi & 36 - 18\varphi + 4\varphi^2 \end{pmatrix}$$

which is not positive for $\varphi = 3$. So, $M$is not (4,3)-1-paranormal.

Theorem 3.9:

An operator $T \in B(H)$ be (k, m)-n-paranormal, then $I \otimes T\epsilon\, H \otimes H$ with $T \otimes I$ are going to be (k, m)-n--paranormal.

Proof:

$$k^{\frac{2}{1+n}}(I \otimes T)^{*m}\,(I \otimes T)^{*\,n}(I \otimes T)^n (I \otimes T)^{\,m} - (1+n)\varphi^n (I \otimes T)^{*m}(I \otimes T)^{\,m}$$
$$= k^{\frac{2}{1+n}}(I \otimes T^{*m})(I \otimes T^{*n})(I \otimes T^n)(I \otimes T^{\,m}) - (1+n)\varphi^n (I \otimes T^{*\,m})\,(I \otimes T^{\,m})$$
$$= I \otimes \left(k^{\frac{2}{1+n}}T^{*m}T^{*n}T^nT^m - (1+n)\varphi^n T^{*m}T^m\right)$$
$$\geq I \otimes \left(k^{\frac{2}{1+n}} - n\varphi^{1+n}I\right)$$
$$= -k^{\frac{2}{1+n}}n\varphi^{1+n}I$$

So: $k^{\frac{2}{1+n}}(I \otimes T)^{*m}\,(I \otimes T)^{*\,n}(I \otimes T)^n(I \otimes T)^{\,m} - (1+n)\varphi^n(I \otimes T)^{*m}\,(I \otimes T)^{\,m} + k^{\frac{2}{1+n}}n\varphi^{1+n}I \geq 0$, so we get, $I \otimes T$ is (k, m)-n-paranormal . By same way, we can prove that $T \otimes I$ is (k, m)-n-paranormal.

4. Conclusions

We can summarize our research by doing the following: The product and sum of $(k,m) - n -$paranormal two operators are not necessarily $(k,m) - n -$paranormal. Scalar multiplication and restriction of $(k,m) - n -$paranormal are $(k,m) - n -$paranormal once more. Furthermore, not all adjoint of $(k,m) - n -$paranormal are $(k,m) - n -$paranormal.

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