Al Hakeem and Al-Mothafar Iraqi Journal of Science, 2023, Vol. 64, No. 8, pp: 4102-4108
DOI: 10.24996/ijs.2023.64.8.32

ISSN: 0067-2904

S-Essentially Compressible Modulesand S-Essentially Retractable Modules

Mohammed Baqer Hashim Al Hakeem *, Nuhad S. Al-Mothafar
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received: 11/9/2022 Accepted: 16/11/2022 Published: 30/8/2023

Abstract

Let R be a commutative ring with 1 and M be a left unitary R-module. In this paper, we give a generalization for the notions of compressible (retractable) Modules. We study s-essentially compressible (s-essentially retractable). We give some of their advantages, properties, characterizations and examples. We also study the relation between s-essentially compressible (s-essentially retractable modules) and some classes of modules.

Keywords: Essentially compressible, Small compressible, S-essentially compressible, Essentially retractable, S-essentially retractable.

مقاسات قابلة للانضغاط جوهرية من النمط S ومقاسات قابلة للسحب جوهرية من النمط S

محمد باقر هاشم الحكيم* ، نهاد سالم المظفر
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة:

لتكن R حلقة إبداليه ذات عنصر محايد وليكن M مقاساً أيسرَ ذات عنصر احادي. في بحثنا هذا اعطينا مفهوم تعميم للمقاسات القابلة للضغط (قابلة للسحب). سندرس المقاسات القابلة للانضغاط جوهرية من النمط – s (مقاسات قابلة للسحب جوهرية من النمط –s). وسنعطي في هذا البحث بعض الخواص والصفات والامثلة. كذلك سندرس العلاقة بين المقاسات القابلة للانضغاط جوهرية من النمط –s (مقاسات قابلة للسحب جوهرية من النمط –s) وبعض من المقاسات الأخرى.

1. Introduction

Let R be a commutative ring with 1 and M be left unitary R-module. A non-zero submodule K is termed an essential submodule of M, if $K \cap L \neq 0$ for every non-zero submodule L of M, [2]. A submodule N of an R −module M is termed a small submodule ($N \ll M$), if $N + L = M$ for every submodule L of M implies $L = M$, [2]. An R-module M is termed a compressible if M can be embedded in every non-zero submodule. In [3], Zhou D.X and Zhang X.R. introduced and study the concept s-essential submodules, where a submodule N of an R-module M is called s-essential ($N \trianglelefteq_s M$) if $N \cap L \neq 0$, for all small submodule L of M. An R-module M is called essentially compressible if M can be embedded in every non-zero essential submodule of M. Equivalently, M is essentially compressible if there exists a monomorphism $f: M \longrightarrow N$;$0 \neq N \leq_e M$. An R-module M is called s-essentially

*Email: mbhmsc2015110@gmail.com

compressible if M can be embedded in every non-zero s-essential submodule of M. Equivalently, M is an s-essentially compressible if there exists a monomorphism $f: M \longrightarrow N$, $0 \neq N \trianglelefteq_s M$. An R-module M is called s-essentially retractable if $Hom(M, N) \neq 0$ for each non-zero s-essential submodules N of M. In this paper we introduce and study the notion of s-essentially compressible module and s-essentially retractable module as a generalization of compressible module and retractable module, respectively. Also, we give some of their properties, examples and some of their advantages characterizations and examples. We noticed that small-essentially compressible, s-uniform and compressible modules are equivalent.

2. Preliminaries:

Definition 2.1: [1] Let M be an $R-$module and $N \leq M$.

(1)M is called an s-uniform if every non-zero submodule of M is an s-essential submodule. If $\{0\}$ is the only small in M, then M is an s-uniform $R-$ module.

(2)M is called compressible if M can be embedded in every non-zero submodule of$M$. Equivalently, M is compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever$0 \neq N \leq M$.

Example 2.2: [3] Let $R = Z$, $M = Z_6$, $N = (\bar{3})$ and $K = (\bar{2})$, then $K \trianglelefteq_s Z_6$.

Remark 2.3: Every essential submodule is an s-essential submodule, but the converse is not true in general, for example $K = (\bar{3})$ is an s-essential submodule of Z_6 as $Z-$module, but N is not essential, see Example 2.2.

Proposition 2.4: [3] Let M be a module,

(1) Assume that N, K, L are submodules of M with $K \subseteq N$.

(a) If $K \trianglelefteq_s M$, then $K \trianglelefteq_s N$ and $N \trianglelefteq_s M$.

(b) $N \cap L \trianglelefteq_s M$ if and only if $N \trianglelefteq_s M$ and $L \trianglelefteq_s M$.

(2) If $K \trianglelefteq_s N$ and $f : M \rightarrow N$ is a homomorphism, then $f^{-1}(K) \trianglelefteq_s M$.

(3) Assume that $K_1 \subseteq M_1 \subseteq M, K_2 \subseteq M_2 \subseteq M$ and $M = M_1 \oplus M_2$, then $K_1 \oplus K_2 \trianglelefteq_s M_1 \oplus M_2$ if and only if $K_1 \trianglelefteq_s M_1$ and $K_2 \trianglelefteq_s M_2$.

The following example shows converse of Proposition 2.4 (1)(a) is not true:

Example 2.5: Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{36}$, $N = 6\mathbb{Z}_{36}$ and $K = 18\mathbb{Z}_{36}$. Then we obtain $K \trianglelefteq_s N$ and $N \trianglelefteq_s M$. But K is not an s-essential in M.

Example 2.6: Assume that $R = Z$, $M = Z_{12}, N = (\bar{3})$ and $K = (\bar{4})$, then we have $N \trianglelefteq_s Z_{12}$. Since $K \ll Z_{12}$ and $N \cap K = 0$ with $K \neq 0$.

3.S-Essentially compressible modules:

In this section, we introduce the concept of s-essentially compressible module as a generalization of compressible module, and some of basic properties, examples and characterized of this concept has been given.

Definition 3.1: An R-module M is said to be an s-essentially compressible if M can be embedded in every non-zero s-essential submodule of M. Equivalently, M is an s-essentially compressible if there exists a monomorphism $f: M \longrightarrow N$ whenever $0 \neq N \trianglelefteq_s M$. A ring R is called an s-essentially compressible if R is an s-essentially compressible as R-module.

Remarks and Examples 3.2:

(1)Each compressible is an s-essentially compressible, but the converse is not true.

(2)The set Z as Z-module is an s-essentially compressible (since Z is compressible).

(3)Every simple R-module is an s-essentially compressible, but the converse is not true in general since Z as Z-module is an s-essentially compressible. However, it is not simple.

(4)The set Q as Z-module is not an s-essentially compressible (since $Z \trianglelefteq_s Q$, $Hom(Q, Z) = 0$)

(5)The set Z_6 as R-module is not an s-essentially compressible (since there is no a monomorphism $f: Z_6 \longrightarrow (\bar{3})$ and $(\bar{3}) \trianglelefteq_s Z_6$).
(6)A homomorphic image of an s-essentially compressible module needs not to be an s-essentially compressible in general, for example Z as Z-module is an s-essentially compressible and $Z/_{6Z} \simeq Z_6$ is not an s-essentially compressible. This is can be shown in Remarks and Examples 3.2 (5).
(7) By Remarks and Examples 3.2 (5), we can obtain that every semisimple R-module needs not to be an s- essentially compressible.

Proposition 3.3: Let M be an s-uniform R-module, then M is an s-essentially compressible module if and only if M is a compressible module.

Proof: $\Longleftarrow$) It is clear by Remarks and Examples 3.2 (1).
$\Longrightarrow$)Suppose that M is an s-essentially compressible and let $0 \neq N \leq M,$ then $N \trianglelefteq_s M$. Since M is an s-uniform and an s-essentially compressible module. Hence, M can be embedded in N. Therefore, M is a compressible.

Remark 3.4: Every s-essentially compressible is essentially compressible. However, the converse is not true.

Proof: Suppose M is an s-essentially compressible module, let $N \leq_e M$, then $N \trianglelefteq_s M$. Since M is an s-essentially compressible module, we have $f: M \longrightarrow N$ is a monomorphism. Thus, M is an essentially compressible. The converse is not true because Z_6 as Z-module is an essentially compressible. However, by Remarks and Examples 3.2 (5), we obtain Z_6 is not an s-essentially compressible.

Proposition 3.5: Every non-zero submodule of an s-essentially compressible R-module M contains a non-zero s-essential submodule of M is also an s-essentially compressible.

Proof: Let $N \leq M$ and $0 \neq K \trianglelefteq_s N \leq M$. But, $M$is an s-essentially compressible, so there exists a monomorphism $f: M \longrightarrow K$ and $i: N \longrightarrow M$ which is the inclusion monomorphism. Therefore, N is an s-essentially compressible module.

Remark 3.6: The direct sum of an s-essentially compressible is not necessarily be an s-essentially
compressible. For example, let $Z_6 = Z_3 \oplus Z_2$ as Z-module Z_3 and Z_2 are s-essentially compressible modules, but by Remarks and Examples 3.2 (5) we obtain Z_6 is not an s-essentially compressible module.

Proposition 3.7: Let $M = M_1 \oplus M_2$ be an R $-$module such that $ann_R M_1 \oplus ann_R M_2 = R$. If M_1 and M_2 are s-essentially compressible modules then M is an s-essentially compressible.

Proof: Let $0 \neq N \trianglelefteq_s M$. Then by Proposition 2.5 we have $N = K_1 \oplus K_2$ for some $0 \neq K_1 \trianglelefteq_s M_1 \leq M$ and $0 \neq K_2 \trianglelefteq_s M_2 \leq M$, hence, by Proposition 2.7(3) we have K_1 and K_2 are s-essential submodules in M_1 and M_2, respectively . But, $M_1$and M_2 are s-essentially compressible modules, so there exists a monomorphisms $f: M_1 \longrightarrow K_1$ and $g: M_2 \longrightarrow K_2$. Define $\hbar: M \longrightarrow N$ by $\hbar(a,b) = (f(a), g(b))$, it is clear that $\hbar$ is a monomorphism. Therefore, M is an s-essentially compressible.

Recall that an R-module M is called an s-essential prime if $ann_R(M) = ann_R(N)$ for each non-zero s-essential submodule N of M [1]

Lemma 3.8: Let M be an s-essential prime R-module then $ann_R(N)$ is a prime ideal of R for every non-zero s-essential submodule N of M in which every submodule N is an s-essential submodule of M.

Proof: Let $a, b \in R$ such that a $.b \in ann_R(N)$, then $abN = 0$. Suppose $bN \neq 0$. Since $0 \neq N \trianglelefteq_s M$ and $bN \subseteq N$, then $bN \trianglelefteq_s M$ by Proposition 2.7(1)b. Since M is an s-essentially prime module then $a \in ann_R(M)$. As $ann_R(N) = ann_R(M)$ this implies, $a \in ann_R(N)$ and hence $ann_R(N)$ is a prime ideal of R.

Proposition 3.9: Every s-essentially compressible $R-$ module is an s-essentially prime module

Proof: Let $0 \neq N \trianglelefteq_s M$. We have to show that $ann_R(M) = ann_R(N)$. Let $r \in ann_R(N)$, then $rN = 0$, but M is an s-essentially compressible R-module, then there exists a monomorphism $f: M \longrightarrow N$, such that $f(rM) = rf(M) \subseteq rN = 0$, and so $rM = 0$, this gives $r \in ann_R(M)$. Therefore, $ann_R(M) = ann_R(N)$ and hence it is an s-essentially prime module.

Definition 3.10 An $R-$ module M is called an s-essentially uniform if every non-zero s-essential submodule of M is an essential submodule in M.

Remark 3.11: Every uniform is an s-essentially uniform. However, the converse needs not to be true in general for example Z_6 as Z-module is an s-essentially uniform module, but it is not uniform.

Remark 3.12: If M is an s-essentially compressible R-module. Then $M$is isomorphic to a submodule s-essential R_x for all $0 \neq x \in M$.

Proof: Let $0 \neq x \in M$, $R_x \trianglelefteq_s M$. Since M is an s-essentially compressible R-module, then there exists a monomorphism $f: M \longrightarrow R_x$ and f is epimorphism, thus $\simeq R_x$.

Remark 3.13: If M is an s-essentially compressible R-module. Then, $M$is isomorphic to R-module $A/_P$ for some s-essential prime ideal P of R and an ideal A of R contains P property.

Proof: Let $0 \neq m \in M$, then $R_m \trianglelefteq_s M$. Then, by Proposition 3.10 we have R_m is an s-essentially prime , so there exists a monomorphism $f: M \longrightarrow R_x$. Hence, M is isomorphic to a submodule of R_x. On the other hand, as $R_x \simeq R / ann_R(M)$ and by Lemma 2.10 we have $ann_R(M)$ is a prime ideal and hence an s-essential prime ideal of R. Put $ann_R(M) = P$, then $M \simeq A / P$ where A is an ideal of R contains P properly.

Proposition 3.14: Let R be an integral domain, then every finitely generated s-uniform $R-$ module is a compressible.

Proof: Let M be a finitely generated s-uniform - module, then $M = R_{x_1} + R_{x_2} \dots .. + R_{x_n}$; $x_i \in M$ for all $i = 1, \dots, n$. Let $0 \neq N \trianglelefteq_s M$. Hence, for each $i = 1, \dots, n$ there exists $t_i \in R$; $t_i \neq 0$ and $0 \neq t_i x_i \in N$. Let $t = t_1.t_2 \dots \dots t_n$, then $t \neq 0$ and $tx_i \in N$ for each i.

Now, for each $m \in M$, we have $m = \sum_{i=1}^{n} r_i x_i$ with $r_i \in R$, and then $tm = \sum_{i=1}^{n} t(r_i x_i) = \sum_{i=1}^{n} r_i(tx_i)$ which gives $tm \in N$ for all $m \in M$. Now define $\phi: M \longrightarrow N$, by $\phi(m) = tm$ for all $m \in M$. It is clear that ϕ is a monomorphism and hence M can be embedded in N. Therefore, M is compressible.

4.S-Essentially retractable modules:

In this section, we introduce the concept of an s-essentially retractable module as a generalization of retractable module and, some of its basic properties, examples and characterizations of this concept has been given.

Definition 4.1: An R-module M is said to be an s-essentially retractable if $Hom(M,N) \neq 0$ for every non-zero s-essential submodules of M. Equivalently, M is an s-essentially retractable if there exists a non-zero homomorphism $f: M \longrightarrow N$, $0 \neq N \trianglelefteq_s M$. A ring R is called an s-essentially retractable if R is an s-essentially retractable as an R-module.

Recall that an R-module M is said to be an essentially retractable if $Hom_R(M,N) \neq 0$ for every essential submodule N of M.

A ring R is said to be an essentially retractable if the R-module R is an essentially retractable. That is $Hom_R(R,I) \neq 0$ for every non-zero essential ideal I of a ring R,[6].

Remarks and Examples 4.2:

1. It's clear that every retractable module is a s-essentially retractable module.
2. Every s-essentially retractable module is essentially retractable module.

Proof: Let $0 \neq N \leq_e M$, then $N \trianglelefteq_s M$. Since M is an s-essentially retractable module, then there exists $0 \neq f: M \longrightarrow N$. Thus M is an essentially retractable.

3. The set Z as Z-module is a s-essentially retractable module, because Z is a retractable.
4. The set Z_6 as Z-module is an s-essentially retractable module.
5. It's obvious that every s-essentially compressible module is an s-essentially retractable module, but the converse is not true in general for example Z_6 as Z-module is an s-essentially retractable module, but not an s-essentially compressible, which can be shown by Examples 4.2 and 3.2.
6. The set Q as Z-module is not an s-essentially retractable module, since $Hom(Q,Z) = 0$ and $Z \trianglelefteq_s Q$.
7. Every compressible module is an s-essentially retractable module, but the converse is not true for example Z_6 as Z-module is an s-essential retractable module, but not compressible.
8. Every simple R-module is an s-essentially retractable module but not conversely, because Z as Z-module is an s-essentially retractable module but not simple.
9. Every semisimple R-module is an s-essentially retractable because it is retractable.

Proposition 4.3: Let M be an s-uniform R-module , then M is an s-essentially retractable module if and only if M is retractable module .

Proof: $\Longleftarrow$) It is clear, by Remark 4.2 (1).

$\Longrightarrow$)Suppose that M is an s-essentially retractable and let $0 \neq N \leq M$, then $N \trianglelefteq_s M$ (since M is an s-uniform), and since M is an s-essentially retractable module. Therefore, M is retractable. Hence, there exists a non-zero homomorphism $f: M \longrightarrow N$ for some $0 \neq N \leq M$.

Proposition 4.4: Every submodule of an s-essentially retractable contains a non-zero s-essential submodule of M is also an s-essentially retractable .

Proof: Let $0 \neq K \trianglelefteq_s N \leq M$ with $K \trianglelefteq_s M$. By Proposition 2.7 (1) (a), $K \trianglelefteq_s N$ and $M$is an s-essentially retractable, so there exists a non-zero homomorphism $f: M \longrightarrow K$ and $i: N \longrightarrow M$ is the inclusion monomorphism. Therefore, N is an s-essentially retractable module.

Remark 4.5: The direct sum of an s-essentially retractable is also an s-essentially retractable.
Proof: Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of an s-essentially retractable module and let K be a non-zero s-essential submodule of M. By Proposition 2.7(1)(b) and for each $i \in I$, $K \cap M_i$ is an s-essential submodule of M_i and we have $Hom_R(M_i , K \cap M_i) \neq 0$. Then $Hom_R(M, K) \neq 0$. Therefore, M is an s-essentially retractable module.

Lemma 4.6: Let M be an s-essentially prime R-module, then $ann_R(N)$ is a prime ideal of R for each non-zero s-essential submodule N of M.

Proof: By using the same technique of Lemma 3.8, we can find the proof.
Proposition 4.7: Every s-essentially retractable R-module is an s-essentially prime module.

Proof: Let $0 \neq N \trianglelefteq_s M$. We have to show that $ann_R(M) = ann_R(N)$. Let $r \in ann_R(N)$, then $rN = 0$, but M is an s-essentially retractable R-module, then there exists $f: M \longrightarrow N$ is a homomorphism , such that $f(rM) = rf(M) \subseteq rN = 0$, and so $rM = 0$, which means $r \in ann_R(M)$. Therefore, $ann_R(M) = ann_R(N)$ and it is an s-essentially prime module.

Recall that an R-module M is called a co-compressible if it is a homomorphism image of each of its non-trivial factor, [7].

And recall that an R-module M is a Hopfian, if for each $f \in End_R(M)$, f is surjective which implies f is injective ($Ker f = 0$), [8].

Proposition 4.9: Let M be a Hopfian and co-compressible R-module . If M is an s-essentially retractable, then M is an s-essentially compressible.

Proof: We want to prove that every $f: M \longrightarrow M$, where $0 \neq f$,then $f \in End_R(M)$ is an epimorphism. Since M is a co-compressible module, then there exists $h: \frac{M}{N} \longrightarrow M$ is an epimorphism, and hence $h \circ \pi: M \longrightarrow M$, where $\pi: M \longrightarrow \frac{M}{N}$ is a projection, thus $f = h \circ \pi \in End_R(M)$ is an epimorphism. Since by our assumption M is a Hopfian, then f is a monomorphism. But M is an s-essentially retractable and M is a Hopfian which implies $Ker f = 0$, then M is an s-essentially compressible.

Recall that A ring R is said to be regular (von-numann) if for each $a \in R$ there exists an element $t \in R$ such that $a = ata$ (if R is a commutative ring, then $a = a^2 t$), [9].

Proposition 4.10: Let M be an indecomposable and an s-essentially retractable R-module. If S is a regular ring, then M is an s-essentially compressible, where $S = End_R(M)$.

Proof: Let $0 \neq N \trianglelefteq_s M$, since M is an s-essentially retractable R-module, then there exists a homomorphism $f: M \longrightarrow N$ such that $0 \neq f$ and $i: N \longrightarrow M$ is an inclusion homomorphism, then $i \circ f: M \longrightarrow M$ is a homomorphism. But S is a regular ring and M is an indecomposable, then by the same proof of the Proposition 4.15 we have $Ker f = Ker\,(i \circ f)$ is direct summand of M and $Ker f = 0$. Therefore, M is an s-essentially compressible.

Conclusion:

In this work, the class of compressible and retractable modules have been generalized to a new concepts called an s-essentially compressible and an s-essentially retractable modules. Several characteristics of this type of modules have been studied. Sufficient conditions under which these modules with compressible and retractable are discuss

Also, we show the relation between s-essentially compressible modules and other related modules as s-essentially retractable module Hopfian, co-compressible and indecomposable modules.

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