Iraqi Journal of Science, 2023, Vol. 64, No. 10, pp: 5228-5240
DOI: 10.24996/ijs.2023.64.10.30

Iraqi Journal of Science

ISSN: 0067-2904

Sandwich Subordinations Imposed by New Generalized Koebe-Type Operator on Holomorphic Function Class

Anwar H. Moureh*, Hiba F. Al-Janaby
Department of Mathematics, College of Science, Baghdad University, Baghdad, Iraq

Received: 23/9/2022 Accepted: 6/2/2023 Published: 30/10/2023

Abstract.
In the complex field, special functions are closely related to geometric holomorphic functions. Koebe function is a notable contribution to the study of the geometric function theory (GFT), which is a univalent function. This sequel introduces a new class that includes a more general Koebe function which is holomorphic in a complex domain. The purpose of this work is to present a new operator correlated with GFT. A new generalized Koebe operator is proposed in terms of the convolution principle. This Koebe operator refers to the generality of a prominent differential operator, namely the Ruscheweyh operator. Theoretical investigations in this effort lead to a number of implementations in the subordination function theory. The tight upper and lower bounds are discussed in the sense of subordinate structure. Consequently, the subordinate sandwich is acquired. Moreover, certain relevant specific cases are examined.

Keywords: Holomorphic function, univalent function; Koebe function; convolution principle; subordinate term.

التبعيات ساندويتش لمؤثر جديد معمم من نوع كوبي على فئة الدوال التحليلية

انوار هاشم مريح*, هبة فوزي الجنابي
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق

الخلاصة:
في المجال المعقد، ترتبط الدوال الخاصة ارتباطًا وثيقًا بالدوال الهندسية التحليلية. وداله كوبي لها مساهمة ملحوظه في دراسة نظرية الداله الهندسية (GFT) ، وهي داله احاديه التكافؤ. يقدم هذا البحث فئة جديدة من داله كوبي الأكثر عمومية والتي تكون تحليلية على المجال العقدي. الغرض من هذا العمل هو تقديم دراسة للمؤثر الجديد المرتبط ب GFT يتم فحص مؤثر كوبي المعمم الجديد من حيث مبدأ الالتواء.يشير مشغل كوبي الى عموميه عامل تفاضلي بارز، و هو مشغل .Ruschewey ..أدت الدراسات النظرية في هذا الجهد إلى عدد من التطبيقات في نظرية دالة التبعية. مؤثر تتم دراسة الحدود العلوية والسفلية الضيقة بمعنى الهيكل الفرعي. وبالتالي ، يتم الحصول التبعية الشطيرة. بالإضافة إلى ذلك ، يتم مناقشة بعض الحالات الخاصة ذات الصلة.

1. Introduction

*Email: anwaranwar97428@gmail.com

The interest in special function theory and its dynamic role in the study of complex analysis [1,2,3], mainly, in geometric function theory (GFT). After employing a hypergeometric function in the verification of a considerable problem in GFT, namely Bieberbach's conjecture [4], this theory motivates the evolution of GFT by captivating numerous scientists to the current research related to operator theory and making worthy contributions [5,6]. Represent by $\mathcal{H}(\Lambda)$ the class of holomorphic functions in in the unit disk $\Lambda = \{z \in \mathbb{C}: |z| < 1\}$ and let $\mathcal{H}[\mathfrak{a}, \mathfrak{n}]$ be the subclass of $\mathcal{H}(\Lambda)$ involving of functions of the formula $\varphi(z) = \mathfrak{a} + \mathfrak{a}_{\mathfrak{n}} z^{\mathfrak{n}} + \mathfrak{a}_{\mathfrak{n}+1} z^{\mathfrak{n}+1} + \cdots$, and let $\mathcal{H}_0 = \mathcal{H}[0,1]$ and $\mathcal{H} = \mathcal{H}[1,1]$. The class $\mathfrak{A}$ of holomorphic functions is stated as:

$$\varphi(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \rho_{\mathfrak{n}}\, z^{\mathfrak{n}}, \tag{1.1}$$

which are normalized (means that $\varphi(0) = \varphi'(0) - 1 = 0$) in Λ, The subclass of $\mathfrak{A}$ that includes univalent functions is symbolized by $\mathcal{S}$, [7]. Following Goodman's notations [7], by $\mathcal{C}_{\mathcal{V}}$ and $\mathcal{S}^*$ the remarkable subclasses of $\mathcal{S}$ which are consecutively, holomorphic convex and holomorphic starlike functions. More precisely, in 1913, the scientist Study [8] initiated an interesting geometric-type function on Λ, namely holomorphic convex, which stated for $\varphi \in \mathfrak{A}$ is called convex provided that image domain $\varphi(\Lambda)$ is convex. The holomorphic starlike function is also a well-constructed notion, and its origin gets back to Nevanlinna's 1921 paper [9], which stated that for $\varphi \in \mathfrak{A}$ is called starlike provided that image domain $\varphi(\Lambda)$ is starlike. Analytically reformulated, convex function φ if and only if φ achieves $\mathcal{R}e\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > 0$ and starlike function φ if and only if φ attains $\mathcal{R}e\left(\frac{z\varphi'(z)}{\varphi(z)}\right) > 0$, consecutively, [10]. Besides, the Koebe function, which belongs to $\mathcal{S}^*$ and is a cornerstone of GFT due to it provides a remarkable extremal function for several problems, gives [7]

$$\mathcal{K}(z) = \frac{z}{(1-z)^2} = z + \sum_{\mathfrak{n}=2}^{\infty} \mathfrak{n}\, z^{\mathfrak{n}}. \tag{1.2}$$

More generally, for $\mu > 0$, the famed geometric function is:

$$\mathcal{K}_{\mu}(z) = \frac{z}{(1-z)^{\mu}} = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{\Gamma(\mu + \mathfrak{n} - 1)}{\Gamma(\mu)\,(\mathfrak{n}-1)!}\, z^{\mathfrak{n}} = \phi(\sigma, 1; z), \tag{1.3}$$

where $\Gamma(\mu)$, indicates to gamma function [11], and worthwhile $\phi(\sigma, \gamma, z)$ is the incomplete beta function given by $\phi(\mu, \gamma, z) = z + \sum_{\mathfrak{n}=2}^{\infty} \frac{\Gamma(\mu+\mathfrak{n}-1)\Gamma(\gamma)}{\Gamma(\mu)\,\Gamma(\gamma+\mathfrak{n}-1)}\, z^{\mathfrak{n}}$, [4].

Further, the term convolution refers to a mathematical operation symbolized by *, was first presented by Hadamard. It is stated as: for $\varphi_1, \varphi_2 \in \mathfrak{A}$ formulated by $\varphi_1(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \rho_{1\mathfrak{n}}\, z^{\mathfrak{n}}$ and $\varphi_2(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \rho_{2\mathfrak{n}}\, z^{\mathfrak{n}}$, the convolution product of φ_1 and φ_2, written by $\varphi_1 * \varphi_2$, produces a holomorphic convolution function described by [7]

$$(\varphi_1 * \varphi_2)(z) = z + \sum_{\mathfrak{n}=2}^{\infty} \rho_{1\mathfrak{n}}\, \rho_{2\mathfrak{n}}\, z^{\mathfrak{n}}. \tag{1.4}$$

On the other hand, Lindeöf [7] in 1909 offered a subordinate principle between holomorphic functions. For $\varphi_1, \varphi_2 \in \mathfrak{A}$, then φ_1 is subordinate to φ_2, symbolization $\varphi_1 \prec \varphi_2$, if there is a function ϖ, holomorphic in Λ, with $\varpi(0) = 0$ and $|\varpi(z)| < 1$ such that $\varphi_1(z) = \varphi_2(\varpi(z))$.

Specifically, if φ_2 is univalent, then $\varphi_1 \prec \varphi_2$, if and only if $\varphi_1(0) = \varphi_2(0)$. and $\varphi_1(\Lambda) \subset \varphi_2(\Lambda)$, [7].

In this relevance, the process of differential subordination and analogous process of differential superordination is a gist study on $\mathfrak{A}$, utilized to tighten upper and lower bounds, called the sandwich problem, which was advanced by the mathematicians Miller and Mocanu [11] in 1981, [12] in 2000, [13] in 2003. Before that time, in 1935, the author Goluzin [14] put forward the first outcome in terms of (first-order) differential subordination. Whereas the investigator Bulboacă [4-15] posed the first outcomes in differential superordination. Accordingly, Miller and Mocanu offered terms [13].

Let, $\mathcal{J}: \mathbb{C}^2 \times \Lambda \longrightarrow \mathbb{C}$ and $\wp$ be univalent in Λ. Postulate that ω is holomorphic in Λ and attains the (first-order) differential subordination:

$$\mathcal{J}(\omega(z), z\omega'(z): z) \prec \wp(z), \tag{1.5}$$

then ω is the so-called a resolution of (1.5). The univalent function $\hbar$ is so-called a dominant of the resolutions of (1.5), merely called dominant, if $\omega \prec \hbar$ for all ω attaining (1.5). A dominant $\tilde{\hbar}$ which attains $\tilde{\hbar} \prec \hbar$, for all dominant $\hbar$ of (1.5) is so-called the best dominant.

Corresponding to the term subordination, let $\mathcal{J}: \mathbb{C}^2 \times \Lambda \longrightarrow \mathbb{C}$ and $\wp$ are holomorphic. Let ω and $\mathcal{J}(\omega(z), z\omega'(z): z)$ are univalent in Λ. If ω attains the (first-order) differential superordination,

$$\wp(z) \prec \mathcal{J}(\omega(z), z\omega'(z): z), \tag{1.6}$$

then ω is so-called a resolution of (1.6). The holomorphic function $\hbar$ is so-called a subordinant of the resolutions of (1.6), merely called subordinant, if $\hbar \prec \omega$ for all ω attains (1.6). A univalent subordinant $\tilde{\hbar}$ that attains $\hbar \prec \tilde{\hbar}$ for all subordinants $\hbar$ of (1.6) is so-called the best subordinant.

In effect, interesting recent studies have emanated about dealing with subordinate and superordinate techniques and sandwiches problems correlated with numerous complex operators, for instance, Atshan et al. [17], Sokół et al. [18], Zayed and Bulboacă [19], Al-Janaby and Ahmad [20], Mishra and Soren [21], Al-Janaby and Darus [22], Al-Janaby et al. ([23], [24]), Ghanim and Al-Janaby ([25], [26]), Attiy ([27], [28]), Lupaş and Oros [29] and Atshan ([30], [31]).

Motivated by these scientific contributions, this paper investigates the new generalized Koebe operator by proposing a more generalized Koebe -type function that is holomorphic in Λ. This is employed to discuss of several interesting subordination and superordination implementations. As a consequence, the subordinate sandwich is derived. In addition, relevance's geometric outcomes are exanimated. In order to illustrate our main outcomes, we include the following concept and central lemmas.

Lemma 1.1 [11]. Let $\hbar$ be univalent in Λ. Let ϕ and ψ be holomorphic in a domain D containing $\hbar(\Lambda)$ with $\phi(\lambda) \neq 0$ when $\lambda \in \hbar(\Lambda)$. Set $\grave{\Omega}(z) = z\hbar'(z)\psi(\hbar(z))$, and $\wp(z) = \phi(\hbar(z)) + \grave{\Omega}(z)$. Assume that

1. $\grave{\Omega}$ is starlike function in Λ
2. $\mathcal{R}e\left\{\frac{z\grave{\wp}(z)}{\grave{\Omega}(z)}\right\} > 0$, $(z \in \Lambda)$.

If ω is holomorphic in Λ whit $\omega(0) = \hbar(0)$, $\omega(\Lambda) \subseteq D$ and the following differential subordination $\phi(\omega(z)) + z\omega'(z)\psi(\omega(z)) \prec \phi(\hbar(z)) + z\hbar'(z)\psi(\hbar(z))$ holds. Then $\omega \prec \hbar$ and $\hbar$ the best dominant.

Definition 1.1 [32] Let Ξ be the set of functions φ that are holomorphic and injective on $\overline{\Lambda}\backslash\mathcal{P}(\varphi)$, where $\mathcal{P}(\varphi)=\left\{\alpha:\alpha\in\partial\Lambda:\lim_{z\to\alpha}\varphi(z)=\infty\right\}$ and such that $\varphi'(\alpha)\neq 0$ for $\alpha\in\partial\Lambda\backslash\mathcal{P}(\varphi)$.

Lemma 1.2. [32] Let $\hbar$ be univalent in Λ and let ϕ and ψ be holomorphic in a domain D containing $\hbar(\Lambda)$. Suppose that

1. $\Omega(z)=z\hbar'(z)\psi(\hbar(z))$ is starlike function in Λ, and
2. $\mathcal{R}e\left\{\frac{z\,\phi'(\hbar(z))}{\psi\,(\hbar(z))}\right\}>0\,, \qquad (z\in\Lambda)$.

If $\omega\in\mathcal{H}[\hbar(0),1]\cap\Xi$, with $\omega(\Lambda)\subset D$, $\phi(\omega(z))+z\omega'(z)\psi(\omega(z))$ is univalent in Λ, and $\phi(\hbar(z))+z\hbar'(z)\psi(\hbar(z))\prec\phi(\omega(z))+z\omega'(z)\psi(\omega(z))$, then $\hbar\prec\omega$ and $\hbar$ is the best subordinate.

2. Proposed Generalized Koebe Operator $\mathcal{L}_{\mu,\sigma}\varphi(z)$

This section imposes the new generalized Koebe function, which is a holomorphic function on Λ. Afterward, a generalized Koebe operator is suggested according to the convolutional structure. This new operator is a generalization of the Ruscheweyh derivative operator.

Corresponding to representations $\mathcal{K}(z)$ and $\mathcal{K}_{\mu}(z)$, given by (1.3) and (1.4) respectively, lead us to consider a new generalized Koebe function as:

$$\mathcal{K}_{\mu,\sigma}(z)=\sum_{\mathrm{n}=0}^{\infty}\frac{\Gamma(\mu+\sigma\mathrm{n})}{\Gamma(\mu)\,\Gamma(\mathrm{n}+1)}\,z^{\mathrm{n}}. \qquad (z,\mu\in\mathbb{C};\quad 0<\mathcal{R}(\sigma)\,) \tag{2.1}$$

Theorem 2.1. For $z,\mu\in\mathbb{C}$ and $0<\mathcal{R}(\sigma)$; the generalized Koebe function (2.1) is a holomorphic function on $\mathbb{C}$.

Proof. Employing the coefficients $\omega_{\mathrm{n}}=\frac{\Gamma(\mu+\sigma\mathrm{n})}{\Gamma(\mathrm{n}+1)}$ of (2.1), Cauchy–Hadamard formula [33] and utilizing $\frac{\Gamma(\zeta+\mathrm{t})}{\Gamma(\xi+\mathrm{t})}\sim\mathrm{t}^{\zeta-\xi}$ [34] as $\mathrm{t}\longrightarrow\infty$, the radius of convergence of (2.1) becomes

$$\frac{1}{\Re}=\lim_{\mathrm{n}\to\infty}sup\left|\frac{\omega_{\mathrm{n}}}{\omega_{\mathrm{n}+1}}\right|=\lim_{\mathrm{n}\to\infty}sup\left|\frac{\Gamma(\mathrm{n}+2)\;\Gamma(\mu+\sigma\mathrm{n})}{\Gamma(\mathrm{n}+1)\,\Gamma(\mu+\sigma+\sigma\mathrm{n})}\right|=\lim_{\mathrm{n}\to\infty}sup\left|\frac{(\mathrm{n}+1)\,\Gamma(\mu+\sigma\mathrm{n})}{\Gamma(\mu+\sigma+\sigma\mathrm{n})}\right|$$
$$=\lim_{\mathrm{n}\to\infty}sup\;(\mathrm{n}+1)|\sigma\mathrm{n}|^{\mu-\mu-\sigma}=0,$$

for $\mu\in\mathbb{C}$, and $0<\mathcal{R}(\sigma)$. Thus, $\Re=\infty$. Hence, the function (2.1) is holomorphic on Λ. For $\mathcal{R}(\sigma)\leq 0$, the formula diverges for everywhere on $\mathbb{C}$.

Remark 2.1. [7] The function (2.1) generalizes some remarkable geometric functions as: for special values of μ and σ.

i. $\mathcal{L}_{2,1}(z)=\mathcal{K}(z)$ as given in (1.3).
ii. $\mathcal{L}_{\mu,1}(z)=\mathcal{K}_{\mu}(z)$ as written in (1.4).
iii. $\mathcal{L}_{1,1}(z)=\frac{z}{1-z}$ is the convex function.

Following this study, the normalization formula for (2.1) is as follows:

$$\mathcal{N}_{\mu,\sigma}(z)\;=\;z\mathcal{K}_{\mu,\sigma}(z)=z+\sum_{\mathrm{n}=2}^{\infty}\frac{\Gamma(\mu+\sigma(\mathrm{n}-1))}{\Gamma(\mu)\,(\mathrm{n}-1)!}\,z^{\mathrm{n}}. \tag{2.2}$$

Then based on the function $\mathcal{N}_{\mu,\sigma}(z)$ written in (2.2), we consider a new complex linear-type operator $\mathcal{L}_{\mu,\sigma}(z)$ maps $\mathfrak{A}$ onto itself, namely Koebe operator, according to convolutional structure as:

$$\mathcal{L}_{\mu,\sigma}\varphi(z) = \mathcal{N}_{\mu,\sigma}(z) * \varphi(z)$$
$$= z + \sum_{\mathfrak{n}=2}^{\infty} \frac{\Gamma(\mu + \sigma(\mathfrak{n} - 1))}{\Gamma(\mu)\ (\mathfrak{n} - 1)!} \rho_{\mathfrak{n}}\, z^{\mathfrak{n}}. \quad (2.3)$$

Remark 2.2. The following specific cases related to the operator $\mathcal{L}_{\mu,\sigma}\varphi(z)$ introduced in (2.3) are also produced by assumption of certain particular values of the parameters as:

1. For $\mu = \sigma = 1$, the operator $\mathcal{L}_{1,1}\varphi(z) = \varphi(z)$.
2. For $\mu = 2$, and $\sigma = 1$, the operator $\mathcal{L}_{2,1}\varphi(z) = z\varphi'(z)$.
3. For $\sigma = 1$, the operator $\mathcal{L}_{\mu,1}\varphi(z)$ coincides with $(\mathcal{K}_a * f)(z)$ obtained from Ozkan [5].
4. For $\mu = \lambda + 1$ and $\sigma = 1$, the operator $\mathcal{L}_{\lambda+1,1}\varphi(z)$ matches $D^{\mathfrak{n}}f(z)$ Ruscheweyh-type operator invented by Ruscheweyh, [6].

Later, utilizing (2.3), we achieve the following recurrence (identity) relation,

$$z\left(\mathcal{L}_{\mu,\sigma}\varphi(z)\right)' = \frac{\mu}{\sigma}\, \mathcal{L}_{\mu+1,\sigma}\varphi(z) - \left(\frac{\mu - \sigma}{\sigma}\right) \mathcal{L}_{\mu,\sigma}\varphi(z). \quad (2.4)$$

3. Outcomes sandwich Properties

This section investigates some sandwich outcomes relating to the Koeba operator for normalized holomorphic function for certain analytic function yielded.

Theorem 3.1. Let $\Upsilon_\iota \in \mathbb{C}$ $(\iota = 1,2)$, $\eta \in \mathbb{C}\backslash(0)$ and $\hbar(z)$ be a convex univalent in Λ, $\hbar(0) = 1$, $\hbar(z) \neq 0$ $(z \in \Lambda)$, and assume that $\hbar$ achieves.

$$\mathcal{R}e\left\{1 + \frac{\Upsilon_2}{\eta}\hbar(z) + \frac{z\hbar''(z)}{\hbar'(z)} - \frac{z\hbar'(z)}{\hbar(z)}\right\} > 0 \quad (3.1)$$

Furthermore, let $\frac{z\hbar'(z)}{\hbar(z)}$ be a starlike univalent in Λ. As well, if $\varphi \in \mathfrak{A}$ achieves the subordination

$$(\Upsilon_1 + \eta) + (\Upsilon_2 - \eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + \eta\, \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$$
$$\prec \Upsilon_1 + \Upsilon_2\, \hbar(z) + \eta \frac{z\hbar'(z)}{\hbar(z)}. \quad (3.2)$$

Then,

$$\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} \prec \hbar(z), \quad (3.3)$$

and $\hbar$ is the best dominant of (3.2).

Proof: Define the function ω by

$$\omega(z) = \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}. \quad (3.4)$$

Clearly, ω is holomorphic in Λ and $\omega(0) = 1$. The computation displays that

$$\frac{z\omega'(z)}{\omega(z)} = \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + 1 - \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}. \quad (3.5)$$

From (3.2), (3.4) and (3.5), we ensure subordination

$$\Upsilon_1 + \Upsilon_2\, \omega(z) + \eta\, \frac{z\omega'(z)}{\omega(z)} = (\Upsilon_1 + \eta) + (\Upsilon_2 - \eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + \eta\, \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$$

$$\prec \Upsilon_1 + \Upsilon_2\, \hbar(z) + \eta \frac{z\hbar'(z)}{\hbar(z)} . \tag{3.6}$$

By setting,

$$\phi(q) = \Upsilon_1 + \Upsilon_2 q , \qquad and \qquad \psi(q) = \frac{\eta}{q}, \quad q \neq 0. \tag{3.8}$$

Evidently, $\phi(q)$ *is a* holomorphic in $\mathbb{C}$, and $\psi(q)$ *is a* holomorphic in $\mathbb{C}\backslash\{0\}$, also, $\psi(q) \neq 0, q \in \mathbb{C}\backslash\{0\}$. In addition, by assuming

$$\Omega(z) = z\hbar'(z)\psi(\hbar(z)) = \eta \frac{z\hbar'(z)}{\hbar(z)}, \tag{3.9}$$

and

$$\wp(z) = \phi(\hbar(z)) + \Omega(z) \quad = \Upsilon_1 + \Upsilon_2\, \hbar(z) + \eta \frac{z\hbar'(z)}{\hbar(z)} . \tag{3.10}$$

Obviously, $\Omega(z)$ is starlike univalent in Λ and in view of (3.1), we gain

$$\mathcal{R}e\left\{\frac{z\wp'(z)}{\Omega(z)}\right\} = \mathcal{R}e\left\{1 + \frac{\Upsilon_2}{\eta}\hbar(z) + \frac{z\hbar''(z)}{\hbar'(z)} - \frac{z\hbar'(z)}{\hbar(z)}\right\} > 0. \tag{3.11}$$

Then, the relation (3.3) follows the implementation of Lemma 1.1.

In view of Theorem 3.1, we yield the following interesting outcomes.

Corollary 3.1. Let the hypotheses of Theorem 3.1 hold. Then, the subordination relation

$$(\Upsilon_1 + \eta) + (\Upsilon_2 - \eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + \eta\, \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$$

$$\prec \Upsilon_1 + \Upsilon_2\left(\frac{1+\mathcal{M}z}{1+\mathcal{N}z}\right) + \eta \frac{(\mathcal{M}-\mathcal{N})z}{(1+\mathcal{M}z)\,(1+\mathcal{N}z)}, \tag{3.12}$$

which implies that $\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$, $-1 \leq \mathcal{N} < \mathcal{M} \leq 1$, and $\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ is the best dominant.

Proof: By letting $\hbar(z) = \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ in Theorem 3.1, we attain the required assertion.

Corollary 3.2. Let the hypotheses of Theorem 3.1 hold. Then, the subordination relation

$$\Upsilon_2\, \frac{z\varphi'(z)}{\varphi(z)} + \eta\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)} - \frac{z\varphi'(z)}{\varphi(z)}\right\}$$

$$\prec \Upsilon_2\hbar + \eta\frac{z\hbar'(z)}{\hbar(z)}, \tag{3.13}$$

which implies that $\frac{z\varphi'(z)}{\varphi(z)} \prec \hbar$, and $\hbar$ is the best dominant.

Proof: By putting $\mu = \sigma = 1$ and $\Upsilon_1 = 0$ in Theorem 3.1, we gain the required assertion.

The next outcome that has been studied by Ravichandran and Jayamala, [35].

Corollary 3.3. If $\varphi \in \mathfrak{A}$ and (3.1) is presumed to hold, then the subordination relation

$$1 + \frac{z\varphi''(z)}{\varphi'(z)} \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z} + \frac{(\mathcal{M}-\mathcal{N})z}{(1+\mathcal{M}z)\,(1+\mathcal{N}z)}, \tag{3.14}$$

which implies that $\frac{z\varphi'(z)}{\varphi(z)} \prec \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$, $-1 \leq \mathcal{N} < \mathcal{M} \leq 1$, and $\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ is the best dominant.

Proof: By setting $\mu = \sigma = 1$, $\Upsilon_1 = 0$, $\Upsilon_2 = \eta = 1$, $\hbar(z) = \frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ in Theorem 3.1, we derive the required assertion.

In view of the recurrence relation (2.4), and by utilizing the technique of proof of Theorem 3.1, we deduce the following main outcome.

Theorem 3.2. Let $\Upsilon_\iota \in \mathbb{C}$ $(\iota = 1,2)$, $\eta \in \mathbb{C}\backslash(0)$ and $\hbar(z)$ be a convex univalent in Λ, $\hbar(0) = 1$, $\hbar(z) \neq 0$ $(z \in \Lambda)$, and assume that $\hbar$ achieves condition (3.1).
Furthermore, let $\frac{z\hbar'(z)}{\hbar(z)}$ be a starlike univalent in Λ. Also, if $\varphi \in \mathfrak{A}$ achieves the subordination

$$\Upsilon_1 + \Upsilon_2\left(\frac{z\left[\mathcal{L}_{\mu-1,\sigma}\varphi(z)\right]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho \prec \Upsilon_1 + \Upsilon_2\,\hbar(z) + \eta\frac{z\hbar'(z)}{\hbar(z)}. \quad (3.15)$$

Then,

$$\left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} \prec \hbar(z), \quad (3.16)$$

and $\hbar$ is the best dominant of (3.15).
Proof: Define the function ω by

$$\omega(z) = \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho}. \quad (3.17)$$

Evidently, ω is holomorphic in Λ and $\omega(0) = 1$. After several computations and utilizing recurrence relation (2.4), we deduce

$$\frac{z\omega'(z)}{\omega(z)} = \frac{\varrho(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho(2\mu-1)}{\sigma} + 2\varrho. \quad (3.18)$$

From (3.15), (3.17) and (3.18), we yield subordination

$$\Upsilon_1 + \Upsilon_2\,\omega(z) + \eta\,\frac{z\omega'(z)}{\omega(z)} = \Upsilon_1 + \Upsilon_2\left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho \prec \Upsilon_1 + \Upsilon_2\,\hbar(z) + \eta\frac{z\hbar'(z)}{\hbar(z)}. \quad (3.19)$$

By setting,

$$\phi(q) = \Upsilon_1 + \Upsilon_2 q, \quad \text{and} \quad \psi(q) = \frac{\eta}{q}, \quad q \neq 0. \quad (3.20)$$

Clearly, $\phi(q)$ is a holomorphic in $\mathbb{C}$, and $\psi(q)$ $is\ a$ holomorphic in $\mathbb{C}\backslash\{0\}$, also, $\psi(q) \neq 0, q \in \mathbb{C}\backslash\{0\}$. In addition, by assuming

$$\Omega(z) = z\hbar'(z)\psi(\hbar(z)) = \eta\frac{z\hbar'(z)}{\hbar(z)}, \quad (3.21)$$

and

$$\wp(z) = \phi(\hbar(z)) + \Omega(z) = \Upsilon_1 + \Upsilon_2\,\hbar(z) + \eta\frac{z\hbar'(z)}{\hbar(z)}. \quad (3.22)$$

Obviously, $\Omega(z)$ is starlike univalent in Λ and from (3.1), we gain

$$\mathcal{R}e\left\{\frac{z\wp'(z)}{\Omega(z)}\right\} = \mathcal{R}e\left\{1 + \frac{\Upsilon_2}{\eta}\hbar(z) + \frac{z\hbar''(z)}{\hbar'(z)} - \frac{z\hbar'(z)}{\hbar(z)}\right\} > 0. \quad (3.23)$$

Then, the relation (3.16) follows the implementation of Lemma 1.1.

In view of Theorem 3.2, we yield the following interesting outcome.
Corollary 3.4. Let the hypotheses of Theorem 3.2 hold. Then, the subordination relation

$$\Upsilon_1+\Upsilon_2\left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho}+\frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)}-\frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}-\frac{\varrho\eta(2\mu-1)}{\sigma}+2\eta\varrho$$

$$\prec \Upsilon_1+\Upsilon_2\left(\frac{1+\mathcal{M}z}{1+\mathcal{N}z}\right)+\eta\frac{(\mathcal{M}-\mathcal{N})z}{(1+\mathcal{M}z)\,(1+\mathcal{N}z)}, \tag{3.24}$$

which implies that $\left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho}\prec\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$, $-1\le\mathcal{N}<\mathcal{M}\le1$, and $\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ is the best dominant.

Proof: By letting $\hbar(z)=\frac{1+\mathcal{M}z}{1+\mathcal{N}z}$ in Theorem 3.2, we attain the required assertion.

Theorem 3.3. Let $\Upsilon_\iota\in\mathbb{C}\ (\iota=1,2)$, $\eta\in\mathbb{C}\backslash(0)$ and $\hbar(z)$ be a convex univalent in Λ, $\hbar(0)=1$, $\hbar(z)\neq0\ (z\in\Lambda)$, and assume that $\hbar$ achieves.

$$\mathcal{R}e\left\{\frac{\Upsilon_2}{\eta}\hbar(z)\right\}>0. \tag{3.25}$$

If $\varphi\in\mathfrak{A}$, $\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\in\mathcal{H}[1,1]\cap\Xi$, and $(\Upsilon_1+\eta)+(\Upsilon_2-\eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}+\eta\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$ is univalent in Λ and

$$\Upsilon_1+\Upsilon_2\,\hbar(z)+\eta\frac{z\hbar'(z)}{\hbar(z)}\prec(\Upsilon_1+\eta)+(\Upsilon_2-\eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}+\eta\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}. \tag{3.26}$$

Then,

$$\hbar(z)\prec\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}, \tag{3.27}$$

and $\hbar$ is the best subordinant of (3.26).

Proof. The purpose is to employ Lemma 1.2. From (3.4), (3.5) and (3.26), we gain superordinate relation

$$\Upsilon_1+\Upsilon_2\,\hbar(z)+\eta\frac{z\hbar'(z)}{\hbar(z)}\prec\Upsilon_1+\Upsilon_2\,\omega(z)+\eta\frac{z\omega'(z)}{\omega(z)}$$

$$=(\Upsilon_1+\eta)+(\Upsilon_2-\eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)}+\eta\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}. \tag{3.28}$$

By utilizing (3.8) and (3.25), we deduce

$$\mathcal{R}e\left\{\frac{z\,\phi'(\hbar(z))}{\psi(\hbar(z))}\right\}=\mathcal{R}e\left\{\frac{\Upsilon_2}{\frac{\eta}{\hbar(z)}}\right\}=\mathcal{R}e\left\{\frac{\Upsilon_2}{\eta}\hbar(z)\right\}>0. \tag{3.29}$$

Relation (3.27) is then followed by an implementation of Lemma 1.2.

Theorem 3.4. Let $\Upsilon_\iota\in\mathbb{C}\ (\iota=1,2)$, $\eta\in\mathbb{C}\backslash(0)$ and $\hbar(z)$ be a convex univalent in Λ, $\hbar(0)=1$, $\hbar(z)\neq0\ (z\in\Lambda)$, and assume that $\hbar$ achieves the inequality given by (3.25).

If $\varphi\in\mathfrak{A}$, $\left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho}\in\mathcal{H}[1,1]\cap\Xi$, and

$$\Upsilon_1 + \Upsilon_2 \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho$$

is univalent in Λ and

$$\Upsilon_1 + \Upsilon_2\, \hbar(z) + \eta\frac{z\hbar'(z)}{\hbar(z)} \prec \Upsilon_1 + \Upsilon_2 \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho. \qquad (3.30)$$

Then,

$$\hbar(z) \prec \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho}, \qquad (3.31)$$

and $\hbar$ is the best subordinant of (3.27).

Proof. The aim is to utilize Lemma 1.2. From (3.17), (3.18) and (3.30), we deduce the superordination relation

$$\Upsilon_1 + \Upsilon_2\, \hbar(z) + \eta\frac{z\hbar'(z)}{\hbar(z)} \prec \Upsilon_1 + \Upsilon_2\, \omega(z) + \eta\frac{z\omega'(z)}{\omega(z)}$$
$$= \Upsilon_1 + \Upsilon_2 \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho. \qquad (3.32)$$

By utilizing (3.8) and (3.25), we gain the relation (3.31), and then it is followed by an implementation of Lemma 1.2.

Remark 3.1. Analogously, by utilizing Corollary 3.1, Corollary 3.2, Corollary 3.3, and Corollary 3.4, we yield superordination Corollaries.

Theorems 3.1 and Theorems 3.3 are combined to attain the ensuing Sandwich Subordinations.

Theorem 3.5. Let $\Upsilon_\iota \in \mathbb{C}$ $(\iota = 1,2)$, $\eta \in \mathbb{C}\backslash(0)$ and $\hbar_1$ and $\hbar_2$ be convex univalent function in Λ, with $\hbar_1(0) = \hbar_2(0) = 1$. Suppose $\hbar_1$ achieves (3.25) and $\hbar_2$ achieves (3.1). Let $\frac{z\hbar'_\iota(z)'}{\hbar_\iota(z)}$ $(\iota = 1,2)$ is starlike univalent in Λ. Let $\varphi \in \mathfrak{A}$ achieves $\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} \in \mathcal{H}\,[1,1] \cap \Xi$ and $(\Upsilon_1 + \eta) + (\Upsilon_2 - \eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + \eta\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$ is univalent in Λ, and the subordination and superordination relation

$$\Upsilon_1 + \Upsilon_2\, \hbar_1(z) + \eta\frac{z\hbar_1'(z)}{\hbar_1(z)} \prec (\Upsilon_1 + \eta) + (\Upsilon_2 - \eta)\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + \eta\frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$$

$$\prec \Upsilon_1 + \Upsilon_2\, \hbar_2(z) + \eta\frac{z\hbar_2'(z)}{\hbar_2(z)} \qquad (3.33)$$

hold. Then $\hbar_1(z) \prec \frac{z[\mathcal{L}_{\mu,\sigma}\varphi(z)]'}{\mathcal{L}_{\mu}\varphi(z)} \prec \hbar_2(z)$, and $\hbar_1$ and $\hbar_2$ are consecutively the best subordinate and best dominate.

In view of Theorem 3.5, we acquire the following exciting outcomes.

Corollary 3.5. Let the hypotheses of Theorem 3.5 hold. Then, the subordination and superordination relation

$$\Upsilon_1 + \Upsilon_2\left(\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}\right) + \eta\frac{(\mathcal{M}_1 - \mathcal{N}_1)z}{(1+\mathcal{M}_1 z)\,(1+\mathcal{N}_1 z)} \prec (\Upsilon_1+\eta) + (\Upsilon_2-\eta)\frac{z\left[\mathcal{L}_{\mu,\sigma}\varphi(z)\right]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} + \eta\,\frac{z\left[\mathcal{L}_{\mu,\sigma}\varphi(z)\right]''}{\mathcal{L}_{\mu,\sigma}\varphi(z)}$$

$$\prec \Upsilon_1 + \Upsilon_2\left(\frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}\right) + \eta\frac{(\mathcal{M}_2 - \mathcal{N}_2)z}{(1+\mathcal{M}_2 z)\,(1+\mathcal{N}_2 z)}, \qquad (3.34)$$

which implies that $\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z} \prec \frac{z\left[\mathcal{L}_{\mu,\sigma}\varphi(z)\right]'}{\mathcal{L}_{\mu,\sigma}\varphi(z)} \prec \frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}$, $-1 \le \mathcal{N}_1 < \mathcal{M}_1 \le 1, -1 \le \mathcal{N}_2 < \mathcal{M}_2 \le 1$, and $\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}$ and $\frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}$ are consecutively the best subordinate and best dominate.

Proof: By letting $\hbar_1(z) = \frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}$ and $\hbar_2(z) = \frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}$ in Theorem 3.5, we attain the required assertion.

Corollary 3.6. Let the hypotheses of Theorem 3.5 hold. Then, the subordination and superordination relation

$$\Upsilon_2\,\hbar_1(z) + \eta\frac{z\hbar_1'(z)}{\hbar_1(z)} \prec \Upsilon_2\,\frac{z\varphi'(z)}{\varphi(z)} + \eta\left\{1 + \frac{z\varphi''(z)}{\varphi'(z)} - \frac{z\varphi'(z)}{\varphi(z)}\right\} \prec \Upsilon_2\,\hbar_2(z) + \eta\frac{z\hbar_2'(z)}{\hbar_2(z)} \qquad (3.35)$$

which implies that $\hbar_1(z) \prec \frac{z\varphi'(z)}{\varphi(z)} \prec \hbar_2(z)$, and $\hbar_1$ and $\hbar_2$ are consecutively the best subordinate and best dominate.

Proof: By putting $\mu = \sigma = 1$ and $\Upsilon_1 = 0$ in Theorem 3.5, we gain the required assertion.

Corollary 3.7. Let the hypotheses of Theorem 3.5 hold. Then, the subordination and superordination relation

$$\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z} + \frac{(\mathcal{M}_1 - \mathcal{N}_1)z}{(1+\mathcal{M}_1 z)\,(1+\mathcal{N}_1 z)} \prec 1 + \frac{z\varphi''(z)}{\varphi'(z)} \prec \frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z} + \frac{(\mathcal{M}_2 - \mathcal{N}_2)z}{(1+\mathcal{M}_2 z)\,(1+\mathcal{N}_2 z)}, \qquad (3.36)$$

which implies that $\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z} \prec \frac{z\varphi'(z)}{\varphi(z)} \prec \frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}$, $-1 \le \mathcal{N}_1 < \mathcal{M}_1 \le 1, -1 \le \mathcal{N}_2 < \mathcal{M}_2 \le 1$, and $\frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}$ and $\frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}$ are consecutively the best subordinate and best dominate.

Proof: By setting $\mu = \sigma = 1$, $\Upsilon_1 = 0$, $\Upsilon_2 = \eta = 1, \hbar_1(z) = \frac{1+\mathcal{M}_1 z}{1+\mathcal{N}_1 z}$ and $\hbar_2(z) = \frac{1+\mathcal{M}_2 z}{1+\mathcal{N}_2 z}$ in Theorem 3.5, we derive the required assertion.

Theorems 3.2 and Theorems 3.4 are combined to attain the ensuing Sandwich Subordinations**.**

Theorem 3.6. Let $\Upsilon_\iota \in \mathbb{C}\ (\iota = 1,2)$, $\eta \in \mathbb{C}\backslash(0)$ and $\hbar_1$and $\hbar_2$ be convex univalent function in Λ, with $\hbar_1(0) = \hbar_2(0) = 1$. Suppose $\hbar_1$ achieves (3.25) and $\hbar_2$ achieves (3.1). Let $\frac{z\hbar'_\iota(z)'}{\hbar_\iota(z)}$ $(\iota = 1,2)$ is starlike univalent in Λ. Let $\varphi \in \mathfrak{A}$ achieves $\left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} \in \mathcal{H}\,[1,1] \cap \Xi$ and

$$\Upsilon_1 + \Upsilon_2 \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho$$

is univalent in Λ, subordination and superordination relation

$$\Upsilon_1 + \Upsilon_2\, \hbar_1(z) + \eta\frac{z\hbar_1'(z)}{\hbar_1(z)} \prec \Upsilon_1 + \Upsilon_2 \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho \prec \Upsilon_1 + \Upsilon_2\, \hbar_2(z) + \eta\frac{z\hbar_2'(z)}{\hbar_2(z)} \quad (3.37)$$

hold. Then $\hbar_1(\mathrm{z}) \prec \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} \prec \hbar_2(\mathrm{z})$, and $\hbar_1$ and $\hbar_2$ are consecutively the best subordinate and best dominate.

In view of Theorem 3.6, we yield the following outcome.

Corollary 3.4. Let the hypotheses of Theorem 3.2 hold. Then, the subordination and superordination relation

$$\Upsilon_1 + \Upsilon_2\left(\frac{1+\mathcal{M}_1\mathrm{z}}{1+\mathcal{N}_1 z}\right) + \eta\frac{(\mathcal{M}_1-\mathcal{N}_1)z}{(1+\mathcal{M}_1 z)\,(1+\mathcal{N}_1 z)} \prec \Upsilon_1 + \Upsilon_2 \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} + \frac{\varrho\eta(\mu-1)}{\sigma}\frac{z\mathcal{L}_{\mu,\sigma}\varphi(z)}{\mathcal{L}_{\mu-1,\sigma}\varphi(z)} - \frac{\varrho\eta\mu}{\sigma}\frac{z\mathcal{L}_{\mu+1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)} - \frac{\varrho\eta(2\mu-1)}{\sigma} + 2\eta\varrho \prec \Upsilon_1 + \Upsilon_2\left(\frac{1+\mathcal{M}_2\mathrm{z}}{1+\mathcal{N}_2 z}\right) + \eta\frac{(\mathcal{M}_2-\mathcal{N}_2)z}{(1+\mathcal{M}_2 z)\,(1+\mathcal{N}_2 z)}, \quad (3.38)$$

which implies that $\frac{1+\mathcal{M}_1\mathrm{z}}{1+\mathcal{N}_1 z} \prec \left(\frac{z\mathcal{L}_{\mu-1,\sigma}\varphi(z)}{\mathcal{L}_{\mu,\sigma}\varphi(z)}\right)^{\varrho} \prec \frac{1+\mathcal{M}_2\mathrm{z}}{1+\mathcal{N}_2 z}$, $-1 \le \mathcal{N}_1 < \mathcal{M}_1 \le 1$, $-1 \le \mathcal{N}_2 < \mathcal{M}_2 \le 1$, and $\frac{1+\mathcal{M}_1\mathrm{z}}{1+\mathcal{N}_1 z}$ and $\frac{1+\mathcal{M}_2\mathrm{z}}{1+\mathcal{N}_2 z}$ are consecutively the best subordinate and best dominate.

Proof: By letting $\hbar_1(z) = \frac{1+\mathcal{M}_1\mathrm{z}}{1+\mathcal{N}_1 z}$ and $\hbar_2(z) = \frac{1+\mathcal{M}_2\mathrm{z}}{1+\mathcal{N}_2 z}$ in Theorem 3.5, we derive the required assertion, we attain the required assertion.

4. Conclusion

During this research, the remarkable contribution is to explore a new special function as an amended and generalized formula of the Koebe function based on the complex gamma function principle so that the Koebe function as a private case can be derived from it. Then, its action on operator theory over a specific complex field is attainable in view of convolution structure. This affords visions of the emergence of a new operator as one of the main consequences of this study. The proposed complex operator is a general formulation of an interesting operator that is the Ruscheweyh operator. Furthermore, by applying subordination and superordination methodology, the validated conclusions are sandwich outcomes that include this innovative operator. For future investigations, discuss by proposing generalizations and modifications to suggest Koebe-type functions and create numerous subclasses of holomorphic functions in the sense of multivalent and harmonic functions. Accordingly, the development offered in this sequel will motivate further attention and discussion in this significant area of mathematics.

References

[1] F. Ghanim, S. Bendak and A. Al-Hawarneh, "Certain implementations in fractional calculus operators involving Mittag-Leffler_confluent hypergeometric functions", *Proceedings of the Royal Society a Mathematical, Physical and Engineering Sciences*, vol. 478, no. 2258, pp. 1-14, 2020.
[2] F. Ghanim and H. F. Al-Janaby, " Some analytical merits of Kummer-Type function associated with Mittag-Leffler parameters", *Arab Journal of Basic and Applied Sciences*, vol. 28, no. 1, pp. 255–263, 2021.
[3] A. Tassaddiq, "Some Difference equations for Srivastava's λ −generalized Hurwitz- Lerch zeta functions with applications", *Symmetry*, vol. 11, no. 311, pp. 1-1, 2019.
[4] L. De Branges, "A proof of the Bieberbach conjecture", *Acta Mathematica*, vol. 154, no. (1-2), pp. 137-152, 1984.
[5] O. Ozkan and O. Atintas, "Applications of differential subordination", *Appl. Math. Lett*., vol. 19, pp. 728–734, 2006.
[6] S. Ruscheweyh, "New criteria for univalent functions", *Proc. Amer. Math. Soc*., vol. 49, pp. 109–115, 1975.
[7] A. W. Goodman, *Univalent functions. I. Tampa*, Florida, USA*:* Mariner, 1983.
[8] E. Study, *Konforme Abbildung Einfachzusammenhangender Bereiche*, Leipzig und Berlin, B. C. Teubner, 1913.
[9] R. Nevanlinna, "Über die konforme Abbildund Sterngebieten", *Oversikt av Finska–Vetenskaps Societen Forhandlingar*, vol. 63A, no 6, pp. 48–403, 1921.
[10] M. I. S. Robertson, "On the theory of univalent functions", *Ann. of Math*., vol. 379, no. 2, pp. 374-408, 1936.
[11] S.S. Miller and P.T. Mocanu, “Differential subordinations and univalent functions", *Michigan Math. J*., vol. 28, PP. 157-171, 1981.
[12] S.S. Miller and P.T. Mocanu, *Differential Subordinations, Theory and applications. A series of Monographs and Textbooks in Pure and Applied Mathematics*, New York, 480 Dekker, 2000.
[13] S.S. Miller and P.T. Mocanu, " Subordinants of differential superordinations", *Complex Variables. Theory and Application*, vol. 48, no. 10, PP. 815-826, 2003.
[14] G.M. Goluzin, " On the majorization principle in function theory (Russian)", *Doklady Akademii Nauk SSSR*, vol. 42, pp. 647-650, 1935.
[15] T. Bulboacă, "A class of superordination-preserving integral operators", *Indagationes Mathematicae (N.S.)*, vol. 13, no. 3, pp. 301-311, 2002.
[16] T. Bulboacă, " Classes of first-order differential superordinations", *Demonstratio Mathematica*, vol. 35, no. 2, pp. 287-292, 2002.
[17] W. G. Atshan, A. H. Majeed, K. A. Jassim, “Some Geometric Properties of a Certain Subclass of Univalent Functions Defined by Differential Subordination Property”, *General Mathematics Notes*, vol. 20, no. 2, pp. 79-94, 2014.
[18] J. Sokół, R. W. Ibrahim, M. Z. Ahmad and H. F. Al-Janaby, "Inequalities of harmonic univalent functions with connections of hypergeometric functions", *Open Mathematics*, vol. 13, no. 1, pp. 1-15, 2015.
[19] H. Zayed and T. Bulboacă, "Sandwich results for higher order fractional derivative operators", *MatematychniStudii*, vol. 49, no. 1, pp. 52-66, 2018.
[20] H. F. Al-Janaby, M. Z. Ahmad, “Differential inequalities related to Salagean type integral operator involving extended generalized Mittag-Leffler function”, *Journal of Physics: Conference Series*, vol. 1132, pp. 1-9, 2018.
[21] A. K. Mishra and M. M. Soren, "Sandwich Results for subclasses of Multivalent meromorphic functions associated with iterations of the Cho-Kwon-Srivastava transform", *Filomat*, vol. 33, no.1, pp. 255-266, 2019.
[22] H. F. Al-Janaby and M. Darus, “Differential subordination results for Mittag-Leffler type functions with bounded turning property”, *Mathematica Slovaca*, vol. 69, no. 3, pp. 573-582, 2019.
[23] H. F. Al-Janaby, F. Ghanim and M. Darus, “Some geometric properties of integral operators proposed by Hurwitz-Lerch Zeta function”, *Journal of Physics: Conference Series*, vol. 1212, pp. 1-8, 2019.
[24] H. F. Al-Janaby, F. Ghanim and M. Darus, “On the third-order complex differential inequalities of ξ-generalized-Hurwitz-Lerch Zeta functions”, *Mathematics*, vol. 8, no. 845, pp. 1-21, 2020.

[25] F. Ghanim and H. F. Al-Janaby, "A certain subclass of univalent meromorphic functions defined by a linear operator associated with the Hurwitz-Lerch zeta function", *Rad Hrvatske akademije znanosti i umjetnosti: Matematičke znanosti*, vol. 538, no. 23, pp. 71-83, 2019.
[26] F. Ghanim, H. F. Al-Janaby, "Inclusion and convolution features of univalent meromorphic functions correlating with Mittag-Leffler function", *Filomat*, vol. 34, no. 7, pp. 2141-2150, 2020.
[27] A. A. Attiy, O. S. Kwon, P. J. Hyang and N. E. Cho, "An Integro differential operator for meromorphic functions associated with the Hurwitz- Lerch zeta function", *Filomat*, vol. 30, no. 7, pp. 2045-2057, 2016.
[28] A. A. Attiy, M. K. Aouf, E. E. Ali and M. F. Yassen, "Differential subordination and superordination results associated with Mittag-Leffler function", *Mathematics*, vol. 9, no. 226, pp. 1-11, 2021.
[29] A. A. Lupaş and G. I. Oros, "Differential Subordination and Superordination Results Using Fractional Integral of Confluent Hypergeometric Function", *Symmetry*, vol. 13, no. 327, pp. 1-10, 2021.
[30] W. G. Atshan, R. Abd Al-sajjad, "Some Applications of Quasi-Subordination for Bi-Univalent Functions Using Jackson's Convolution Operator", *Iraqi Journal of Science*, vol. 63, no. 10, pp. 4417-4428, 2022.
[31] S. D. Theyab, W. G. Atshan, H. K. Abdullah, "On Some Sandwich Results of Univalent Functions Related by Differential Operator", *Iraqi Journal of Science*, vol. 63, no. 11, pp. 4928-4936, 2022.
[32] S. S. Miller and P.T. Mocanu, "Briot-Boquet differential superordinations and sandwich theorems", *J. Math. Anal. Appl*., vol. 329, pp. 327-335, 2007.
[33] E. C. Titchmarsh, *The Theory of Functions. Oxford University Press*, London, Amen House, 1964.
[34] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions. Dover Publications*, New York, Inc., 1968.
[35] V. Ravichandran and M. Jayamala, "On sufficient conditions for Caratheodory functions", *Far East Journal of Mathematical Sciences*, vol. 12, pp. 191-201, 2004.